Faculty of Engineering
Civil Engineering

Numerical Methods

Chapter 2

Special Matrices and Gauss-Siedel

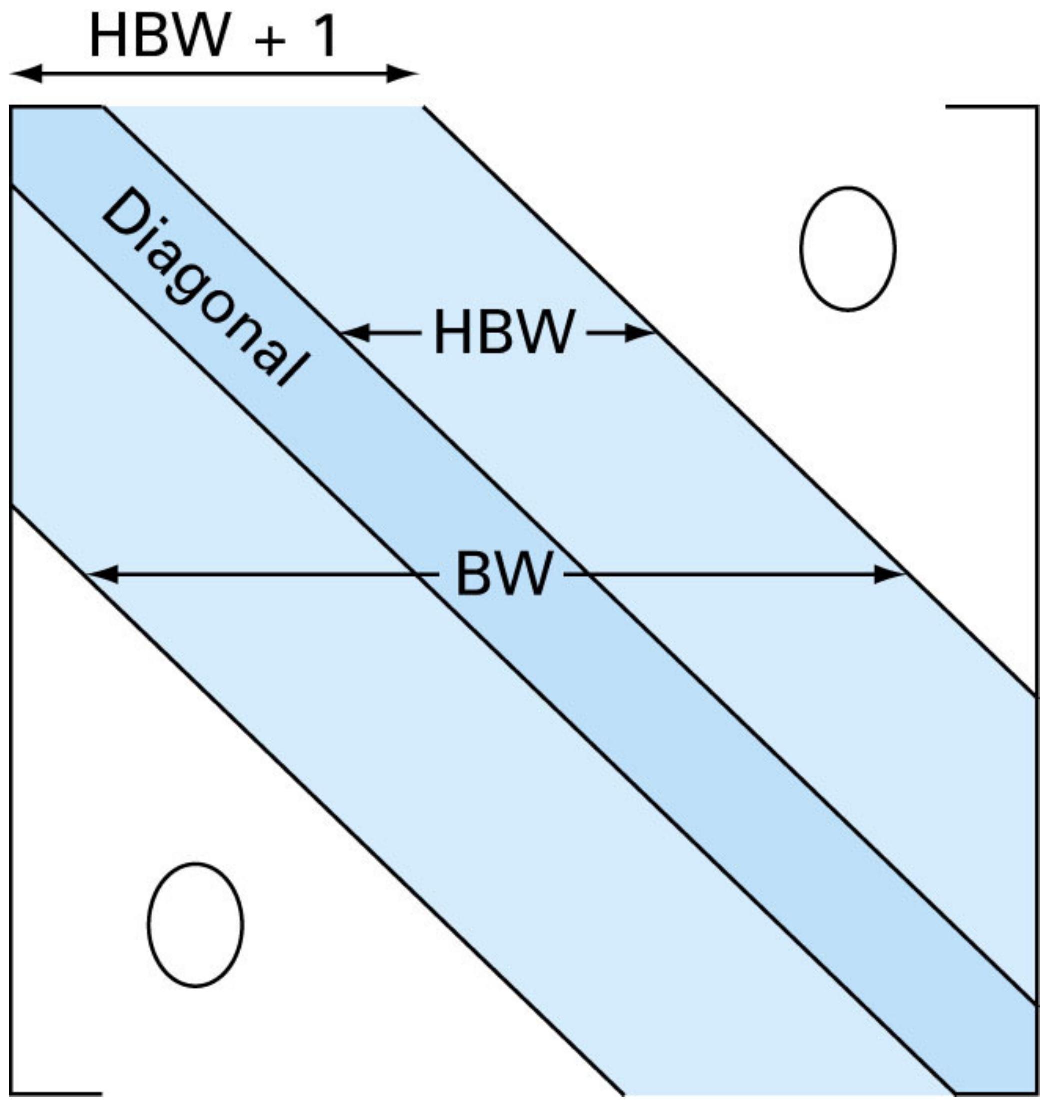
Introduction

- Certain matrices have particular structures that can be exploited to develop efficient solution schemes.

- A ***banded matrix*** is a square matrix that has all elements equal to zero, with the exception of a band centered on the main diagonal. These matrices typically occur in solution of differential equations.

- The dimensions of a banded system can be quantified by two parameters: the band width BW and half-bandwidth HBW. These two values are related by BW=2HBW+1.

