1.3 Subspaces of Vector Spaces

- Subspace:

$(V,+,\cdot)$: a vector space

$$\left.\begin{array}{l} W \neq \Phi \\ W \subseteq V \end{array}\right\} : \text{a nonempty subset of } V$$

$(W,+,\cdot)$: The nonempty subset W is called a subspace **if** W **is a vector space** under the operations of vector addition and scalar multiplication defined on V

- Trivial subspace:

Every vector space V has at least two subspaces

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V (It satisfies the ten axioms)

(2) V is a subspace of V

※ Any subspaces other than these two are called proper (or nontrivial) subspaces

- Examination of whether W being a subspace
  - Since the vector operations defined on W are the same as those defined on V, and most of the ten axioms inherit the properties for vector operations, it is not needed to verify those axioms
  - Therefore, the following theorem tells us it is sufficient to test for the closure conditions under vector addition and scalar multiplication to identify that a nonempty subset of a vector space is a subspace

- Theorem 1.4: Test whether a nonempty subset being a subspace

  If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold

  (1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W

  (2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W

Pf:

1. Note that if **u**, **v**, and **w** are in W, then they are also in V. Furthermore, W and V share the same operations. Consequently, vector space axioms 2, 3, 7, 8, 9, and 10 are satisfied automatically

2. Suppose that the closure conditions hold in Theorem 1.4, i.e., the axioms 1 and 6 for vector spaces are satisfied

3. Since the axiom 6 is satisfied (i.e., $c\mathbf{u}$ is in W if **u** is in W), we can obtain

3.1. for a scalar $c=0,\ c\mathbf{u}=\mathbf{0}\in W \ \Rightarrow\ \exists$ zero vector in W

$\Rightarrow$ axiom 4 is satisfied

3.2. for a scalar $c=-1,\ (-1)\mathbf{u}\in W \Rightarrow \exists\ -\mathbf{u}\equiv(-1)\mathbf{u}$

st. $\mathbf{u}+(-\mathbf{u})=\mathbf{u}+(-1)\mathbf{u}=\mathbf{0}$

$\Rightarrow$ axiom 5 is satisfied

- Ex 2: A subspace of $M_{2\times2}$

  Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times2}$, with the standard operations of matrix addition and scalar multiplication

Sol:

First, we knon that W, the set of all 2×2 symmetric matrices, is an nonempty subset of the vector space $M_{2\times2}$

Second,

$$A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \quad (A_1 + A_2 \in W)$$

$$c \in R,\ A \in W \Rightarrow (cA)^T = cA^T = cA \quad (cA \in W)$$

The definition of a symmetric matrix A is that $A^T = A$

- Ex 3: The set of singular matrices is not a subspace of $M_{2\times 2}$

  Let W be the set of singular (noninvertible) matrices of order 2. Show that W is not a subspace of $M_{2\times 2}$ with the standard matrix operations

Sol:

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in W,\ B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\in W$$

$$\because\ A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I\notin W$$ (W is not closed under vector addition)

$\therefore W$ is not a subspace of $M_{2\times 2}$

- Ex 4: The set of first-quadrant vectors is not a subspace of R^2

  Show that $W = \{(x_1, x_2) : x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2

Sol:

$$\text{Let } \mathbf{u} = (1,\ 1) \in W$$

$$\because (-1)\mathbf{u} = (-1)(1, 1) = (-1, -1) \notin W$$

(W is not closed under scalar multiplication)

$$\therefore W \text{ is not a subspace of } R^2$$

- Ex 6: Identify subspaces of R^2

Which of the following two subsets is a subspace of R^2?

