Faculty of Engineering

Civil Engineering

Numerical Methods

Chapter 2

LU Decomposition and Matrix Inversion

Introduction

- Gauss elimination solves [A] {x} ={B}

- It becomes insufficient when solving these equations for different values of {B}

- LU decomposition works on the matrix [A] and the vector {B} separately.

- LU decomposition is very useful when the vector of variables {x} is estimated for different parameter vectors {B} since the forward elimination process is not performed on {B}.

LU Decomposition

If:

L: lower triangular matrix

U: upper triangular matrix

Then,

[A]{X}={B} can be decomposed into two matrices [L] and [U] such that:

1. [L][U] = [A] ➔ ([L][U]){X} = {B}

LU Decomposition

Consider:

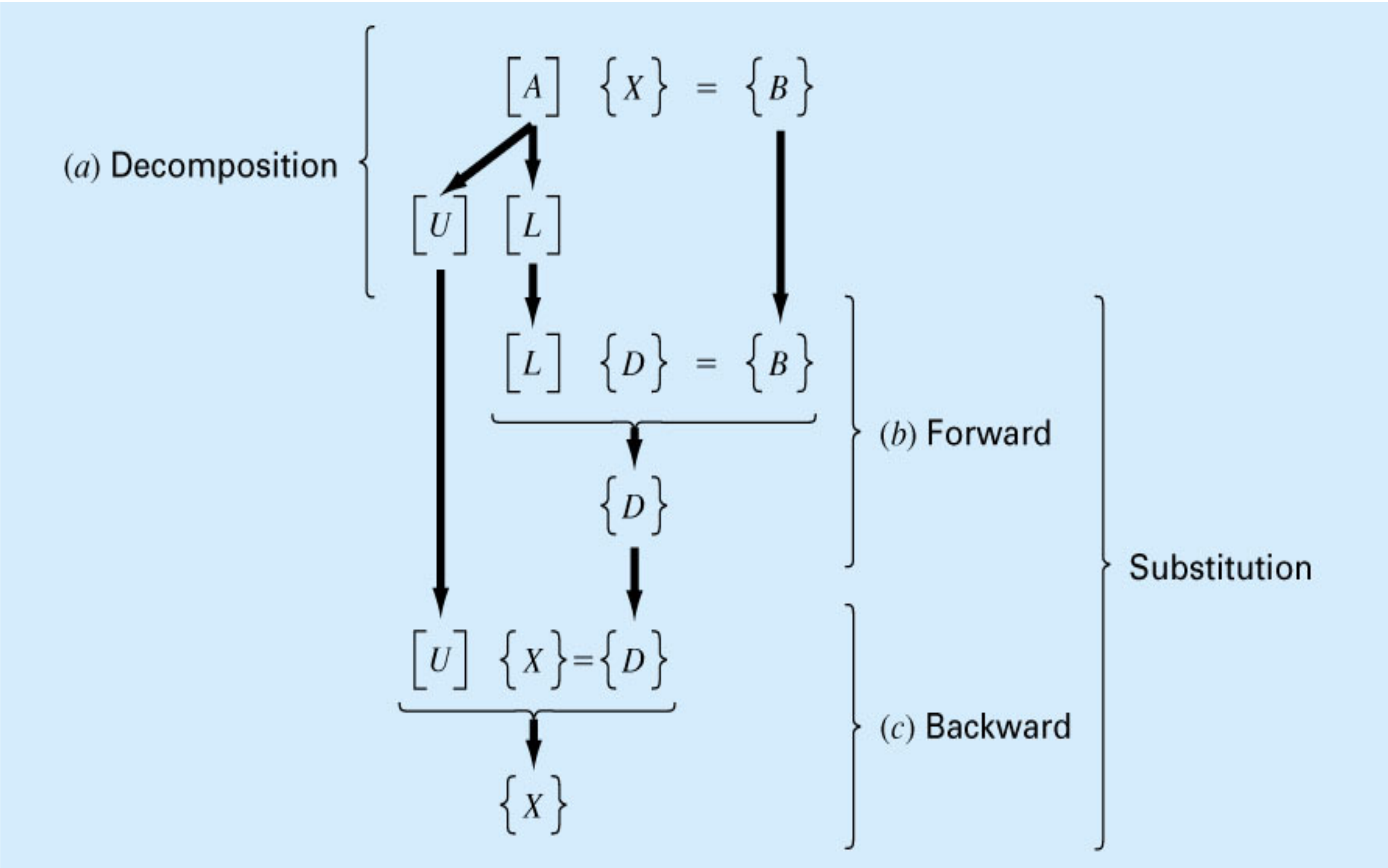
$[U]\{X\} = \{D\}$

So, $[L]\{D\} = \{B\}$

2. $[L]\{D\} = \{B\}$ is used to generate an intermediate vector $\{D\}$ by **forward substitution**.

3. Then, $[U]\{X\}=\{D\}$ is used to get $\{X\}$ by **back substitution.**

Summary of LU Decomposition



LU Decomposition

As in Gauss elimination, LU decomposition must employ pivoting to avoid division by zero and to minimize round off errors. The pivoting is done immediately after computing each column.

