Cálculo de inversas de matrices usando la factorización LU

A es invertible → se calcula el determinante y sea diferente de 0

2×2 → $A^{-1} = ?$ $A.A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A. 1era columna de A^{-1}

1era columna de $A.A^{-1}$ → (1,1) de $A.A^{-1}$ → 1era fila de A. 1era colum. A^{-1}

(2,1) de $A.A^{-1}$ → $\underbrace{\text{2do fila de A.}}_{A}$ 1era colum. A^{-1}

1era columna de $A.A^{-1}$ = A. 1era columna de A^{-1} → ¿1

2da columna de $A.A^{-1}$ = A. 2da columna de A^{-1} → ¿2

¿1 $A.\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x_1, x_2$

¿2 $A.\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow x_1, x_2$

Ejemplo

$A = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ $\Delta = 1$ $A^{-1} = ?$

$\boxed{\text{1era columna de } A^{-1}}$

$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow A = LU$ → Tener la factorización LU de A

$L.U\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

→ Matriz triangular inferior

$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$

$\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{pmatrix}$

$u_{11} = -1 \quad , \quad u_{12} = 2 \qquad \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$

$l_{21} u_{11} = -2 \Rightarrow l_{21}(-1) = -2 \Rightarrow l_{21} = 2$

$l_{21} u_{12} + u_{22} = 3 \Rightarrow 2 \cdot 2 + u_{22} = 3 \Rightarrow 4 + u_{22} = 3 \Rightarrow u_{22} = -1$

$L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} , \quad U = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

$L \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} z_1 = 1 \\ 2z_1 + z_2 = 0 \end{cases}$

$2 \cdot 1 + z_2 = 0 \Rightarrow z_2 = -2$

$z = U \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$U \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$-x_1 + 2x_2 = 1 \qquad -x_1 + 2(2) = 1$

$-x_2 = -2 \qquad x_2 = 2 \qquad x_1 = 3$

2ª columna de A^{-1}

$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$L \cdot \underbrace{U \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad L \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{cases} z_1 = 0 \\ 2z_1 + z_2 = 1 \Rightarrow z_2 = 1 \end{cases}$

$U \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{cases} -x_1 + 2x_2 = 0 \\ -x_2 = 1 \end{cases} \qquad -x_1 - 2 = 0 \quad x_1 = -2 \qquad x_2 = -1$

$A^{-1} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \qquad A^{-1} = ? \qquad A \cdot A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1ra columna de A^{-1}

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A = L \cdot U$$

$$\begin{pmatrix} -1 & 2 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix}$$

$\underline{u_{11}} = -1 \qquad \underline{u_{12}} = 2 \qquad \underline{u_{13}} = 3$

$l_{21}(-1) = 2 \Rightarrow \underline{l_{21}} = -2$

$l_{21}u_{12} + u_{22} = -1 \qquad (-2)(2) + u_{22} = -1 \Rightarrow \underline{u_{22}} = 3$

$l_{21}u_{13} + u_{23} = 1 \qquad (-2)(3) + u_{23} = 1 \Rightarrow \underline{u_{23}} = 7$

$l_{31}u_{11} = -1 \qquad l_{31}(-1) = -1 \Rightarrow \underline{l_{31}} = 1$

$l_{31}u_{12} + l_{32}u_{22} = 1 \qquad (1)(2) + l_{32}(3) = 1 \Rightarrow \underline{l_{32}} = -1/3$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \qquad (1)(3) + (-1/3)(7) + u_{33} = 1 \Rightarrow \underline{u_{33}} = 1/3$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1/3 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 3 & 7 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$$L \cdot \underbrace{U \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{Z} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1/3 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} z_1 = 1 \\ -2z_1 + z_2 = 0 & z_2 = 2 \\ z_1 - 1/3 z_2 + z_3 = 0 \end{cases}$$

$z_3 = -1/3$

$$\begin{pmatrix} -1 & 2 & 3 \\ 0 & 3 & 7 \\ 0 & 0 & 1/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1/3 \end{pmatrix} \Rightarrow \begin{cases} -x_1 + 2x_2 + 3x_3 = 1 \\ 3x_2 + 7x_3 = 2 \\ 1/3 x_3 = -1/3 \end{cases} \quad \begin{matrix} x_1 = 2 \\ x_2 = 3 \\ x_3 = -1 \end{matrix}$$

2da Colomna de A^{-1}

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$L \cdot \underbrace{U \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{Z} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1/3 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} z_1 = 0 \\ -2z_1 + z_2 = 1 \quad z_2 = 1 \\ z_1 - 1/3 z_2 + z_3 = 0 \end{cases}$$

$$z_3 = 1/3$$

$$\begin{pmatrix} -1 & 2 & 3 \\ 0 & 3 & 7 \\ 0 & 0 & 1/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1/3 \end{pmatrix} = \begin{cases} -x_1 + 2x_2 + 3x_3 = 0 \\ 3x_2 + 7x_3 = 1 \\ 1/3 x_3 = 1/3 \end{cases} \quad \begin{matrix} x_1 = -1 \\ x_2 = -2 \\ x_3 = 1 \end{matrix}$$

3ra columna de A^{-1}

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$L \cdot \underbrace{U \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{Z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1/3 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} z_1 = 0 \\ -2z_1 + z_2 = 0 \quad z_2 = 0 \\ z_1 - 1/3 z_2 + z_3 = 1 \end{cases}$$

$$z_3 = 1$$

$$\begin{pmatrix} -1 & 2 & 3 \\ 0 & 3 & 7 \\ 0 & 0 & 1/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{cases} -x_1 + 2x_2 + 3x_3 = 0 \\ 3x_2 + 7x_3 = 0 \\ 1/3 x_3 = 1 \end{cases} \quad \begin{matrix} x_1 = -5 \\ x_2 = -7 \\ x_3 = 3 \end{matrix}$$

$$A^{-1} = \begin{pmatrix} 2 & -1 & -5 \\ 3 & -2 & -7 \\ -1 & 1 & 3 \end{pmatrix}$$