1.1 Vectors in R^n

- An ordered n-tuple:

  a sequence of n real numbers $(x_1, x_2, \cdots, x_n)$

- R^n-space:

  the set of all ordered n-tuples

$n = 1$ $\quad$ R^1-space = set of all real numbers
(R^1-space can be represented geometrically by the x-axis)

$n = 2$ $\quad$ R^2-space = set of all ordered pair of real numbers (x_1, x_2)
(R^2-space can be represented geometrically by the xy-plane)

$n = 3$ $\quad$ R^3-space = set of all ordered triple of real numbers (x_1, x_2, x_3)
(R^3-space can be represented geometrically by the xyz-space)

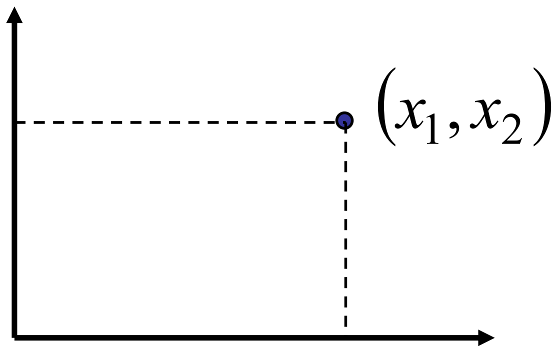
$n = 4$ $\quad$ R^4-space = set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

- Notes:

(1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as a point in R^n with the x_i's as its coordinates

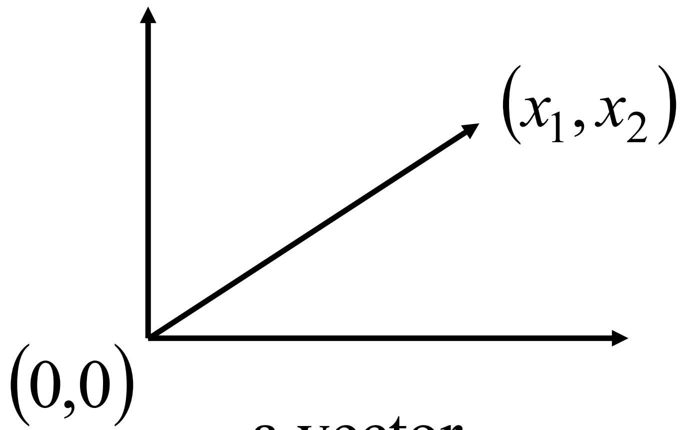
(2) An n-tuple $(x_1, x_2, \cdots, x_n)$ also can be viewed as a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components

- Ex:



a point

or



a vector

※ A vector on the plane is expressed geometrically by a directed line segment whose initial point is the origin and whose terminal point is the point (x_1, x_2)

$$\mathbf{u} = (u_1, u_2, \cdots, u_n), \quad \mathbf{v} = (v_1, v_2, \cdots, v_n) \qquad \text{(two vectors in } R^n\text{)}$$

- Equality:

$\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1,\ u_2 = v_2,\ \cdots,\ u_n = v_n$

- Vector addition (the sum of **u** and **v**):

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$

- Scalar multiplication (the scalar multiple of **u** by *c*):

$$c\mathbf{u} = (cu_1, cu_2, \cdots, cu_n)$$

- Notes:

The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n

- Difference between **u** and **v**:

$$\mathbf{u}-\mathbf{v} \equiv \mathbf{u}+(-1)\mathbf{v} = (u_1 - v_1,\ u_2 - v_2,\ u_3 - v_3, \ldots,\ u_n - v_n)$$

- Zero vector:

$$\mathbf{0} = (0, 0, \ldots, 0)$$

- Notes:

A vector $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ in R^n can be viewed as:

Use comma to separate components

a $1\times n$ row matrix (row vector): $\mathbf{u}=[u_1\ u_2\ \cdots\ u_n]$

Use blank space to separate entries

or

a $n\times 1$ column matrix (column vector): $\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

※ Therefore, the operations of matrix addition and scalar multiplication generate the same results as the corresponding vector operations (see the next slide)

Vector addition

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

Scalar multiplication

$$c\mathbf{u}=c(u_1,\ u_2,\cdots,\ u_n)$$
$$=(cu_1,cu_2,\cdots,\ cu_n)$$

Regarded as $1\times n$ row matrix

$$\mathbf{u}+\mathbf{v}=[u_1\ \ u_2\ \ \cdots\ \ u_n]+[v_1\ \ v_2\ \ \cdots\ \ v_n]$$
$$=[u_1+v_1\ \ u_2+v_2\ \ \cdots\ \ u_n+v_n]$$

$$c\mathbf{u}=c[u_1\ u_2\cdots u_n]$$
$$=[cu_1\ \ cu_2\ \ \cdots\ \ cu_n]$$

Regarded as $n\times 1$ column matrix

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}$$

$$c\mathbf{u}=c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$$

Theorem 1.1: Properties of vector addition and scalar multiplication

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars

(1) $\mathbf{u}+\mathbf{v}$ is a vector in R^n (closure under vector addition)

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ (commutative property of vector addition)

