3

INTERPOLATION AND CURVE FITTING

- Introduction
- Newton Interpolation: Finite Divided Difference
- Lagrange Interpolation
- Spline Interpolation
- Polynomial Regression
- Multivariable Interpolation

3.1 Introduction

- **Interpolation** is a technique to estimate the value between a set of data.
- A general approach is to map the data into an *n*-th order polynomial:

$$f_n(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i \tag{3.1}$$

- This chapter covers three types of techniques, i.e. the Newton interpolation, the Lagrange interpolation and the Spline interpolation.
- The resulting equation can be used for curve fitting.

3.2 Newton Interpolation: Finite Divided Difference

- For the linear interpolation, consider Fig. 3.1 and the following relation:

$$\frac{f_1(x)-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

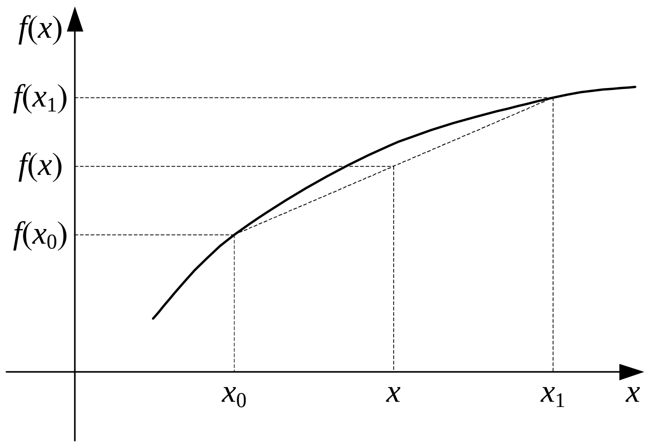

FIGURE 3.1 Linear interpolation

Rearranging the above relation leads to

$$f_1(x)=f(x_0)+\frac{f(x_1)-f(x_0)}{x_1-x_0}\cdot(x-x_0) \tag{3.2}$$

The term $[f(x_1)-f(x_0)]/(x_1-x_0)$ is referred to as the *first-order finite divided difference*.

Example 3.1

Calculate an approximate value of ln 2 using the linear interpolation using the following ranges: (1,4) and (1,6). Use ln 1 = 0, ln 4 = 1.3862944 and ln 6 = 1.7917595 as the source data. Estimate the relative error if the actual value is ln 2 = 0.69314718.

Solution

For the linear interpolation between $x_0 = 1$ and $x_1 = 6$:

$$f_1(2)=0+\frac{1.7917595-0}{6-1}\cdot(2-1)=0.35835190$$

This produces a relative error of $\varepsilon_t = 48.3\%$.

For the linear interpolation between $x_0 = 1$ dan $x_1 = 4$:

$$f_1(2) = 0 + \frac{1.3862944 - 0}{4 - 1} \cdot (2 - 1) = 0.46209812$$

This produces a relative error of $\varepsilon_t = 33.3\%$, which smaller than the previous case.

- For the quadratic interpolation, consider Fig. 3.2 and the following relation:

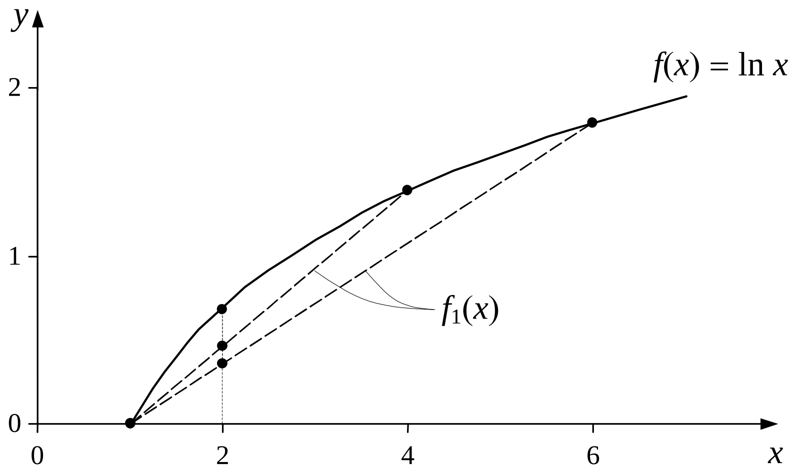

FIGURE 3.2 The result for the linear interpolation for ln 2

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \tag{3.3}$$

By taking the values of x equal to x_0, x_1 dan x_2 in Eq. (3.3), the coefficients b_i can be calculated as followed:

$$b_0 = f(x_0) \tag{3.4a}$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{3.4b}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \tag{3.4c}$$

Example 3.2

Repeat Example 3.1 using the quadratic interpolation.