(a) The set of points on the line given by $x+2y=0$

(b) The set of points on the line given by $x+2y=1$

Sol:

(a) $W=\{(x,y) \mid x+2y=0\}=\{(-2t,t) \mid t\in R\}$ (Note: the zero vector (0,0) is on this line)

Let $\mathbf{v}_1=\left(-2t_1,t_1\right)\in W$ and $\mathbf{v}_2=\left(-2t_2,t_2\right)\in W$

$\because\ \mathbf{v}_1+\mathbf{v}_2=\left(-2\left(t_1+t_2\right),t_1+t_2\right)\in W$ (closed under vector addition)

$c\mathbf{v}_1=\left(-2\left(ct_1\right),ct_1\right)\in W$ (closed under scalar multiplication)

$\therefore W$ is a subspace of R^2

(b) $W = \{(x, y) \mid x + 2y = 1\}$ (Note: the zero vector (0, 0) is not on this line)

Consider $\mathbf{v} = (1, 0) \in W$

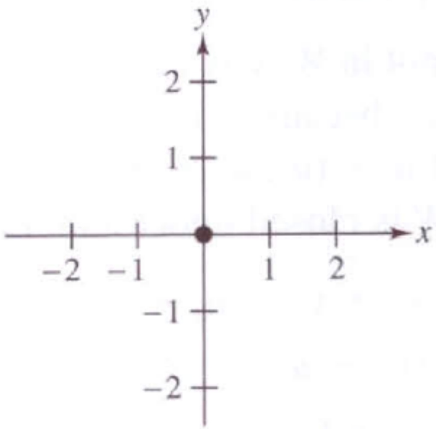
$\because (-1)\mathbf{v} = (-1, 0) \notin W \quad \therefore W$ is not a subspace of R^2

- Note: Subspaces of R^2

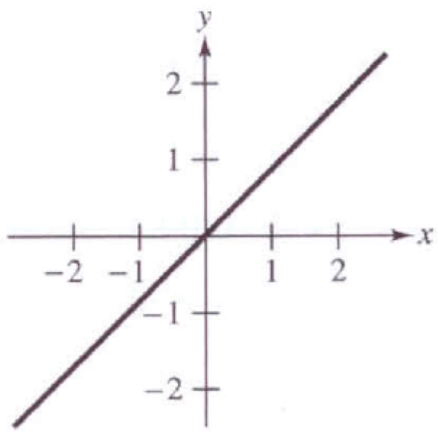
(1) W consists of the *single point* $\mathbf{0} = (0, 0)$ (trivial subspace)

(2) W consists of all points on a *line* passing through the origin

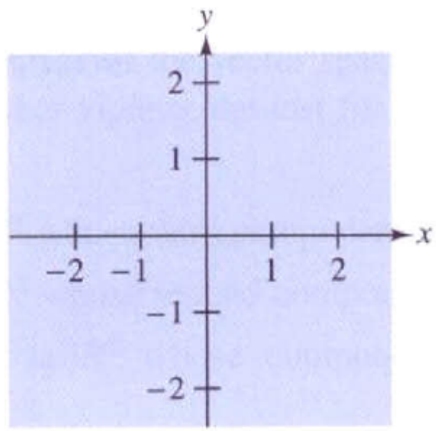
(3) R^2 (trivial subspace)



$W = \{(0, 0)\}$



$W =$ all points on a line passing through the origin



$W = R^2$

- Ex 8: Identify subspaces of R^3

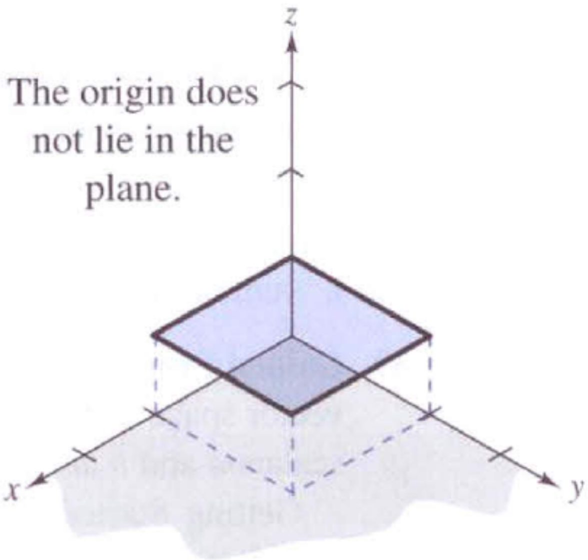
Which of the following subsets is a subspace of R^3?

