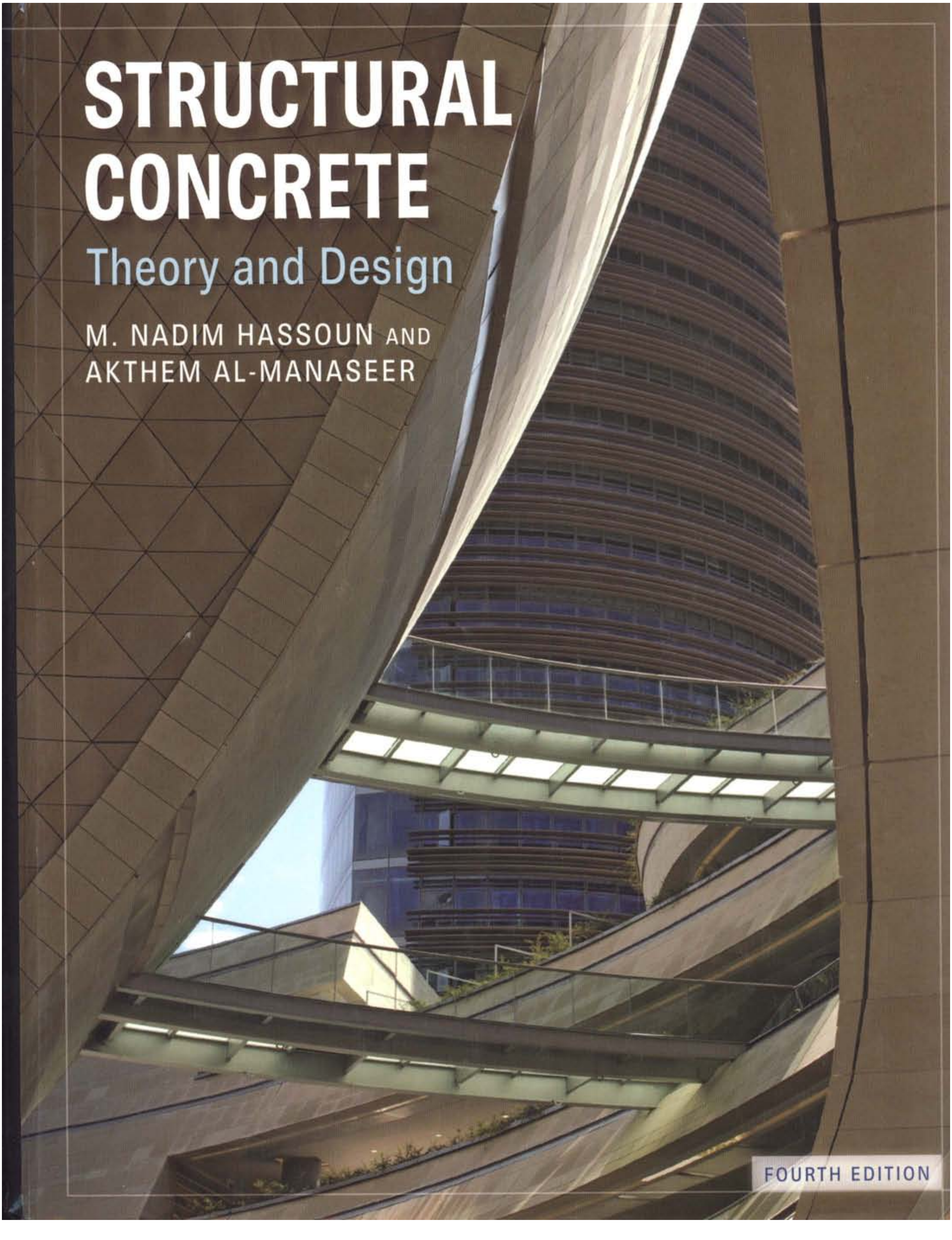
STRUCTURAL CONCRETE

Theory and Design

M. NADIM HASSOUN AND
AKTHEM AL-MANASEER



FOURTH EDITION

Structural Concrete

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Theory and Design

Fourth Edition

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A companion Web site for the book is available at www.wiley.com/college/hassoun. This Web site contains MSExcel spreadsheets that enable students to evaluate different design aspects of concrete members in an interactive environment, and a solutions manual for instructors.

PREFACE

The main objective of a course on structural concrete design is to develop, in the engineering student, the ability to analyze and design a reinforced concrete member subjected to different types of forces in a simple and logical manner using the basic principles of statistics and some empirical formulas based on experimental results. Once the analysis and design procedure is fully understood, its application to different types of structures becomes simple and direct, provided that the student has a good background in structural analysis.

The material presented in this book is based on the requirements of the American Concrete Institute (ACI) Building Code (318–08). Also, information has been presented on material properties, including volume changes of concrete, stress–strain behavior, creep, and elastic and nonlinear behavior or reinforced concrete.

Concrete structures are widely used in the United States and almost all over the world. The progress in the design concept has increased in the last few decades, emphasizing safety, serviceability, and economy. To achieve economical design of a reinforced concrete member, specific restrictions, rules, and formulas are presented in the codes to ensure both safety and reliability of the structure. Engineering firms expect civil engineering graduates to understand the code rules and, consequently, to be able to design a concrete structure effectively and economically with minimum training period or overhead costs. Taking this into consideration, this book is written to achieve the following objectives:

1. To present the material for the design of reinforced concrete members in a simple and logical approach.
2. To arrange the sequence of chapters in a way compatible with the design procedure of actual structures.
3. To provide a large number of examples in each chapter in clear steps to explain the analysis and design of each type of structural member.
4. To provide an adequate number of practical problems at the end of each chapter to achieve a high level of comprehension.
5. To explain the failure mechanism of a reinforced concrete beam due to flexure and to develop the necessary relationships and formulas for design.