Solution

From Example 3.1, the data can be rewritten as followed:

$$x_0 = 1: \quad f(x_0) = 0$$
$$x_1 = 4: \quad f(x_1) = 1.3862944$$
$$x_2 = 6: \quad f(x_2) = 1.7917595$$

Using Eq. (3.4):

$$b_0 = 0$$
$$b_1 = \frac{1.3862944 - 0}{4 - 1} = 0.46209812$$
$$b_2 = \frac{\dfrac{1.7917595 - 1.3862944}{6 - 4} - \dfrac{1.3862944 - 0}{4 - 1}}{6 - 1} = -0.051873113$$

Hence, Eq. (3.3) yields

$$f_2(x) = 0 + 0.46209812(x - 1) - 0.051873113(x - 1)(x - 4)$$

For $x = 2$:

$$f_2(2) = 0 + 0.46209812(2 - 1) - 0.051873113(2 - 1)(2 - 4),$$
$$= 0.56584436.$$

This produces the relative error $\varepsilon_t = 18.4\%$ as shown below.

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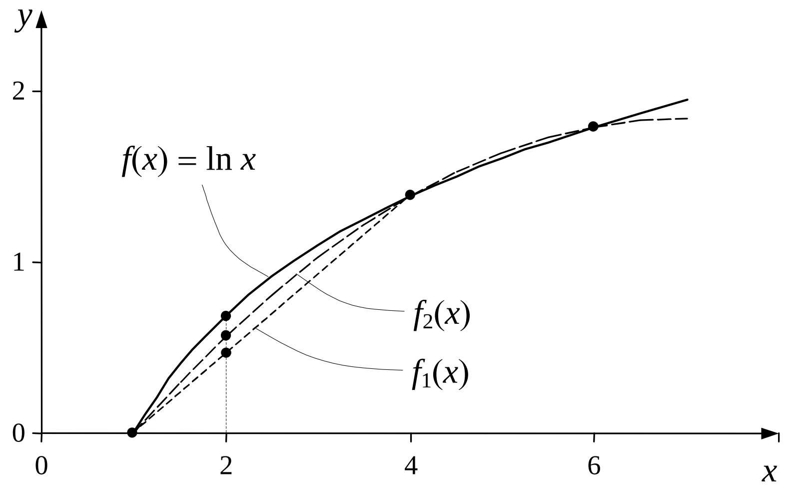

FIGURE 3.3 The result for the quadratic interpolation for ln 2

- The general form representing an n-th order interpolation function (requires n+1 pairs of data) is:

$$f_n(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \qquad (3.5)$$

where the coefficients b_i can be obtained via

$$\begin{aligned} b_0 &= f(x_0) \\ b_1 &= f[x_1, x_0] \\ b_2 &= f[x_2, x_1, x_0] \\ &\vdots \\ b_n &= f[x_n, x_{n-1}, \ldots, x_1, x_0] \end{aligned}$$

where the terms […] are *finite divided differences* and are defined as:

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j} \qquad (3.6a)$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k} \qquad (3.6b)$$

$$f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_2, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_1, x_0]}{x_n - x_0} \qquad (3.6c)$$

Hence, Eq. (3.5) forms the *Newton interpolation polynomial*:

$$\begin{aligned} f_n(x) = {} & f(x_0) + (x - x_0) f[x_1, x_0] + (x - x_0)(x - x_1) f[x_2, x_1, x_0] \\ & + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1}) f[x_n, x_{n-1}, \ldots, x_0] \end{aligned} \qquad (3.7)$$

TABLE 3.1 Newton finite divided difference interpolation polynomial

i	x_i	$f(x_i)$	**I**	**II**	**III**
0	x_0	$f(x_0)$	$f[x_1,x_0]$	$f[x_2,x_1,x_0]$	$f[x_3,x_2,x_1,x_0]$
1	x_1	$f(x_1)$	$f[x_2,x_1]$	$f[x_3,x_2,x_1]$	
2	x_2	$f(x_2)$	$f[x_3,x_2]$		
3	x_3	$f(x_3)$			

- The relation for error for the *n*-th order interpolation function can be estimated using:

$$R_n \approx f\left[x_{n+1}, x_n, x_{n-1}, \ldots, x_0\right](x-x_0)(x-x_1)\cdots(x-x_n) \tag{3.8}$$

Example 3.3

Repeat Example 3.1 with an additional fourth point of $x_3 = 5$, $f(x_3) = 1.6094379$ using the third order Newton interpolation polynomial.

