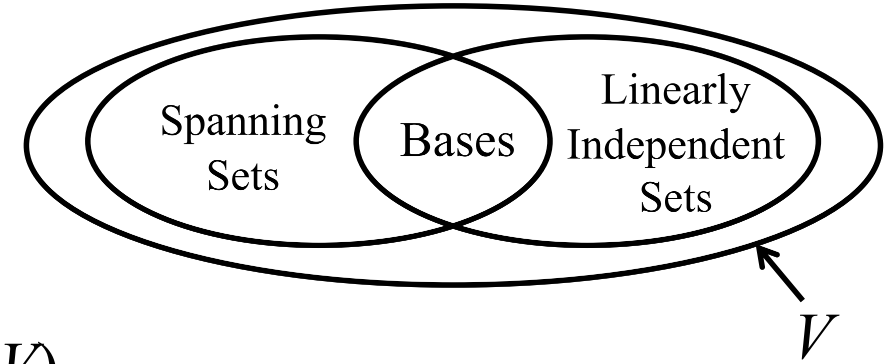
1.5 Basis and Dimension

- **Basis:**



V: a vector space

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$

$$\begin{cases} (1) & S \text{ spans } V \text{ (i.e., span}(S) = V) \\ (2) & S \text{ is linearly independent} \end{cases}$$

(For any $\mathbf{u} \in V$, $\sum c_i \mathbf{v}_i = A\mathbf{x} = \mathbf{u}$ has at least one solution (If there is exact one solution ($\det(A) \neq 0$) or if there are infinitely many solutions ($\det(A) = 0$))

(For $\sum c_i \mathbf{v}_i = A\mathbf{x} = \mathbf{0}$, there is only the trivial solution ($\det(A) \neq 0$). See the definition on Slide 4.49 and Ex 8 on Slide 4.50)

$\Rightarrow S$ is called a basis for V ($\Rightarrow$ For $\sum c_i \mathbf{v}_i = A\mathbf{x}$, $\det(A) \neq 0$)

- **Notes:**

A basis S must have enough vectors to span V, but not so many vectors that one of them could be written as a linear combination of the other vectors in S

- Notes:

(1) the **standard basis** for R^3:

$\{i, j, k\}$, for $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

(2) the **standard basis** for R^n :

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, for $\mathbf{e}_1=(1,0,\ldots,0)$, $\mathbf{e}_2=(0,1,\ldots,0),\ldots,$ $\mathbf{e}_n=(0,0,\ldots,1)$

Ex: For R^4, $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

※Express any vector in R^n as the linear combination of the vectors in the standard basis: the coefficient for each vector in the standard basis is the value of the corresponding component of the examined vector, e.g., (1, 3, 2) can be expressed as $1\cdot(1, 0, 0) + 3\cdot(0, 1, 0) + 2\cdot(0, 0, 1)$

(3) the **standard basis** for $m\times n$ matrix space:

$\{ E_{ij} \mid 1\le i\le m ,\ 1\le j\le n \}$, and in E_{ij} $\begin{cases} a_{ij} = 1 \\ \text{other entries are zero} \end{cases}$

Ex: 2×2 matrix space:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(4) the **standard basis** for $P_n(x)$:

$\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x)$ $\{1, x, x^2, x^3\}$

- Ex 2: The nonstandard basis for R^2

Show that $S=\{\mathbf{v}_1,\mathbf{v}_2\}=\{(1,1),\ (1,-1)\}$ is a basis for R^2

(1) For any $\mathbf{u}=(u_1,u_2)\in R^2$, $c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{u}\ \Rightarrow\ \begin{cases} c_1+c_2=u_1 \\ c_1-c_2=u_2 \end{cases}$

Because the coefficient matrix of this system has a **nonzero determinant**, the system has a unique solution for each **u**. Thus you can conclude that S spans R^2

(2) For $c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{0}\ \Rightarrow\ \begin{cases} c_1+c_2=0 \\ c_1-c_2=0 \end{cases}$

Because the coefficient matrix of this system has a **nonzero determinant**, you know that the system has only the trivial solution. Thus you can conclude that S is linearly independent

According to the above two arguments, we can conclude that S is a (nonstandard) basis for R^2

- Theorem 1.8: Uniqueness of basis representation for any vectors

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space *V*, then every vector in *V* can be written in one and only one way as a linear combination of vectors in *S*

Pf:

$$\because S \text{ is a basis} \Rightarrow \begin{cases} (1)\ \text{span}(S) = V \\ (2)\ S \text{ is linearly independent} \end{cases}$$