6. To explain *why* the code used specific equations and specific restrictions on the design approach based either on a mathematical model or experimental results. This approach will improve the design ability of the student.
7. To provide adequate number of design aids to help the student in reducing the repetitive computations of specific commonly used values.
8. To enhance the student's ability to use a total quality and economical approach in the design of concrete structures and to help the student to design reinforced concrete members with confidence.
9. To explain the nonlinear behavior and the development of plastic hinges and plastic rotations in continuous reinforced concrete structures.
10. To provide a summary at the end of each chapter to help the student to review the materials of each chapter separately.
11. To provide new information on the design of special members, such as beams with variable depth (Chapter 8), stairs (Chapter 18), seismic design utilizing IBC 2006 (Chapter 20), and beams curved in plan (Chapter 21), that are not covered in other books on concrete.
12. To present information on the design of reinforced concrete frames, principles of limit design, and moment redistribution in continuous reinforced concrete structures.
13. To provide examples in SI units in all chapters of the book. Equivalent conversion factors from customary units to SI units are also presented. Design tables in SI units are given in Appendix B.
14. References are presented at the end of most chapters.

The book is an outgrowth of the author's lecture notes, which represent their teaching and industrial experience over the past 28 years. The industrial experience of the authors includes the design and construction supervision and management of many reinforced, prestressed, and precast concrete structures. This is in addition to the consulting work they performed for international design and construction firms, professional registration in the United Kingdom, Canada, and other countries, and a comprehensive knowledge of other European codes on the design of concrete structures.

The book is written to cover two courses in reinforced concrete design. Depending on the proficiency required, the first course may cover Chapters 1 through 11 and part of Chapter 13, whereas the second course may cover the remaining chapters. Parts of the late chapters may also be taught in the first course as needed. A number of optional sections have been included in various chapters. These sections are indicated by an asterisk (*) in the Table of Contents and may easily be distinguished from those that form the basic requirements of the first course. The optional sections may be covered in the second course or relegated to a reading assignment. Brief descriptions of the chapters are given below.

The first chapter of the book presents information on the historical development of concrete, codes of practice, loads and safety provisions, and design philosophy and concepts. The second chapter deals with the properties of concrete as well as steel reinforcement used in the design of reinforced concrete structures, including stress–strain relationships, modulus of elasticity and shear modulus of concrete, shrinkage, creep, fire resistance, high-performance concrete, and fibrous concrete. Because the current ACI Code emphasizes the strength approach based on strain limits, this approach has been adopted throughout the text. Chapters 3 and 4 cover the analysis and design of reinforced concrete sections based on strain limits. The behavior of reinforced concrete beams loaded to failure, the types of flexural failure, and failure mechanism are explained very clearly. It is essential for the student to understand the failure concept and the inherent reserve strength and ductility before using the necessary design formulas.

Chapter 5 covers alternative design methods based on methods described in Appendix A, B, and C of the ACI code. It explains the alternative load factors with the relative strength reduction factors and describes the strut and tie provisions.

Chapter 6 deals with the serviceability of reinforced concrete beams, including deflection and control of cracking. Chapters 7 and 8 cover the bond, development length, shear, and diagonal tension. In Chapter 8, expressions are presented for the design of members of variable depth in addition to prismatic sections and deep beams. It is quite common sometimes to design members with variable depth in actual structures. An example is introduced to explain the design of deep beams using the strut and tie approach.

Chapter 9 covers the design of one-way slabs, including joist-floor systems. Distributions of loads from slabs to beams and columns are also presented in this chapter to enhance the student's understanding of the design loads on each structural component. Chapter 10, 11, and 12 cover the design of axially loaded, eccentrically loaded, and long columns, respectively. Chapter 10 allows the student to understand the behavior of columns, failure conditions, ties and spirals, and other code limitations. Absorbing basic information, the student is introduced in Chapter 11 to the design of columns subjected to compression and bending. New mathematical models are introduced to analyze column sections controlled by compression or tension stresses. Biaxial bending for rectangular and circular columns are introduced using Bresler, PCA, and Hsu methods. Design of long columns is presented in Chapter 12 using the ACI moment-magnifier method.

Chapter 13 and 14 cover the design of footings and retaining walls, whereas Chapter 15 covers the design of reinforced concrete sections for shear and torsion. Torsional theories as well as ACI Code design procedure are explained. Chapter 16 deals with continuous beams and frames. A unique feature of this chapter is the introduction of the design of frames, frame hinges, limit state design collapse mechanism, rotation and plastic hinges, and moment redistribution. Adequate examples are presented to explain these concepts.

Design of two-way slabs introduced in Chapter 17. All types of two-way slabs, including waffle slabs, are presented with adequate examples. Summary of the design procedure is introduced with tables and diagrams. Chapter 18 covers the design of reinforced concrete stairs. Slabtype as well as stepped-type stairs are explained. The second type, although quite common, has not been covered in any text. Chapter 19 covers an introduction to prestressed concrete. Methods of prestressing, fully and partially prestressed concrete design, losses, and shear design are presented with examples. Chapter 20 presents the seismic design and analysis of members utilizing the IBC 2006 and the ACI code. Chapter 21 deals with the design of curved beams. In actual structures curved beams are used frequently. These beams are subjected to flexure, shear, and torsion.

In Appendix A and B of this book, design tables using customary units and SI units are presented.

The photos shown in this book were taken by the authors. We wish to express appreciation to John Gardner and Murat Saatcioglu from the University of Ottawa, Canada, for the photos provided in the seismic chapter.