Solution

The third order Newton interpolation polynomial can be written as

$$f_3(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2)$$

i	x_i	$f(x_i)$	**I**	**II**	**III**
0	1	0	0.46209813	–0.0597386	0.00786553
1	4	1.3862944	0.22314355	–0.0204110	
2	5	1.6094379	0.18232156		
3	6	1.7917595			

Thus,

$$\begin{aligned} f_3(x) &= 0 + 0.46209812(x-1) - 0.0597386(x-1)(x-4) \\ &\quad + 0.00786553(x-1)(x-4)(x-5), \end{aligned}$$

$$\begin{aligned} f_3(2) &= 0 + 0.46209812(2-1) - 0.0597386(2-1)(2-4) \\ &\quad + 0.00786553(2-1)(2-4)(2-5), \\ &= 0.6287691. \end{aligned}$$

and producing a relative error of $\varepsilon_t = 9.29\%$.

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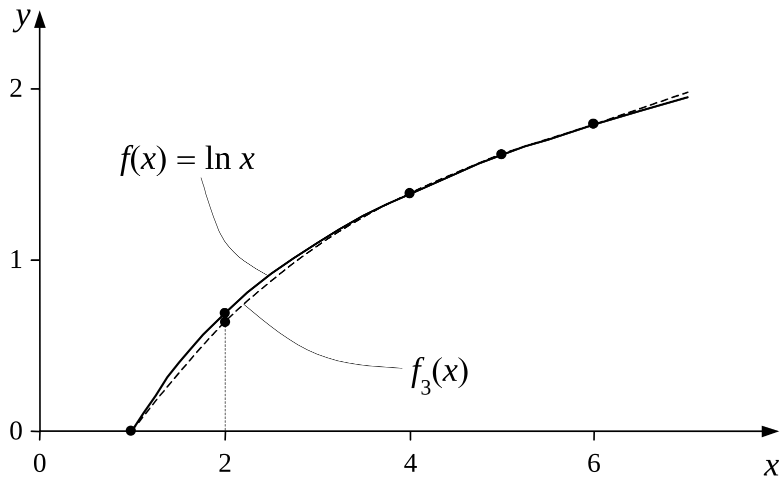

TABLE 3.4 The result for the cubic interpolation for ln 2

3.3 Lagrange Interpolation

- The formula for the Lagrange interpolation is:

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) \tag{3.9}$$

where the parameter $L_i(x)$ is defined as

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} \tag{3.10}$$

- For $n = 1$, the first order Lagrange interpolation function can be written as:

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

- For $n = 2$, the second order Lagrange interpolation can be written as:

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

- The error term for the Lagrange interpolation can be estimated using:

$$R_n = f[x_{n+1}, x_n, x_{n-1}, \ldots, x_0] \prod_{i=0}^{n} (x - x_i) \tag{3.11}$$

Example 3.5

Repeat Examples 3.1 and 3.2 using the first and second order Lagrange interpolation functions, respectively.

Solution

From Examples 3.1:

$$x_0 = 1: \quad f(x_0) = 0$$
$$x_1 = 4: \quad f(x_1) = 1.3862944$$
$$x_2 = 6: \quad f(x_2) = 1.7917595$$

For the first order Lagrange interpolation:

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$\therefore \quad f_1(2) = \frac{2-4}{1-4}(0) + \frac{2-1}{4-1}(1.3862944)$$

$$= 0.4620981$$

For the second order Lagrange interpolation:

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

$$f_2(2) = \frac{(2-4)(2-6)}{(1-4)(1-6)}(0) + \frac{(2-1)(2-6)}{(4-1)(4-6)}(1.3862944) + \frac{(2-1)(2-4)}{(6-1)(6-4)}(1.7917595)$$

$$= 0.5658444$$

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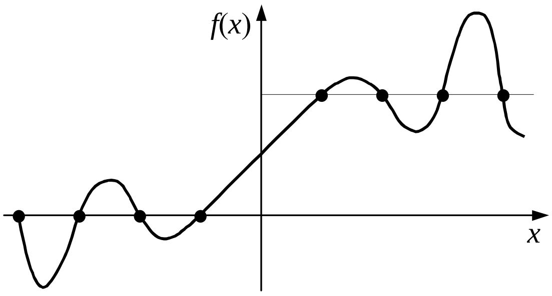
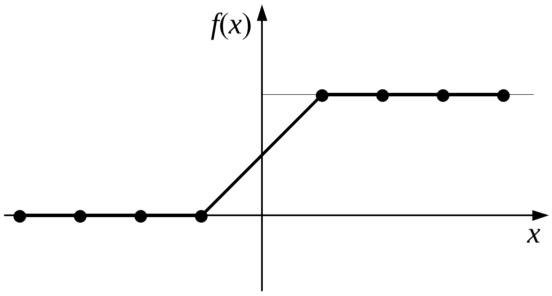
3.4 Spline Interpolation

- The polynomial interpolation can cause oscillation to the function and this can be remedied by using the *spline interpolation*.



(a) polynomial



(b) Linear spline

FIGURE 3.5 Comparison between polynomial and spline interpolation

- The formula for the *linear spline* interpolation for n+1 data between points x_0 and x_n (n intervals) is

$$\begin{aligned} f_{11}(x) &= f(x_0) + m_0(x - x_0) && \text{untuk } x_0 \le x \le x_1 \\ f_{12}(x) &= f(x_1) + m_1(x - x_1) && \text{untuk } x_1 \le x \le x_2 \\ &\vdots && \vdots \\ f_{1n}(x) &= f(x_{n-1}) + m_{n-1}(x - x_{n-1}) && \text{untuk } x_{n-1} \le x \le x_n \end{aligned} \tag{3.12}$$

dengan m_i is the gradient for the (i+1)-th interval:

$$m_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \tag{3.13}$$

Example 3.6

Use the following data to estimate the function $f(x)$ at $x = 5$ using the linear spline interpolation.

x	$f(x)$
3.0	2.5
4.5	1.0
7.0	2.5
9.0	0.5

Solution

The value of $x = 5$ is in the range $4.5 \le x \le 7$, where the gradient is

$$m_1 = \frac{2.5 - 1.0}{7.0 - 4.5} = 0.60$$

From Eq. (3.12):

$$f_{12}(x) = f(x_1) + m_1(x - x_1),$$
$$f_{12}(5) = 1.0 + 0.60(5 - 4.5) = 1.3.$$

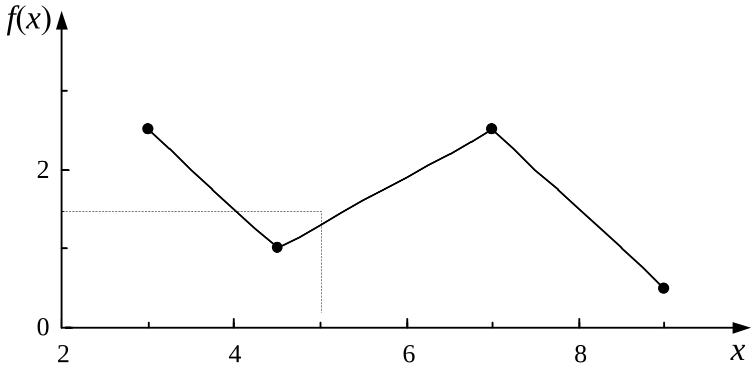


FIGURE 3.6 Linear spline interpolation for Example 3.6

- For the *quadratic spline*, the general polynomial for each interval is

$$f_{2i}(x) = a_i x^2 + b_i x + c_i \tag{3.14}$$

The coefficient a_i, b_i and c_i for each interval can be evaluated using:

1. Continuity at vertices ($2n$–2 equations):

$$a_{i-1}x_{i-1}^2 + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1}) \tag{3.15}$$

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1}) \tag{3.16}$$

2. Conditions at end points (2 equations):

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0) \tag{3.17}$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n) \tag{3.18}$$