$\because \text{span}(S) = V$ Let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$

$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n$

$\Rightarrow \mathbf{v} + (-1)\mathbf{v} = \mathbf{0} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \ldots + (c_n - b_n)\mathbf{v}_n$

$\because S$ is linearly independent $\Rightarrow$ with only the trivial solution

$\Rightarrow$ coefficients for $\mathbf{v}_i$ are all zero

$\Rightarrow c_1 = b_1,\ c_2 = b_2, \ldots, c_n = b_n$ (i.e., unique basis representation)

Theorem 1.9: Bases and linear dependence

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent (In other words, every linearly independent set contains at most n vectors)

Pf:

Let $S_1=\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, $m>n$

$\because \text{span}(S)=V$

$$\mathbf{u}_i \in V \quad \Rightarrow \quad \begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1+c_{21}\mathbf{v}_2+\cdots+c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1+c_{22}\mathbf{v}_2+\cdots+c_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1+c_{2m}\mathbf{v}_2+\cdots+c_{nm}\mathbf{v}_n \end{aligned}$$

Consider $k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$

(if k_i's are not all zero, S_1 is linearly dependent)

$\Rightarrow d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_n\mathbf{v}_n=\mathbf{0}$ $(d_i=c_{i1}k_1+c_{i2}k_2+\ldots+c_{im}k_m)$

$\because S$ is L.I. $\Rightarrow$ $d_i=0 \quad \forall i$ i.e.,

$$\begin{aligned}
c_{11}k_1+c_{12}k_2+\cdots+c_{1m}k_m&=0\\
c_{21}k_1+c_{22}k_2+\cdots+c_{2m}k_m&=0\\
&\vdots\\
c_{n1}k_1+c_{n2}k_2+\cdots+c_{nm}k_m&=0
\end{aligned}$$

$\because$ Theorem 1.1: If the homogeneous system has fewer equations (n equations) than variables (k_1, k_2, …, k_m), then it must have infinitely many solutions

$\therefore$ $m>n\Rightarrow k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$ has nontrivial (nonzero) solution

$\Rightarrow S_1$ is linearly dependent

Theorem 1.10: Number of vectors in a basis

If a vector space V has one basis with n vectors, then every basis for V has n vectors

Pf: ※ According to Thm. 1.9, every linearly independent set contains at most n vectors in a vector space if there is a basis of n vectors spanning that vector space

$$\left.\begin{array}{l} S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \\ S'=\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \end{array}\right\} \text{are two bases with different sizes for } V$$

$$\left.\begin{array}{l} \left.\begin{array}{l} S \text{ is a basis spanning } V \\ S' \text{ is a set of L.I. vectors} \end{array}\right\} \Rightarrow m \leq n \\ \left.\begin{array}{l} S' \text{ is a basis spanning } V \\ S \text{ is a set of L.I. vectors} \end{array}\right\} \Rightarrow n \leq m \end{array}\right\} \Rightarrow n = m$$

※ For R^3, since the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ can span this vector space, you can infer any basis that can span R^3 must have exactly 3 vectors

※ For example, $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ in Ex 5 on Slide 4.44 is another basis of R^3 (because S can span R^3 and S is linearly independent), and S has 3 vectors

- Dimension:

The dimension of a vector space V is defined to be the number of vectors in a basis for V

V: a vector space $\qquad$ S: a basis for V

$\Rightarrow \dim(V) = \#(S)$ (the number of vectors in a basis S)

- Finite dimensional:

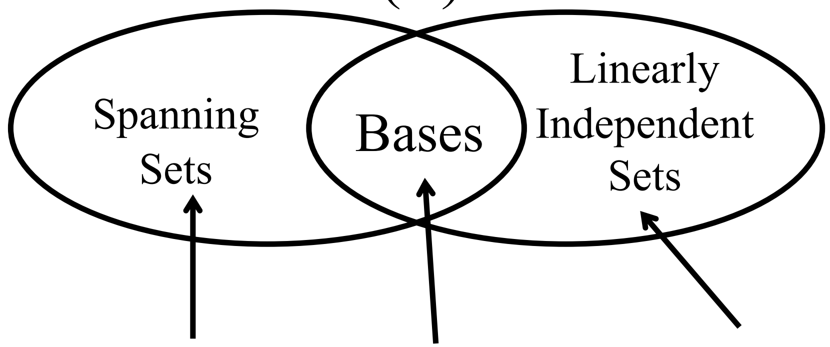
A vector space V is finite dimensional if it has a basis consisting of a finite number of elements

- Infinite dimensional:

If a vector space V is not finite dimensional, then it is called infinite dimensional

- Notes:

$\dim(V) = n$



$\#(S) \geq n$ $\quad \#(S) = n$ $\quad \#(S) \leq n$

(1) $\dim(\{\mathbf{0}\}) = 0$

(If V consists of the zero vector alone, the dimension of V is defined as zero)

(2) Given $\dim(V) = n$, for $S \subseteq V$

S: a spanning set $\Rightarrow$ $\#(S) \geq n$ (from Ex 5 on Slides 4.44 and 4.45)

S: a L.I. set $\Rightarrow$ $\#(S) \leq n$ (from Theorem 4.10)

S: a basis $\Rightarrow$ $\#(S) = n$ (Since a basis is defined to be a set of L.I. vectors that can spans V, $\#(S) = \dim(V) = n$) (see the above figure)

(3) Given $\dim(V) = n$, if W is a subspace of $V \Rightarrow \dim(W) \leq n$

※ For example, if $V = R^3$, you can infer the $\dim(V)$ is 3, which is the number of vectors in the standard basis

※ Considering $W = R^2$, which is a subspace of R^3, due to the number of vectors in the standard basis, we know that the $\dim(W)$ is 2, that is smaller than $\dim(V)$=3

- Ex: Find the dimension of a vector space according to the standard basis

※ The simplest way to find the dimension of a vector space is to count the number of vectors in the "standard" basis for that vector space

(1) Vector space R^n $\Rightarrow$ standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$$\Rightarrow \dim(R^n) = n$$

(2) Vector space $M_{m\times n}$ $\Rightarrow$ standard basis $\{E_{ij} \mid 1\le i\le m,\ 1\le j\le n\}$

and in E_{ij} $\begin{cases} a_{ij} = 1 \\ \text{other entries are zero} \end{cases}$

$$\Rightarrow \dim(M_{m\times n}) = mn$$

(3) Vector space $P_n(x)$ $\Rightarrow$ standard basis $\{1, x, x^2, \ldots, x^n\}$

$$\Rightarrow \dim(P_n(x)) = n+1$$

(4) Vector space $P(x)$ $\Rightarrow$ standard basis $\{1, x, x^2, \ldots\}$

$$\Rightarrow \dim(P(x)) = \infty$$

- Ex 9: Determining the dimension of a subspace of R^3

(a) $W = \{(d, c-d, c)$: c and d are real numbers$\}$

(b) $W = \{(2b, b, 0)$: b is a real number$\}$

Sol: (Hint: find a set of L.I. vectors that spans the subspace, i.e., find a basis for the subspace.)

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S spans W)

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2$

(b) $\because (2b, b, 0) = b(2, 1, 0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$

- Ex 11: Finding the dimension of a subspace of $M_{2\times 2}$

  Let W be the subspace of all symmetric matrices in $M_{2\times 2}$.

  What is the dimension of W?

Sol:

$$W=\left\{\begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a,b,c\in R\right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix}=a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow S=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W $\quad\Rightarrow \dim(W) = \#(S) = 3$

- Theorem 1.11: Methods to identify a basis in an n-dimensional space

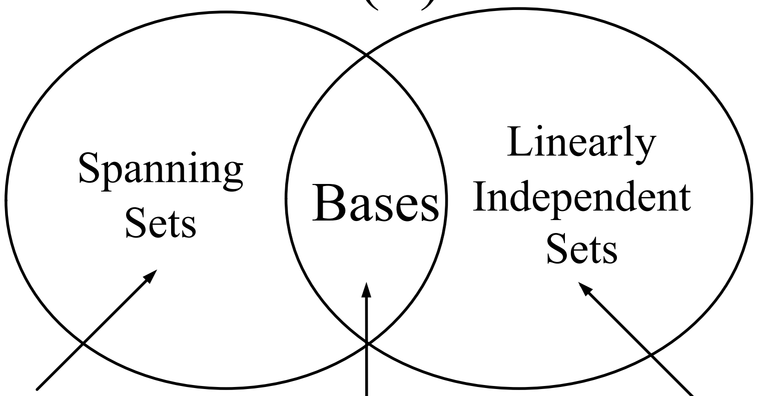
  Let V be a vector space of dimension n

  (1) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V

  (2) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ spans V, then S is a basis for V

  (Both results are due to the fact that $\#(S) = n$)

$\dim(V) = n$



$\#(S) \geq n$

$\#(S) = n$

$\#(S) \leq n$

Keywords in Section 1.5:

- basis
- dimension
- finite dimension
- infinite dimension