Our sincere thanks go out to Nadim Wehbe, South Dakota State University, Ahmet Pamuk, Florida A&M University, M. Issa, University of Illinois Chicago, and Faisal Wafa, King Abdul-Aziz University, for their constructive comments to this edition. Our thanks to Basile Rabbat of the Portland Cement Association, Skokie, Illinois, for many discussions on the code interpretation.

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Finally, the book is written to provide basic reference materials on the analysis and design of structural concrete members in a simple, practical, and logical approach. Because this is a required course for seniors in civil engineering, we believe this book will be accepted by reinforced concrete instructors at different universities as well as designers who can make use of the information in their practical design of reinforced concrete structures.

M. Nadim Hassoun *Akthem Al-Manaseer*

NOTATION

c	Distance from extreme compression fiber to neutral axis
c_2	Side of rectangular column measured transverse to the span
C	Cross-sectional constant $\sum (1 - 0.63x/y)x^3 y/3$; compression force
C_c	Compression force in a concrete section with a depth equal to a
C_m	Correction factor applied to the maximum end moment in columns
C_r	Creep coefficient = creep strain per unit stress per unit length
C_s	Force in compression steel
C_t	Factor relating shear and torsional stress properties $= b_w d/\sum x^2 y$
C_w	Compression force in web
C_1	Force in the compression steel
d	Distance from extreme compression fiber to centroid of tension steel
d'	Distance from extreme compression fiber to centroid of compression steel
d_b	Nominal diameter of reinforcing bar
d_c	Distance from tension extreme fiber to center of bar closest to that fiber, used for crack control
d_t	Distance from extreme compression fibers to extreme tension steel
D	Dead load, diameter of a circular section
e	Eccentricity of load
e'	Eccentricity of load with respect to centroid of tension steel
E	Modulus of elasticity, force created by earthquake
E_c	Modulus of elasticity of concrete $= 33w^{1.5}\sqrt{f'_c}$
E_{cb}	Modulus of elasticity of beam concrete
E_{cc}	Modulus of elasticity of column concrete
E_{cs}	Modulus of elasticity of slab concrete
EI	Flexural stiffness of compression member
E_s	Modulus of elasticity of steel $= 29 \times 10^6$ psi $= 2 \times 10^5$ MPa
f	Flexural stress

f_c	Maximum flexural compressive stress in concrete due to service loads
f_{ca}	Allowable compressive stress in concrete (alternate design method)
f_c'	28-day compressive strength of concrete (standard cylinder strength)
f_{ci}	Compressive strength of concrete at transfer (initial prestress)
f_{pc}	Compressive stress in concrete due to prestress after all losses
f_{pe}	Compressive stress in concrete at extreme fiber due to the effective prestressing force after all losses
f_{ps}	Stress in prestress steel at nominal strength
f_{pu}	Tensile strength of prestressing tendons
f_{py}	Yield strength of prestressing tendons
f_r	Modulus of rupture of concrete $= 7.5\lambda\sqrt{f_c'}$ psi
f_s	Stress in tension steel due to service load
f_s'	Stress in the compression steel due to service load
f_{se}	Effective stress in prestressing steel after all losses
f_t	Tensile stress in concrete
f_y	Yield strength of steel reinforcement
F	Lateral pressure of liquids
F_n	Nominal strength of a strut, tie, or nodal zone
F_{ns}	Nominal strength of a strut
F_{nt}	Nominal strength of a tie
G	Shear modulus of concrete (in torsion) $= 0.45E_c$
h	Total depth of beam or slab or column
h_f	Depth of flange in flanged sections
h_p	Total depth of shearhead cross section
H	Lateral earth pressure
I	Moment of inertia
I_b	Moment of inertia of gross section of beam about its centroidal axis
I_c	Moment of inertia of gross section of column
I_{cr}	Moment of inertia of cracked transformed section
I_e	Effective moment of inertia, used in deflection
I_g	Moment of inertia of gross section neglecting steel
I_s	Moment of inertia of gross section of slab
I_{se}	Moment of inertia of steel reinforcement about centroidal axis of section
J	Polar moment of inertia
K	Kip = 1000 lb, a factor used to calculate effective column length
K_b	Flexural stiffness of beam
K_c	Flexural stiffness of column
K_{ec}	Flexural stiffness of equivalent column
K_s	Flexural stiffness of slab
K_t	Torsional stiffness of torsional member
KN	Kilonewton
Ksi	Kip per square inch
ℓ_n	Clear span
ℓ_u	Unsupported length of column
L	Live load, span length
l_d	Development length

l_{dh}	l_{hb} times the applicable modification factor
l_{hb}	Basic development length of a standard hook
l_n	Clear span
l_u	Unsupported length of compression member
l_v	Length of shearhead arm
l_1	Span length in the direction of moment
l_2	Span length in direction transverse to span l_1
M	Bending moment
M_1	Smaller end moment at end of column
M_2	Larger end moment at end of column
M_a	Maximum service load moment
M_b	Balanced moment in columns, used with P_b
M_{cr}	Cracking moment
M_m	Modified moment
M_n	Nominal moment strength $= M_u/\phi$
M'_n	Nominal moment strength using an eccentricity e'
M_o	Total factored moment
M_p	Plastic moment
M_u	Moment strength due to factored loads
M_{u1}	Part of M_u when calculated as singly reinforced
M_{u2}	Part of M_u due to compression reinforcement or overhanging flanges in T- or L-sections
M'_u	Moment strength using an eccentricity e'
M_v	Shearhead moment resistance
n	Modular ratio $= E_s/E_c$
N	Normal force
N_u	Factored normal load
N_1	Normal force in bearing at base of column
NA	Neutral axis
psi	Pounds per square inch
P_{cp}	Outside perimeter of gross area $= 2(x_0 + y_0)$
P_o	Perimeter of shear flow in area A_o
P	Unfactored concentrated load
P_b	Balanced load in column (at failure)
P_c	Euler buckling load
P_n	Nominal axial strength of column for a given e
P_o	Axial strength of a concentrically loaded column
P_s	Prestressing force in the tendon at the jacking end
P_u	Factored load $= \phi P_n$
P_x	Prestressing force in the tendon at any point x
q	Soil-bearing capacity
q_a	Allowable bearing capacity of soil
q_u	Ultimate bearing capacity of soil using factored loads
r	Radius of gyration, radius of a circle
R	Resultant of force system, reduction factor for long columns, or $R = R_u/\phi$
R_u	A factor $= M_u/bd^2$
s	Spacing between bars, stirrups, or ties