3. Differentiability at vertices (n–1 equations):

$$f'(x) = 2ax + b$$

$$2a_{i-1}x_{i-1} + b_{i-1} = 2a_i x_{i-1} + b_i \tag{3.19}$$

4. Assumption at the first interval (1 persamaan):

$$a_1 = 0 \tag{3.20}$$

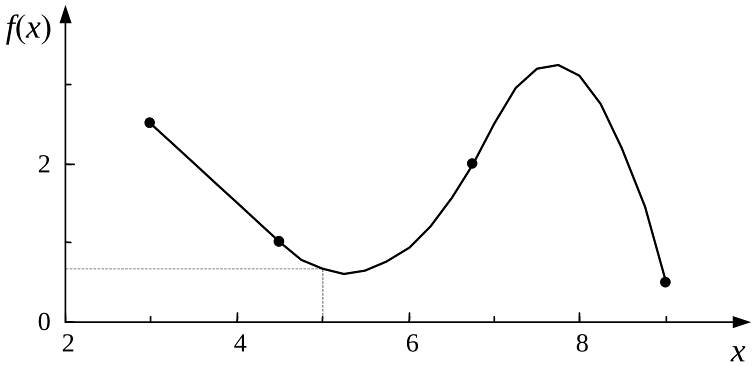


FIGURE 3.7 The quadratic spline interpolasi for example 3.7

- For the *cubic spline*, the general polynomial for each interval is

$$f_{3i}(x) = a_i x^3 + b_i x^2 + c_i x + d_i \tag{3.21}$$

- The formulation for the *cubic spline* can be derived as followed:

  For interval (x_i–1, x_i), the second derivative can be written as:

$$f_i''(x) = f_i''(x_{i-1})\frac{x - x_i}{x_{i-1} - x_i} + f_i''(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}} \tag{3.22}$$

  The integration of Eq. (3.22) twice produces two integration constants, which can be evaluated using $f(x) = f(x_{i-1})$ at x_{i-1} and $f(x) = f(x_i)$ at x_i, thus produces

$$\begin{aligned} f_{3i}(x) = {} & \frac{f''(x_{i-1})}{6(x_i - x_{i-1})}(x_i - x)^3 + \frac{f''(x_i)}{6(x_i - x_{i-1})}(x - x_{i-1})^3 \\ & + \left[\frac{f(x_{i-1})}{(x_i - x_{i-1})} - \frac{f''(x_{i-1})(x_i - x_{i-1})}{6}\right](x_i - x) \\ & + \left[\frac{f(x_i)}{(x_i - x_{i-1})} - \frac{f''(x_i)(x_i - x_{i-1})}{6}\right](x - x_{i-1}) \end{aligned} \tag{3.23}$$

The differentiability can be maintained troughout the function via

$$f'_{i-1}(x_i) = f'_i(x_i) \tag{3.24}$$

Eq. (3.23) can be differentiated for both the (i–1)-th and i-th intervals and following Eq. (3.24):

$$\begin{aligned}&(x_i - x_{i-1})f''(x_{i-1}) + 2(x_{i+1} - x_{i-1})f''(x_i) + (x_{i+1} - x_i)f''(x_{i+1})\\ &= \frac{6}{(x_{i+1} - x_i)}\left[f(x_{i+1}) - f(x_i)\right] + \frac{6}{(x_i - x_{i-1})}\left[f(x_{i-1}) - f(x_i)\right]\end{aligned} \tag{3.25}$$

Example 3.8

Repeat Example 3.6 using the cubic spline interpolation.

Solution

Using Eq. (3.25) for the first interval (i = 1):

$$\begin{aligned}&(4.5-3)f''(3) + 2(7-3)f''(4.5) + (7-4.5)f''(7)\\ &= \frac{6}{(7-4.5)}(2.5-1) + \frac{6}{(4.5-3)}(2.5-1)\end{aligned}$$

It is known that $f''(3) = 0$. Hence

$$8f''(4.5) + 2.5f''(7) = 9.6$$

For the second interval (i = 2):

$$2.5f''(4.5) + 9f''(7) = -9.6$$

Both equations is solved simultaneously to yield

$$f''(4.5) = 1.67909$$
$$f''(7) = 1.53308$$

Using Eq. (3.23) for the first interval:

$$\begin{aligned}f_{31}(x) &= \frac{1.67909}{6(4.5-3)}(x-3)^3 + \frac{2.5}{4.5-3}(4.5-x)\\ &\quad + \left[\frac{1}{4.5-3} - \frac{1.67909(4.5-3)}{6}\right](x-3),\\ &= 0.186566(x-3)^3 + 1.666667(4.5-x) + 0.246894(x-3).\end{aligned}$$