LU Decomposition

System of linear equations [A]{x}={B}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$l_{21} = \frac{a_{12}}{a_{11}}; \qquad l_{31} = \frac{a_{31}}{a_{11}}$$

$$l_{32} = \frac{a_{32}^{\backslash}}{a_{22}^{\backslash}}$$

Step 1: **Decomposition**

$$[L][U] = [A]$$

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{/} & a_{23}^{/} \\ 0 & 0 & a_{33}^{//} \end{bmatrix} \qquad [L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

LU Decomposition

Step 2: Generate an intermediate vector **{D}** by forward substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Step 3: Get **{X}** by back substitution.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

LU Decomposition-Example

$$[A]=\begin{Bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{Bmatrix} \Rightarrow \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.003 & -0.293 \\ 0 & -0.19 & 10.02 \end{bmatrix}$$

$$l_{21}=\frac{0.1}{3}=0.03333; \qquad l_{31}=\frac{0.3}{3}=0.1000$$

$$l_{32}=\frac{a_{32}^{\backslash}}{a_{22}^{\backslash}}=\frac{-0.19}{7.003}=-0.02713$$

$$[U]=\begin{Bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.003 & -0.293 \\ 0 & 0 & 10.012 \end{Bmatrix} \qquad [L]=\begin{bmatrix} 1 & 0 & 0 \\ 0.03333 & 1 & 0 \\ 0.1000 & -.02713 & 1 \end{bmatrix}$$

LU Decomposition-Example (cont'd)

Use previous L and D matrices to solve the system:

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.31 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

Step 2: Find the intermediate vector {D} by forward substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & -0.02713 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix} \Rightarrow \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

LU Decomposition-Example (cont'd)

Step 3: Get {X} by back substitution.

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.012 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -2.5 \\ 7.00003 \end{Bmatrix}$$

Decomposition Step

- % Decomposition Step

```
for k=1:n-1
   [a,o]= pivot(a,o,k,n);
   for i = k+1:n
      a(i,k) = a(i,k)/a(k,k);
      a(i,k+1:n)= a(i,k+1:n)-a(i,k)*a(k,k+1:n);
   end
end
```

Partial Pivoting

- %Partial Pivoting

```
function [a,o] = pivot(a,o,k,n)
[big piv]=max(abs(a(k:n,k)));
piv=piv+(k-1);
if piv ~= k
   temp = a(piv,:);
   a(piv,:)= a(k,:);
   a(k,:)=temp;
   temp = o(piv);
   o(piv)=o(k);
   o(k)=temp;
end
```

Substitution Steps

```
%Forward Substitution
d(1)=bn(1);
for i=2:n
  d(i)=bn(i)-a(i,1:i-1)*(d(1:i-1))';
 end
```

- % Back Substitution

```
x(n)=d(n)/a(n,n);
for i=n-1:-1:1
    x(i)=(d(i)-a(i,i+1:n)*x(i+1:n)')/a(i,i);
end
```

Matrix Inverse Using the LU Decomposition

- LU decomposition can be used to obtain the inverse of the original coefficient matrix [A].
- Each column j of the inverse is determined by using a unit vector (with 1 in the j^{th} raw).

Matrix Inverse: LU Decomposition

$[A]\,[A]^{-1} = [A]^{-1}[A] = I$

$$[A]\{x\}_1 = \{b\}_1 \qquad [A]\{x\}_2 = \{b\}_2 \qquad [A]\{x\}_3 = \{b\}_3$$

$$[A]\{x\}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad [A]\{x\}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \qquad [A]\{x\}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

1st column of $[A]^{-1}$ **2nd column of $[A]^{-1}$** **3rd column of $[A]^{-1}$**

$$[A]^{-1} = \left[\{x\}_1 \quad \{x\}_2 \quad \{x\}_3\right]$$

Matrix inverse using LU decomposition Example

$$[A]=\begin{Bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{Bmatrix} \quad [L]=\begin{bmatrix} 1 & 0 & 0 \\ 0.03333 & 1 & 0 \\ 0.1000 & -.02713 & 1 \end{bmatrix} \quad [U]=\begin{Bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.003 & -0.293 \\ 0 & 0 & 10.012 \end{Bmatrix}$$

1A. [L]{d}$_1$ = {b}$_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & -0.02713 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}=\begin{Bmatrix} 1 \\ -0.03333 \\ -0.1009 \end{Bmatrix}$$

1B. Then, [U]{X}$_1$={d}$_1$

1st column of $[A]^{-1}$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.012 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} 1 \\ -0.03333 \\ -0.1009 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} 0.33249 \\ -0.00518 \\ -0.01008 \end{Bmatrix}$$

Matrix inverse using LU decomposition Example (cont'd)

2A. **[L]{d}$_2$ = {b}$_2$**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & -0.02713 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0.02713 \end{Bmatrix}$$

2B. **Then, [U]{X}$_2$={d}$_2$**

2nd column of $[A]^{-1}$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.012 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0.02713 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0.004944 \\ 0.142903 \\ 0.00271 \end{Bmatrix}$$

Matrix inverse using LU decomposition Example (cont'd)

3A. [L]{d}$_3$ = {b}$_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1000 & -0.02713 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

3B. Then, [U]{X}$_3$={d}$_3$

3rd column of $[A]^{-1}$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.012 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0.006798 \\ 0.004183 \\ 0.09988 \end{Bmatrix}$$

Matrix inverse using LU decomposition Example (cont'd)

$$[A]^{-1} = \begin{bmatrix} 0.33249 & 0.004944 & 0.006798 \\ -0.00518 & 0.142903 & 0.004183 \\ -0.01008 & 0.00271 & 0.09988 \end{bmatrix}$$