SI	International system of units
t	Thickness of a slab
T	Torque, tension force
T_c	Nominal torsional strength provided by concrete
T_{cr}	Cracking torsional moment
T_n	Nominal torsional strength provided by concrete and steel
T_s	Nominal torsional strength provided by reinforcement
T_u	Torque provided by factored load $= \phi T_n$
u	Bond stress
U	Design strength required to resist factored loads
V	Shear stress produced by working loads
v_c	Shear stress of concrete
v_{cr}	Shear stress at which diagonal cracks develop
v_h	Horizontal shear stress
v_t	Shear stress produced by a torque
v_u	Shear stress produced by factored loads
V	Unfactored shear force
V_c	Shear strength of concrete
V_{ci}	Nominal shear strength of concrete when diagonal cracking results from combined shear and moment
V_{cw}	Nominal shear strength of concrete when diagonal cracking results from excessive principal tensile stress in web
V_d	Shear force at section due to unfactored dead load (d = distance from the face of support)
V_n	Nominal shear strength $= V_c + V_s$
V_p	Vertical component of effective prestress force at section
V_s	Shear strength carried by reinforcement
V_u	Shear force due to factored loads
w	Width of crack at the extreme tension fiber, unit weight of concrete
w_u	Factored load per unit length of beam or per unit area of slab
W	Wind load or total load
x_o	Length of the short side of a rectangular section
x_1	Length of the short side of a rectangular closed stirrup
y_b	Same as y_t, except to extreme bottom fibers
y_o	Length of the long side of a rectangular section
y_t	Distance from centroidal axis of gross section, neglecting reinforcement, to extreme top fiber
y_1	Length of the long side of a rectangular closed stirrup
α	Angle of inclined stirrups with respect to longitudinal axis of beam, ratio of stiffness of beam to that of slab at a joint
α_c	Ratio of flexural stiffness of columns to combined flexural stiffness of the slabs and beams at a joint; $(\Sigma\ K_c)/\Sigma(K_s + K_b)$
α_{ec}	Ratio of flexural stiffness of equivalent column to combined flexural stiffness of the slabs and beams at a joint: $(K_{ec})/\Sigma(K_s + K_b)$
α_m	Average value of a for all beams on edges of a panel
α_v	Ratio of stiffness of shearhead arm to surrounding composite slab section
β	Ratio of long to short side of rectangular footing, measure of curvature in biaxial bending

β_1	Ratio of a/c, where a = depth of stress block and c = distance between neutral axis and extreme compression fibers (This factor is 0.85 for $f_c' \leq 4000$ psi and decreases by 0.05 for each 1000 psi in excess of 4000 psi but is at least 0.65.)
β_a	Ratio of unfactored dead load to unfactored live load per unit area
β_c	Ratio of long to short sides of column or loaded area
β_{dns}	Ratio of maximum factored dead load moment to maximum factored total moment
β_t	Ratio of torsional stiffness of edge beam section to flexural stiffness of slab: $E_{cb}C/2E_{cs}I_s$
γ	Distance between rows of reinforcement on opposite sides of columns to total depth of column h
γ_f	Fraction of unbalanced moment transferred by flexure at slab-column connections
γ_p	Factor for type of prestressing tendon (0.4 or 0.28)
γ_v	Fraction of unbalanced moment transferred by eccentricity of shear at slab-column connections
δ	Magnification factor
δ_{ns}	Moment magnification factor for frames braced against sidesway
δ_s	Moment magnification factor for frames not braced against sidesway
Δ	Deflection
ε	Strain
ε_c	Strain in concrete
ε_s	Strain in steel
ε_s'	Strain in compression steel
ε_y	Yield strain = f_y/E_s
θ	Slope angle
λ	Multiplier factor for reduced mechanical properties of lightweight concrete
λ_Δ	Multiplier for additional long-time deflection
μ	Poisson's ratio; coefficient of friction
ζ	Parameter for evaluating capacity of standard hook
π	A constant equal to approximately 3.1416
ρ	Ratio of the tension steel area to the effective concrete area = A_s/bd
ρ'	Ratio of compression steel area to effective concrete area = A_s'/bd
ρ_1	$(\rho - \rho')$
ρ_b	Balanced steel ratio
ρ_g	Ratio of total steel area to total concrete area
ρ_p	Ratio of prestressed reinforcement A_{ps}/bd
ρ_s	Ratio of volume of spiral steel to volume of core
ρ_w	$A_s/b_w d$
ϕ	Strength-reduction factor
ψ_e	Factor used to modify development length based on reinforcement coating
ψ_s	Factor used to modify development length based on reinforcing size
ψ_t	Factor used to modify development length based on reinforcement location
ω	Tension reinforcing index = $\rho f_y/f'c$
ω'	Compression reinforcing index = $\rho' f_y/f_c'$
ω_p	Prestressed steel index = $\rho_p f_{ps}/f_c'$
ω_{pw}	Prestressed steel index for flanged sections
ω_w	Tension reinforcing index for flanged sections
ω_w'	Compression reinforcing index for flanged sections computed as for ω, ω_p and ω'