For the second and third intervals:

$$f_{32}(x) = 0.111939(7-x)^3 - 0.102205(x-4.5)^3 - 0.299621(7-x)$$
$$+1.638783(x-4.5)$$
$$f_{33}(x) = -0.127757(9-x)^3 + 1.761027(9-x) + 0.25(x-7)$$

At $x = 5$, use the function for the second interval:

$$f_{32}(5) = 0.111939(7-5)^3 - 0.102205(5-4.5)^3 - 0.299621(7-5)$$
$$+1.638783(5-4.5),$$
$$= 1.102886.$$

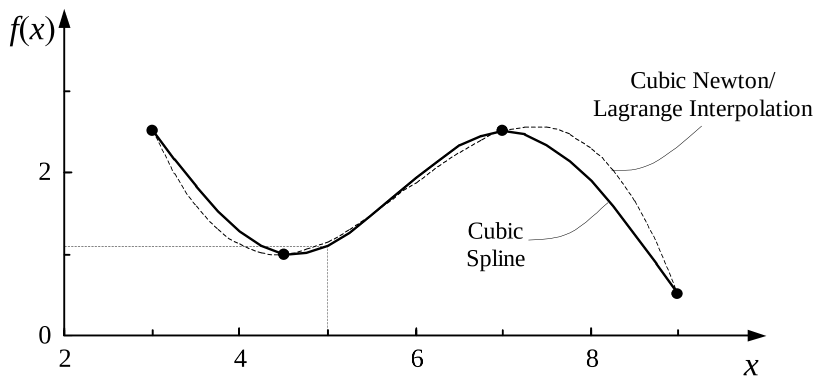


FIGURE 3.8 The cubic spline interpolation for Example 3.8

3.5 Polynomial Regression

- A "best fit" polynomial of an order lower than the number of intervals is sometimes required to represent the data and can be evaluated via *polynomial regression*.

- Consider the following form of polynomial:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad (3.26)$$

where the error for the i-th data (x_i, y_i) is

$$e_i = y_i - f(x_i) = y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n$$

The sum of the squares of error for N data is

$$S = \sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n \right)^2 \qquad (3.27)$$

These errors can be minimised as followed:

$$\frac{\partial S}{\partial a_0} = 0 = \sum_{i=1}^{N} 2\left(y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n \right)(-1)$$
$$\frac{\partial S}{\partial a_1} = 0 = \sum_{i=1}^{N} 2\left(y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n \right)(-x_i)$$
$$\vdots$$
$$\frac{\partial S}{\partial a_n} = 0 = \sum_{i=1}^{N} 2\left(y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_n x_i^n \right)(-x_i^n)$$

producing the following system of equations:

$$\begin{aligned}
a_0 N + a_1 \sum x_i + a_2 \sum x_i^2 + \cdots + a_n \sum x_i^n &= \sum y_i \\
a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 + \cdots + a_n \sum x_i^{n+1} &= \sum x_i y_i \\
a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 + \cdots + a_n \sum x_i^{n+2} &= \sum x_i^2 y_i \\
&\vdots \\
a_0 \sum x_i^n + a_1 \sum x_i^{n+1} + a_2 \sum x_i^{n+2} + \cdots + a_n \sum x_i^{2n} &= \sum x_i^n y_i
\end{aligned} \qquad (3.28)$$

or, in a form of matrix equation:

$$\begin{bmatrix} N & \sum x_i & \sum x_i^2 & \cdots & \sum x_i^n \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \cdots & \sum x_i^{n+1} \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \cdots & \sum x_i^{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum x_i^n & \sum x_i^{n+1} & \sum x_i^{n+2} & \cdots & \sum x_i^{2n} \end{bmatrix} \cdot \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \\ \vdots \\ \sum x_i^n y_i \end{Bmatrix} \tag{3.29}$$

- The standard deviation σ for this case can be evaluated as followed:

$$\sigma = \sqrt{\frac{S}{N-n+1}} = \sqrt{\frac{\sum_{i=1}^{N} e_i^2}{N-n+1}} \tag{3.30}$$

- The case for $n = 1$ is referred to as the *linear regression*:

$$y = a_0 + a_1 x$$

$$\begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \cdot \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \end{Bmatrix} \tag{3.31}$$