(3) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ (associative property of vector addition)

(4) $\mathbf{u}+\mathbf{0} = \mathbf{u}$ (additive identity property)

(5) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$ (additive inverse property) (Note that $-\mathbf{u}$ is just the notation of the additive inverse of $\mathbf{u}$, and $-\mathbf{u} = (-1)\mathbf{u}$ will be proved in Thm. 4.4)

(6) $c\mathbf{u}$ is a vector in R^n (closure under scalar multiplication)

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$ (distributive property of scalar multiplication over vector addition)

(8) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$ (distributive property of scalar multiplication over real-number addition)

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$ (associative property of multiplication)

(10) $1(\mathbf{u}) = \mathbf{u}$ (multiplicative identity property)

※ Except Properties (1) and (6), these properties of vector addition and scalar multiplication actually inherit the properties of matrix addition and scalar multiplication in Ch 2 because we can regard vectors in R^n as special cases of matrices

- Ex 5: Practice standard vector operations in R^4

Let $\mathbf{u} = (2, -1, 5, 0)$, $\mathbf{v} = (4, 3, 1, -1)$, and $\mathbf{w} = (-6, 2, 0, 3)$ be vectors in R^4. Solve $\mathbf{x}$ in each of the following cases.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u} - \mathbf{v}+\mathbf{x}$

Sol: (a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

$$= 2\mathbf{u} + (-1)(\mathbf{v} + 3\mathbf{w})$$

$$= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w}$$ (distributive property of scalar multiplication over vector addition)

$$= (4,\ -2,\ 10,\ 0) - (4,\ 3,\ 1,\ -1) - (-18,\ 6,\ 0,\ 9)$$

$$= (4-4+18,\ -2-3-6,\ 10-1-0,\ 0+1-9)$$

$$= (18,\ -11,\ 9,\ -8)$$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u} - \mathbf{v} + \mathbf{x}$

$3\mathbf{x}+3\mathbf{w} = 2\mathbf{u} - \mathbf{v} + \mathbf{x}$ (distributive property of scalar multiplication over vector addition)

$3\mathbf{x}-\mathbf{x} = 2\mathbf{u} - \mathbf{v} - 3\mathbf{w}$ (subtract (3**w**+**x**) from both sides)

$2\mathbf{x} = 2\mathbf{u} - \mathbf{v} - 3\mathbf{w}$

$\mathbf{x} = \mathbf{u} - \tfrac{1}{2}\mathbf{v} - \tfrac{3}{2}\mathbf{w}$ (scalar multiplication for the both sides with a scalar to be 1/2)

$= \left(2,-1,5,0\right) + \left(-2,\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{1}{2}\right) + \left(9,-3,0,\tfrac{-9}{2}\right)$

$= \left(9,\tfrac{-11}{2},\tfrac{9}{2},-4\right)$

- Notes:

(1) The zero vector **0** in R^n is called the additive identity in R^n (see Property 4)

(2) The vector –**u** is called the additive inverse of **u** (see Property 5)

- Theorem 1.2: (Properties of additive identity and additive inverse)

Let **v** be a vector in R^n and c be a scalar. Then the following properties are true

(1) The additive identity is unique, i.e., if $\mathbf{v}+\mathbf{u} = \mathbf{v}$, $\mathbf{u}$ must be $\mathbf{0}$

(2) The additive inverse of **v** is unique, i.e., if $\mathbf{v}+\mathbf{u} = \mathbf{0}$, $\mathbf{u}$ must be $-\mathbf{v}$

(3) $0\mathbf{v} = \mathbf{0}$

(4) $c\mathbf{0} = \mathbf{0}$

(5) If $c\mathbf{v} = \mathbf{0}$, either $c = 0$ or $\mathbf{v} = \mathbf{0}$

These three properties are valid for any vector space and will be proved on Slides 4.22-4.23

(6) $-(-\mathbf{v}) = \mathbf{v}$ (Since $-\mathbf{v} + \mathbf{v} = \mathbf{0}$, the additive inverse of $-\mathbf{v}$ is $\mathbf{v}$, i.e., $\mathbf{v}$ can be expressed as $-(-\mathbf{v})$. Note that $\mathbf{v}$ and $-\mathbf{v}$ are the additive inverses for each other)

- Linear combination in R^n:

  The vector **x** is called a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, if it can be expressed in the form

  $$\mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v}_n, \text{ where } c_1,\ c_2, \ldots,\ c_n \text{ are real numbers}$$

- Ex 6:

  Given **x** = (–1, –2, –2), **u** = (0,1,4), **v** = (– 1,1,2), and **w** = (3,1,2) in R^3, find *a*, *b*, and *c* such that **x** = *a***u**+*b***v**+*c***w**.

Sol:

$$\begin{aligned} -b + 3c &= -1 \\ a + b + c &= -2 \\ 4a + 2b + 2c &= -2 \end{aligned}$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1$$

Thus $\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$

Keywords in Section 1.1:

- ordered *n*-tuple
- R^n-space
- equal
- vector addition
- scalar multiplication
- zero vector
- additive identity
- additive inverse
- linear combination