CONVERSION FACTORS

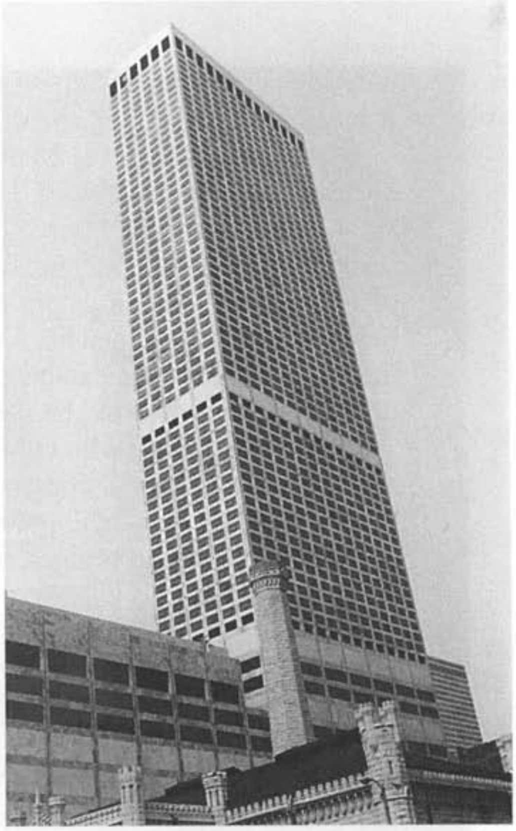
To Convert	to	Multiply By
1. Length		
Inch	Millimeter	25.4
Foot	Millimeter	304.8
Yard	Meter	0.9144
Meter	Foot	3.281
Meter	Inch	39.37
2. Area		
Square inch	Square millimeter	645
Square foot	Square meter	0.0929
Square yard	Square meter	0.836
Square meter	Square foot	10.76
3. Volume		
Cubic inch	Cubic millimeter	16390
Cubic foot	Cubic meter	0.02832
Cubic yard	Cubic meter	0.765
Cubic foot	Liter	28.3
Cubic meter	Cubic foot	35.31
Cubic meter	Cubic yard	1.308
4. Mass		
Ounce	Gram	28.35
Pound (lb)	Kilogram	0.454
Pound	Gallon	0.12

To Convert	to	Multiply By
Short ton (2000 lb)	Kilogram	907
Long ton (2240 lb)	Kilogram	1016
Kilogram	Pound (lb)	2.205
Slug	Kilogram	14.59
5. Density		
Pound/cubic foot	Kilogram/cubic meter	16.02
Kilogram/cubic meter	Pound/cubic foot	0.06243
6. Force		
Pound (lb)	Newton (N)	4.448
Kip (1000 lb)	Kilonewton (kN)	4.448
Newton (N)	Pound	0.2248
Kilonewton (kN)	Kip (K)	0.225
7. Force/length		
Kip/foot	Kilonewton/meter	14.59
Kilonewton/meter	Pound/foot	68.52
Kilonewton/meter	Kip/foot	0.06852
8. Force/area (stress)		
Pound/square inch (psi)	Newton/square centimeter	0.6895
Pound/square inch (psi)	Newton/square millimeter (MPa)	0.0069
Kip/square inch (Ksi)	Meganewton/square meter	6.895
Kip/square inch (Ksi)	Newton/square millimeter	6.895
Pound/square foot	Kilonewton/square meter	0.04788
Pound/square foot	Newton/square meter	47.88
Kip/square foot	Kilonewton/square meter	47.88
Newton/square millimeter	Kip/square inch (Ksi)	0.145
Kilonewton/square meter	Kip/square foot	0.0208
Kilonewton/square meter	Pound/square foot	20.8
9. Moments		
Foot·Kip	Kilonewton·meter	1.356
Inch·Kip	Kilonewton·meter	0.113
Inch·Kip	Kilogram force·meter	11.52
Kilonewton·meter	Foot·Kip	0.7375

Structural Concrete

CHAPTER 1

INTRODUCTION



Water Tower Place, Chicago, 74 stories, tallest concrete building in the United States.

1.1 STRUCTURAL CONCRETE

The design of different structures is achieved by performing, in general, two main steps: (1) determining the different forces acting on the structure using proper methods of structural analysis, and (2) proportioning all structural members economically, considering the safety, stability, serviceability, and functionality of the structure. Structural concrete is one of the materials commonly used to design all types of buildings. Its two component materials, concrete and steel, work together to form structural members that can resist many types of loadings. The key to its performance lies in strengths that are complementary: Concrete resists compression and steel reinforcement resists tension forces.

The term *structural concrete* indicates all types of concrete used in structural applications. Structural concrete may be plain, reinforced, prestressed, or partially prestressed concrete; in addition, concrete is used in composite design. Composite design is used for any structural member, such as beams or columns, when the member contains a combination of concrete and steel shapes.

1.2 HISTORICAL BACKGROUND

The first modern record of concrete is as early as 1760, when John Smeaton used it in Britain in the first lock on the river Calder [1]. The walls of the lock were made of stones filled in with concrete. In 1796, J. Parker discovered Roman natural cement, and 15 years later Vicat burned a mixture of clay and lime to produce cement. In 1824, Joseph Aspdin manufactured portland cement in Wakefield, Britain. It was called portland cement because when it hardened, it resembled stone from the quarries of the Isle of Portland.