Banded matrix



Tridiagonal Systems

- A tridiagonal system has a bandwidth of 3:

$$\begin{bmatrix} f_1 & g_1 & & \\ e_2 & f_2 & g_2 & \\ & e_3 & f_3 & g_3 \\ & & e_4 & f_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix}$$

- An efficient LU decomposition method, called *Thomas algorithm*, can be used to solve such an equation. The algorithm consists of three steps: decomposition, forward and back substitution, and has all the advantages of LU decomposition.

(a) Decomposition

```
DO k = 2, n
   e_k = e_k/f_{k-1}
   f_k = f_k - e_k · g_{k-1}
END DO
```

$$\begin{aligned}&\text{DO } k = 2,\ n\\&\quad e_k = e_k/f_{k-1}\\&\quad f_k = f_k - e_k \cdot g_{k-1}\\&\text{END DO}\end{aligned}$$

(b) Forward substitution

$$\begin{aligned}&\text{DO } k = 2,\ n\\&\quad r_k = r_k - e_k \cdot r_{k-1}\\&\text{END DO}\end{aligned}$$

(c) Back substitution

$$\begin{aligned}&x_n = r_n/f_n\\&\text{DO } k = n-1,\ 1,\ -1\\&\quad x_k = (r_k - g_k \cdot x_{k+1})/f_k\\&\text{END DO}\end{aligned}$$

Cholesky Decomposition

- This method is suitable for only symmetric systems where:

$$a_{ij} = a_{ji} \quad and \quad A = A^T$$

$$[L] = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$A = L * L^T$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} * \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Cholesky Decomposition

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} \cdot l_{kj}}{l_{ii}} \quad for \quad i = 1, 2, \cdots, k-1$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

Pseudocode for Cholesky's LU Decomposition algorithm (cont'd)

$$
\begin{aligned}
&DO\ k = 1, n\\
&\quad DO\ i = 1,\ k - 1\\
&\qquad sum = 0.\\
&\qquad DO\ j = 1,\ i - 1\\
&\qquad\quad sum = sum + a_{ij} \cdot a_{kj}\\
&\qquad END\ DO\\
&\qquad a_{ki} = (a_{ki} - sum)/a_{ii}\\
&\quad END\ DO\\
&\quad sum = 0.\\
&\quad DO\ j = 1,\ k - 1\\
&\qquad sum = sum + a^2_{kj}\\
&\quad END\ DO\\
&\quad a_{kk} = \sqrt{a_{kk} - sum}\\
&END\ DO
\end{aligned}
$$

Gauss-Siedel

- Iterative or approximate methods provide an alternative to the elimination methods. The Gauss-Seidel method is the most commonly used iterative method.
- The system *[A]{X}={B}* is reshaped by solving the first equation for x_1, the second equation for x_2, and the third for x_3, …and n^th equation for x_n. We will limit ourselves to a 3x3 set of equations.

Gauss-Siedel

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\Rightarrow$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_1 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Now we can start the solution process by choosing guesses for the x's. A simple way to obtain initial guesses is to assume that they are zero. These zeros can be substituted into x_1 equation to calculate a new $x_1 = b_1/a_{11}$.

Gauss-Siedel

- New x_1 is substituted to calculate x_2 and x_3. The procedure is repeated until the convergence criterion is satisfied:

$$\left|\varepsilon_{a,i}\right| = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| 100\% < \varepsilon_s$$

Jacobi iteration Method

An alternative approach, called ***Jacobi iteration,*** utilizes a somewhat different technique. This technique includes computing a set of new x's on the basis of a set of old x's. Thus, as the new values are generated, they are not immediately used but are retained for the next iteration.

First Iteration

(a)	(b)
$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$	$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$
$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$	$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$
$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$	$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$

Second Interation

(a)	(b)
$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$	$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$
$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$	$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$
$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$	$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$

(a) The Gauss-Seidel method

(b) The Jacobi iteration method

Convergence Criterion for Gauss-Seidel Method

- The gauss-siedel method is similar to the technique of fixed-point iteration.
- The Gauss-Seidel method has two fundamental problems as any iterative method:
  1. It is sometimes non-convergent, and
  2. If it converges, converges very slowly.
- Sufficient conditions for convergence of two linear equations, *u(x,y)* and *v(x,y)* are:

$$\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right| < 1$$

$$\left|\frac{\partial v}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| < 1$$

Convergence Criterion for Gauss-Seidel Method (cont'd)

- Similarly, in case of two simultaneous equations, the Gauss-Seidel algorithm can be expressed as:

$$u(x_1, x_2) = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2$$

$$v(x_1, x_2) = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1$$

$$\frac{\partial u}{\partial x_1} = 0 \qquad \frac{\partial u}{\partial x_2} = -\frac{a_{12}}{a_{11}}$$

$$\frac{\partial v}{\partial x_1} = -\frac{a_{21}}{a_{22}} \qquad \frac{\partial v}{\partial x_2} = 0$$