Example 3.9

Determine a best-fit quadratic equation which can represent the following experimental data:

x_i	0.05	0.11	0.15	0.31	0.46	0.52
y_i	0.956	0.890	0.832	0.717	0.571	0.539

x_i	0.70	0.74	0.82	0.98	1.17
y_i	0.378	0.370	0.306	0.242	0.104

Solution

For $n = 2$, the quadratic polynomial can be written as

$$f(x) = a_0 + a_1 x + a_2 x^2$$

From Eq. (3.28):

$$\begin{bmatrix} N & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \cdot \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{Bmatrix}$$

From the following table:

$$\sum x_i = 6.01 \qquad N = 11$$
$$\sum x_i^2 = 4.6545 \qquad \sum y_i = 5.905$$
$$\sum x_i^3 = 4.1150 \qquad \sum x_i y_i = 2.1839$$
$$\sum x_i^4 = 3.9161 \qquad \sum x_i^2 y_i = 1.3357$$

Hence the matrix equation becomes:

$$\begin{bmatrix} 11 & 6.01 & 4.6545 \\ 6.01 & 4.6545 & 4.1150 \\ 4.6545 & 4.1150 & 3.9161 \end{bmatrix} \cdot \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 5.905 \\ 2.1839 \\ 1.3357 \end{Bmatrix}$$

$$a_0 = 0.998 \qquad a_1 = -1.018 \qquad a_2 = 0.225$$

Hence, the quadratic function which can represent the data is

$$f(x) = 0.998 - 1.018x + 0.225x^2$$

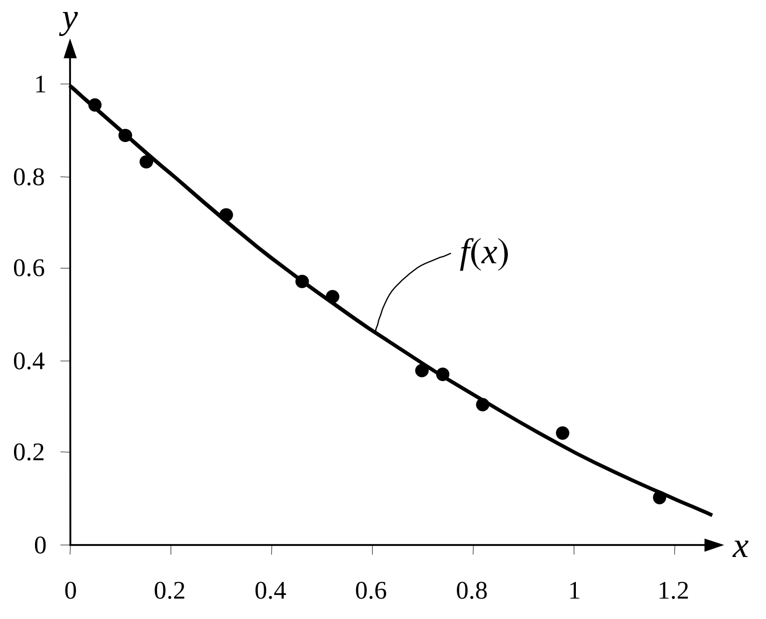


FIGURE 3.9 The fitting of data using a quadratic function

3.6 Multivariable Interpolation

- The Newton, Lagrange and spline interpolation method can be applied for multivariable cases, e.g. for two and three variables:

$$f_{pq}(x,y)=\sum_{i=0}^{p}\sum_{j=0}^{q}L_{x_i}(x)\cdot L_{y_j}(y)f(x_i,y_j) \tag{3.32}$$

$$f_{pqr}(x,y,z)=\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r}L_{x_i}(x)\cdot L_{y_j}(y)\cdot L_{z_k}(z)f(x_i,y_j,z_k) \tag{3.33}$$

where

$$L_{x_i}(x)=\prod_{\substack{l=0\\l\neq i}}^{p}\frac{x-x_l}{x_i-x_l}\qquad L_{y_j}(y)=\prod_{\substack{l=0\\l\neq j}}^{q}\frac{y-y_l}{y_j-y_l}\qquad L_{z_k}(z)=\prod_{\substack{l=0\\l\neq k}}^{r}\frac{z-z_l}{z_k-z_l}$$