In France, François Marte Le Brun built a concrete house in 1832 in Moissac, in which he used concrete arches of 18-ft span. He used concrete to build a school in St. Aignan in 1834 and a church in Corbarièce in 1835. Joseph Louis Lambot [2] exhibited a small rowboat made of reinforced concrete at the Paris Exposition in 1854. In the same year, W. B. Wilkinson of England obtained a patent for a concrete floor reinforced by twisted cables. The Frenchman François Cignet obtained his first patent in 1855 for his system of iron bars, which were embedded in concrete floors and extended to the supports. One year later, he added nuts at the screw ends of the bars, and in 1869, he published a book describing the applications of reinforced concrete.

Joseph Monier, who obtained his patent in Paris on July 16, 1867, was given credit for the invention of reinforced concrete [3]. He made garden tubs and pots of concrete reinforced with iron mesh, which he exhibited in Paris in 1867. In 1873, he registered a patent to use reinforced concrete in tanks and bridges, and four years later, he registered another patent to use it in beams and columns [1].

In the United States, Thaddeus Hyatt conducted flexural tests on 50 beams that contained iron bars as tension reinforcement and published the results in 1877. He found that both concrete and steel can be assumed to behave in a homogeneous manner for all practical purposes. This assumption was important for the design of reinforced concrete members using elastic theory. He used prefabricated slabs in his experiments and considered prefabricated units to be best cast in T-sections and placed side by side to form a floor slab. Hyatt is generally credited with developing the principles upon which the analysis and design of reinforced concrete are now based.

A reinforced concrete house was built by W. E. Ward near Port Chester, New York, in 1875. It used reinforced concrete for walls, beams, slabs, and staircases. P. B. In 1877, Write described in the *American Architect and Building News* the applications of reinforced concrete in Ward's house as a new method in building construction.

E. L. Ransome, head of the Concrete Steel Company in San Francisco, used reinforced concrete in 1879 and deformed bars for the first time in 1884. During 1889–1891, he built the two-story Leland Stanford Museum in San Francisco using reinforced concrete. He also built a reinforced concrete bridge in San Francisco. In 1900, after Ransome introduced the reinforced concrete skeleton, the thick wall system started to disappear in construction. He registered the skeleton type of structure in 1902, using spiral reinforcement in the columns as was suggested by Armand Considére of France. A. N. Talbot, of the University of Illinois, and F. E. Turneaure and M. O. Withney, of the University of Wisconsin, conducted extensive tests on concrete to determine its behavior, compressive strength, and modulus of elasticity.

In Germany, G. A. Wayass bought the French Monier patent in 1879 and published his book on Monier methods of construction in 1887. Rudolph Schuster bought the patent rights in Austria, and the name of Monier spread throughout Europe, which is the main reason for crediting Monier as the inventor of reinforced concrete.

In 1900, the Ministry of Public Works in France called for a committee headed by Armand Considére, chief engineer of roads and bridges, to establish specifications for reinforced concrete, which were published in 1906.

Reinforced concrete was further refined by introducing some precompression in the tension zone to decrease the excessive cracks. This refinement was the preliminary introduction of partial and full prestressing. In 1928, Eugene Freyssinet established the practical technique of using prestressed concrete [4].

The Barwick House, a three-story concrete building built in 1905, Montreal, Canada.

From 1915 to 1935, research was conducted on axially loaded columns and creep effects on concrete; in 1940, eccentrically loaded columns were investigated. Ultimate-strength design started to receive special attention, in addition to diagonal tension and prestressed concrete. The American Concrete Institute Code (ACI Code) specified the use of ultimate-strength design in 1963 and included this method in all later codes. Building codes and specifications for the design of reinforced concrete structures are established in most countries, and research continues on developing new applications and more economical designs.

1.3 ADVANTAGES AND DISADVANTAGES OF REINFORCED CONCRETE

Reinforced concrete, as a structural material, is widely used in many types of structures. It is competitive with steel if economically designed and executed.

The advantages of reinforced concrete can be summarized as follows:

1. It has a relatively high compressive strength.
2. It has better resistance to fire than steel.
3. It has a long service life with low maintenance cost.
4. In some types of structures, such as dams, piers, and footings, it is the most economical structural material.
5. It can be cast to take the shape required, making it widely used in precast structural components. It yields rigid members with minimum apparent deflection.

The disadvantages of reinforced concrete can be summarized as follows:

1. It has a low tensile strength of about one-tenth of its compressive strength.
2. It needs mixing, casting, and curing, all of which affect the final strength of concrete.
3. The cost of the forms used to cast concrete is relatively high. The cost of form material and artisanry may equal the cost of concrete placed in the forms.
4. It has a low compressive strength as compared to steel (the ratio is about 1:10, depending on materials), which leads to large sections in columns of multistory buildings.
5. Cracks develop in concrete due to shrinkage and the application of live loads.

1.4 CODES OF PRACTICE

The design engineer is usually guided by specifications called the codes of practice. Engineering specifications are set up by various organizations to represent the minimum requirements necessary for the safety of the public, although they are not necessarily for the purpose of restricting engineers.

Most codes specify design loads, allowable stresses, material quality, construction types, and other requirements for building construction. The most significant code for structural concrete design in the United States is the Building Code Requirements for Structural Concrete, ACI 318, or the ACI Code. Most of the design examples of this book are based on this code. Other codes of practice and material specifications in the United States include the International Code, the Uniform Building Code, Standard Building Code, National Building Code, Basic Building Code, South Florida Building Code, American Association of State Highway and Transportation Officials (AASHTO) specifications, and specifications issued by the American Society for Testing and Materials (ASTM), American Railway Engineering Association (AREA), and Bureau of Reclamation, Department of the Interior.