(a) $W = \{(x_1, x_2, 1) \mid x_1, x_2 \in R\}$ (Note: the zero vector is not in W)

(b) $W = \{(x_1, x_1 + x_3, x_3) \mid x_1, x_3 \in R\}$ (Note: the zero vector is in W)

Sol:

(a)

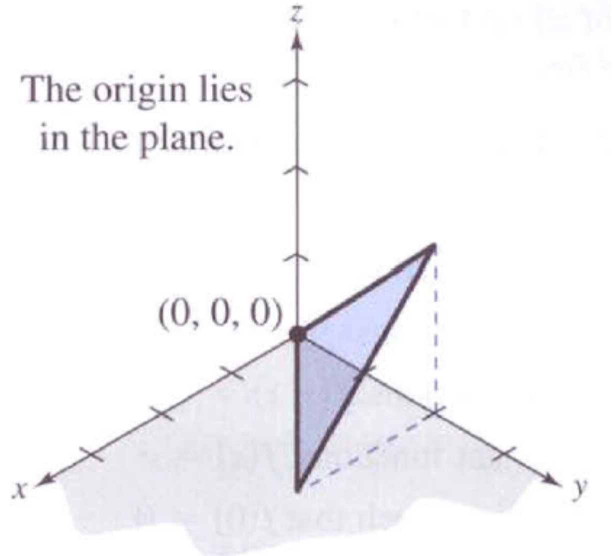


Consider $\mathbf{v} = (0, 0, 1) \in W$

$\because (-1)\mathbf{v} = (0, 0, -1) \notin W$

$\therefore W$ is not a subspace of R^3

(b)



Consider $\mathbf{v}=(v_1, v_1+v_3, v_3)\in W$ and $\mathbf{u}=(u_1, u_1+u_3, u_3)\in W$

$\because\ \mathbf{v}+\mathbf{u}=\left(v_1+u_1, \left(v_1+u_1\right)+\left(v_3+u_3\right), v_3+u_3\right)\in W$

$c\mathbf{v}=\left(cv_1, \left(cv_1\right)+\left(cv_3\right), cv_3\right)\in W$

$\therefore W$ is closed under vector addition and scalar multiplication,

so W is a subspace of R^3

- Note: Subspaces of R^3

(1) W consists of the *single point* $\mathbf{0}=(0,0,0)$ (trivial subspace)

(2) W consists of all points on a *line* passing through the origin

(3) W consists of all points on a *plane* passing through the origin

(The W in problem (b) is a plane passing through the origin)

(4) R^3 (trivial subspace)

※ According to Ex. 6 and Ex. 8, we can infer that if W is a subspace of a vector space V, then both W and V must contain the same zero vector $\mathbf{0}$

- Theorem 1.5: The intersection of two subspaces is a subspace

If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U

Pf:

(1) For $\mathbf{v}_1$ and $\mathbf{v}_2$ in $V \cap W$, since $\mathbf{v}_1$ and $\mathbf{v}_2$ are in V (and W), $\mathbf{v}_1 + \mathbf{v}_2$ is in V (and W). Therefore, $\mathbf{v}_1 + \mathbf{v}_2$ is in $V \cap W$

(2) For $\mathbf{v}_1$ in $V \cap W$, since $\mathbf{v}_1$ is in V (and W), $c\mathbf{v}_1$ is in V (and W). Therefore, $c\mathbf{v}_1$ is in $V \cap W$

Consequently, we can conclude the intersection of V and W ($V \cap W$) is also a subspace of U

※ However, the union of two subspaces is not a subspace

Keywords in Section 1.3:

- subspace
- trivial subspace