- For example, Eq. (3.32) can be expanded as

$$\begin{aligned}f_{1,1}(x,y)=&\frac{(x-x_1)(y-y_1)}{(x_0-x_1)(y_0-y_1)}f(x_0,y_0)+\frac{(x-x_0)(y-y_1)}{(x_1-x_0)(y_0-y_1)}f(x_1,y_0)\\&+\frac{(x-x_1)(y-y_0)}{(x_0-x_1)(y_1-x_0)}f(x_0,y_1)+\frac{(x-x_0)(y-y_0)}{(x_1-x_0)(y_1-y_0)}f(x_1,y_1)\end{aligned}$$

Example 3.10

The following table show the topology of a 3-D non-planar surface:

$z = f(x,y)$		y			
		0.2	**0.3**	**0.4**	**0.5**
	1.0	0.640	1.003	1.359	1.703
x	**1.5**	0.990	1.524	2.045	2.549
	2.0	1.568	2.384	3.177	3.943

Use the Newton interpolation to estimate the value of z at the coordinat (1.6, 0.33). As a comparison, the above data follows the function $f(x,y)=e^x\sin y+y-0.1$, thus $f(1.6, 0.33) = 1.8350$.

Solution

At $x = 1.0$:

j	y_j	$f(1.0, y_j)$	**I**	**II**	**III**
0	0.2	0.640	0.363	–0.007	0.005
1	0.3	1.003	0.356	–0.012	
2	0.4	1.359	0.344		
3	0.5	1.703			

$$f_1(1.0, y) = 0.640 + 0.363(y-0.2) - 0.007(y-0.2)(y-0.3) + 0.005(y-0.2)(y-0.3)(y-0.4),$$

$$f_1(1.0, 0.33) = 1.1108.$$

At $x = 1.5$:

j	y_j	$f(1.5, y_j)$	**I**	**II**	**III**
0	0.2	0.990	0.534	–0.013	–0.004
1	0.3	1.524	0.521	–0.017	
2	0.4	2.045	0.504		
3	0.5	2.549			

$$f_1(1.5, y) = 0.990 + 0.534(y-0.2) - 0.013(y-0.2)(y-0.3) - 0.004(y-0.2)(y-0.3)(y-0.4),$$

$$f_1(1.5, 0.33) = 1.6818.$$

At $x = 2.0$:

j	y_j	$f(2.0, y_j)$	**I**	**II**	**III**
0	0.2	1.568	0.816	–0.023	–0.004
1	0.3	2.384	0.793	–0.027	
2	0.4	3.177	0.766		
3	0.5	3.943			

$$f_1(2.0, y) = 1.568 + 0.816(y-0.2) - 0.023(y-0.2)(y-0.3) - 0.004(y-0.2)(y-0.3)(y-0.4),$$

$$f_1(2.0, 0.33) = 2.6245.$$

Next, at $y = 0.33$:

i	x_i	$f(x_i, 0.33)$	**I**	**II**
0	1.0	1.1108	0.5710	0.3717
1	1.5	1.6818	0.9427	
2	2.0	2.6245		

$$f_{1,1}(x, 0.33) = 1.1108 + 0.5710(x - 1.0) + 0.3717(x - 1.0)(x - 1.5),$$

$$f_{1,1}(1.6, 0.33) = 1.8406\,.$$

This produces a relative error of $\varepsilon_t = 0.305\%$.

Exercise

1. The fuel consumption of an engine has been recorded as shown in the following table.

Time, hour	Fuel, liter
1.2	0.33201
1.7	0.54739
1.8	0.60496
2.0	0.73891

If a user runs the engine for 1.55 hours, determine the estimated fuel consumption using the Newton and Lagrange interpolation methods.

2. The following data shows the height function of a hill a a distance x from a reference. Form a cubic polynomial via regression.

x_i	0	1	2	3	4	5	6	7	8
h_i	4	5	10	17	21	16	11	3	1

Also, calculate the corresponding standard deviation.

3. The following data represents temperature distribution $T(x,y)$ in a metal plate:

Temperature $T(x,y)$		y coordinate			
		0.5	1.0	1.5	2.0
x coordinate	0.5	7.51	10.05	12.70	15.67
	1.0	10.00	10.00	10.00	10.00
	1.5	12.51	9.95	7.32	4.33
	2.0	15.00	10.00	5.00	0.00

Estimate the temperature at the coordinate $(x, y) = (1.15, 1.42)$ using:

a. the Newton linear interpolation,
b. the Newton cubic interpolation.