Different codes other than those of the United States include the British Standard (BS) Code of Practice for Reinforced Concrete, CP 110 and BS 8110; the National Building Code of Canada; the German Code of Practice for Reinforced Concrete, DIN 1045; Specifications for Steel Reinforcement (U.S.S.R.); and Technical Specifications for the Theory and Design of Reinforced Concrete Structures, CC-BA (France), and the CEB Code (Comitè European Du Beton).

1.5 DESIGN PHILOSOPHY AND CONCEPTS

The design of a structure may be regarded as the process of selecting the proper materials and proportioning the different elements of the structure according to state-of-the-art engineering science and technology. In order to fulfill its purpose, the structure must meet the conditions of safety, serviceability, economy, and functionality. This can be achieved using design approach-based strain limits in concrete and steel reinforcement.

The unified design method (UDM) is based on the strength of structural members assuming a failure condition, whether due to the crushing of the concrete or to the yield of the reinforcing steel bars. Although there is some additional strength in the bars after yielding (due to strain hardening), this additional strength is not considered in the analysis of reinforced concrete members. In this approach, the actual loads, or working loads, are multiplied by load factors to obtain the factored design loads. The load factors represent a high percentage of the factor for safety required in the design. Details of this method are presented in Chapters 3, 4, and 11. The

ACI Code emphasizes this method of design, and its provisions are presented in the body of the Code. The reason for introducing this approach by the ACI Code relates to the fact that different design methods were developed for reinforced and prestressed concrete beams and columns. Also, design procedures for prestressed concrete were different from reinforced concrete. The purpose of the Code approach is to simplify and unify the design requirements for reinforced and prestressed flexural members and compression members.

A second approach for the design of reinforced and prestressed concrete flexural and compression members is called the strength design method, or the alternative provisions (ADM), as introduced in the ACI Code, Appendix B. When this method is used in the design, the designer must adhere to all sections of Appendixes B and C and substitute accordingly for the corresponding sections of the Code. Reinforcement limits, strength reduction factors, load factors, and moment redistribution are affected. The provisions of this method satisfy the Code and are equally acceptable.

A third approach for the design of concrete members is called the strut and tie method (STM). The provisions of this method are introduced in the ACI Code, Appendix A. It applies effectively in regions of discontinuity such as support and load applications on beams. Consequently, the structural element is divided into segments and then analyzed using the truss analogy approach, where the concrete resists compression forces as a strut, while the steel reinforcement resists tensile forces as a tie.

A basic method that is not commonly used is called the working stress design or the elastic design method. The design concept is based on the elastic theory assuming a straight line stress distribution along the depth of the concrete section under service loads. The members are proportioned on the basis of certain allowable stresses in concrete and steel. The allowable stresses are fractions of the crushing strength of concrete and yield strength of steel. This method has been deleted from the ACI Code. The application of this approach is still used in the design of prestressed concrete members under service load conditions, as shown in Chapter 19.

Limit state design is a further step in the strength design method. It indicates the state of the member in which it ceases to meet the service requirements such as losing its ability to withstand external loads, or suffering excessive deformation, cracking, or local damage. According to the limit state design, reinforced concrete members have to be analyzed with regard to three limiting states:

1. Load carrying capacity (safety, stability, and durability)
2. Deformation (deflections, vibrations, and impact)
3. The formation of cracks.

The aim of this analysis is to ensure that no limiting state will appear in the structural member during its service life.

1.6 UNITS OF MEASUREMENT

Two units of measurement are commonly used in the design of structural concrete. The first is the U.S. customary system (lying mostly in its human scale and its ingenious use of simple numerical proportions), and the second is the SI (Le Système International d'Unités), or metric, system.

The metric system is planned to be in universal use within the coming few years. The United States is committed to change to SI units. Great Britain, Canada, Australia, and other countries have been using SI units for several years.

The base units in the SI system are the units of length, mass, and time, which are the meter (m), the kilogram (kg), and the second (s), respectively. The unit of force, a derived unit called the newton (N), is defined as the force that gives the acceleration of 1 meter per second per second (1 m/s^2) to a mass of 1 kg, or $1 \text{ N} = 1 \text{ kg} \times \text{m/s}^2$.

The weight of a body, W, which is equal to the mass, m, multiplied by the local gravitational acceleration, g (9.81 m/s^2), is expressed in newtons (N). The weight of a body of 1 kg mass is $W = mg = 1 \text{ kg} \times 9.81 \text{ m/s}^2 = 9.81 \text{ N}$.

Multiples and submultiples of the base SI units can be expressed through the use of prefixes. The prefixes most frequently used in structural calculations are the kilo (k), mega (M), milli (m), and micro (μ). For example,

$$1 \text{ km} = 1000 \text{ m} \quad 1 \text{ mm} = 0.001 \text{ m} \quad 1 \, \mu\text{m} = 10^{-6} \text{ m}$$

$$1 \text{ kN} = 1000 \text{ N} \quad 1 \text{ Mg} = 1000 \text{ kg} = 10^6 \text{ g}$$

1.7 LOADS

Structural members must be designed to support specific loads.

Loads are those forces for which a given structure should be proportioned. In general, loads may be classified as dead or live.

Dead loads include the weight of the structure (its self-weight) and any permanent material placed on the structure, such as tiles, roofing materials, and walls. Dead loads can be determined with a high degree of accuracy from the dimensions of the elements and the unit weight of materials.