Convergence Criterion for Gauss-Seidel Method (cont'd)

Substitution into convergence criterion of two linear equations yield:

$$\left|\frac{a_{12}}{a_{11}}\right| < 1, \qquad \left|\frac{a_{21}}{a_{22}}\right| < 1$$

In other words, the absolute values of the slopes must be less than unity for convergence:

$$|a_{11}| > |a_{12}|$$
$$|a_{22}| > |a_{21}|$$

For n equations

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}|$$

That is, the diagonal element must be greater than the off-diagonal element for each row.

Gauss-Siedel Method- Example 1

$$\begin{bmatrix} 3 & -0.1 & 0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix} \Rightarrow \begin{aligned} x_1 &= \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \\ x_2 &= \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} \\ x_3 &= \frac{71.4 - 0.3x_1 + 0.2x_2}{10} \end{aligned}$$

- **Guess** $x_1, x_2, x_3 =$ zero for the first guess

Iter.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\lvert\varepsilon_{a,1}\rvert$ **(%)**	$\lvert\varepsilon_{a,2}\rvert$ **(%)**	$\lvert\varepsilon_{a,3}\rvert$ **(%)**
0	0	0	0	-	-	-
1	2.6167	-2.7945	7.005610	100	100	100
2	2.990557	-2.499625	7.000291	12.5	11.8	0.076

Improvement of Convergence Using Relaxation

$$x_i^{new} = \lambda \cdot x_i^{new} + (1-\lambda) \cdot x_i^{old}$$

- Where λ is a weighting factor that is assigned a value between [0, 2]
- If λ = 1 the method is unmodified.
- If λ is between 0 and 1 (under relaxation) this is employed to make a non convergent system to converge.
- If λ is between 1 and 2 (over relaxation) this is employed to accelerate the convergence.

Gauss-Siedel Method- Example 2

$$-8x_1 + x_2 - 2x_3 = -20$$
$$-3x_1 - x_2 + 7x_3 = -34$$
$$2x_1 - 6x_2 - x_3 = -38$$

Rearrange so that the equations are diagonally dominant

$$-8x_1 + x_2 - 2x_3 = -20$$
$$2x_1 - 6x_2 - x_3 = -38$$
$$-3x_1 - x_2 + 7x_3 = -34$$

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} \qquad x_2 = \frac{-38 - 2x_1 + x_3}{-6} \qquad x_3 = \frac{-34 + 3x_1 + x_2}{7}$$

Gauss-Siedel Method- Example 2

iteration	unknown	value	ε_a	maximum ε_a
0	x_1	0		
	x_2	0		
	x_3	0		
1	x_1	2.5	100.00%	
	x_2	7.166667	100.00%	
	x_3	-2.7619	100.00%	100.00%
2	x_1	4.08631	38.82%	
	x_2	8.155754	12.13%	
	x_3	-1.94076	42.31%	42.31%
3	x_1	4.004659	2.04%	
	x_2	7.99168	2.05%	
	x_3	-1.99919	2.92%	2.92%

Gauss-Siedel Method- Example 2

The same computation can be developed with relaxation where $\lambda = 1.2$

<u>First iteration:</u>

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

Relaxation yields: $x_1 = 1.2(2.5) - 0.2(0) = 3$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(3) + 0}{-6} = 7.333333$$

Relaxation yields: $x_2 = 1.2(7.333333) - 0.2(0) = 8.8$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(3) + 8.8}{7} = -2.3142857$$

Relaxation yields: $x_3 = 1.2(-2.3142857) - 0.2(0) = -2.7771429$

Gauss-Siedel Method- Example 2

Iter.	unknown	value	relaxation	ε_a	maximum ε_a
1	x_1	2.5	3	100.00%	
	x_2	7.3333333	8.8	100.00%	
	x_3	-2.314286	-2.777143	100.00%	100.000%
2	x_1	4.2942857	4.5531429	34.11%	
	x_2	8.3139048	8.2166857	7.10%	
	x_3	-1.731984	-1.522952	82.35%	82.353%
3	x_1	3.9078237	3.7787598	20.49%	
	x_2	7.8467453	7.7727572	5.71%	
	x_3	-2.12728	-2.248146	32.26%	32.257%
4	x_1	4.0336312	4.0846055	7.49%	
	x_2	8.0695595	8.12892	4.38%	
	x_3	-1.945323	-1.884759	19.28%	19.280%