Live loads are all other loads that are not dead loads. They may be steady or unsteady or movable or moving; they may be applied slowly, suddenly, vertically, or laterally, and their magnitudes may fluctuate with time. In general, live loads include the following:

- Occupancy loads caused by the weight of the people, furniture, and goods
- Forces resulting from wind action and temperature changes
- The weight of snow if accumulation is probable
- The pressure of liquids or earth on retaining structures
- The weight of traffic on a bridge
- Dynamic forces resulting from moving loads (impact), earthquakes, or blast loading

The ACI Code does not specify loads on structures; however, occupancy loads on different types of buildings are prescribed by the American National Standards Institute (ANSI) [5]. Some typical values are shown in Table 1.1. Table 1.2 on page 7 shows weights and specific gravity of various materials.

AASHTO and AREA specifications prescribe vehicle loadings on highway and railway bridges, respectively. These loads are given in Refs. 6 and 7.

Snow loads on structures may vary between 10 and 40 lb/ft^2 (0.5 and 2 kN/m^2), depending on the local climate.

Wind loads may vary between 15 and 30 lb/ft^2, depending on the velocity of wind. The wind pressure of a structure, F, can be estimated from the following equation:

$$F = 0.00256 C_s V^2 \tag{1.1}$$

where

V = velocity of air (mi/h)

C_s = shape factor of the structure

F = the dynamic wind pressure (lb/ft^2)

Table 1.1 Typical Uniformly Distributed Design Loads

Occupancy	Contents	Design Live Load lb/ft^2	kN/m^2
Assembly hall	Fixed seats	60	2.9
	Movable seats	100	4.8
Hospital	Operating rooms	60	2.9
	Private rooms	40	1.9
Hotel	Guest rooms	40	1.9
	Public rooms	100	4.8
	Balconies	100	4.8
Housing	Private houses and apartments	40	1.9
	Public rooms	100	4.8
Institution	Classrooms	40	1.9
	Corridors	100	4.8
Library	Reading rooms	60	2.9
	Stack rooms	150	7.2
Office building	Offices	50	2.4
	Lobbies	100	4.8
Stairs (or balconies)		100	4.8
Storage warehouses	Light	100	4.8
	Heavy	250	12.0
Yards and terraces		100	4.8

Table 1.2 Density and Specific Gravity of Various Materials

Material	Density lb/ft^3	kg/m^3	Specific Gravity
Building materials			
Bricks	120	1,924	1.8–2.0
Cement, portland, loose	90	1,443	—
Cement, portland, set	183	2,933	2.7–3.2
Earth, dry, packed	95	1,523	—
Sand or gravel, dry, packed	100–120	1,600–1,924	—
Sand or gravel, wet	118–120	1,892–1,924	—
Liquids			
Oils	58	930	0.9–0.94
Water (at 4 °C)	62.4	1,000	1.0
Ice	56	898	0.88–0.92
Metals and minerals			
Aluminum	165	2,645	2.55–2.75
Copper	556	8,913	9.0
Iron	450	7,214	7.2
Lead	710	11,380	11.38
Steel, rolled	490	7,855	7.85
Limestone or marble	165	2,645	2.5–2.8
Sandstone	147	2,356	2.2–2.5
Shale or slate	175	2,805	2.7–2.9
Normal-weight concrete			
Plain	145	2,324	2.2–2.4
Reinforced or prestressed	150	2,405	2.3–2.5

As an example, for a wind of 100 mi/h with $C_s = 1$, the wind pressure is equal to 25.6 lb/ft^2. It is sometimes necessary to consider the effect of gusts in computing the wind pressure by multiplying the wind velocity in Eq. 1.1 by a gust factor, which generally varies between 1.1 and 1.3.

The shape factor, C_s, varies with the horizontal angle of incidence of the wind. On vertical surfaces of rectangular buildings, C_s may vary between 1.2 and 1.3. Detailed information on wind loads can be found in Ref. 5.

1.8 SAFETY PROVISIONS

Structural members must always be proportioned to resist loads greater than the service or actual load in order to provide proper safety against failure. In the strength design method, the member is designed to resist factored loads, which are obtained by multiplying the service loads by load factors. Different factors are used for different loadings. Because dead loads can be estimated quite accurately, their load factors are smaller than those of live loads, which have a high degree of uncertainty. Several load combinations must be considered in the design to compute the maximum and minimum design forces. Reduction factors are used for some combinations of loads to reflect the low probability of their simultaneous occurrences. The ACI Code presents specific values of load factors to be used in the design of concrete structures (see Chapter 3, Section 3.5).

In addition to load factors, the ACI Code specifies another factor to allow an additional reserve in the capacity of the structural member. The nominal strength is generally calculated using accepted analytical procedure based on statistics and equilibrium; however, in order to account for the degree of accuracy within which the nominal strength can be calculated, and for adverse variations in materials and dimensions, a strength reduction factor, ϕ, should be used in the strength design method. Values of the strength reduction factors are given in Chapter 3, Section 3.6.

To summarize the above discussion, the ACI Code has separated the safety provision into an overload or load factor and to an undercapacity (or strength reduction) factor, ϕ. A safe design is achieved when the structure's strength, obtained by multiplying the nominal strength by the reduction factor, ϕ, exceeds or equals the strength needed to withstand the factored loadings (service loads times their load factors). For example,

$$M_u \leq \phi M_n \quad \text{and} \quad V_u \leq \phi V_n \tag{1.2}$$

where

M_u and V_u = external factored moment and shear forces

M_n and V_n = nominal flexural strength and shear strength of the member, respectively

Given a load factor of 1.2 for dead load and a load factor of 1.6 for live load, the overall safety factor for a structure loaded by a dead load, D, and a live load, L, is

$$\text{Factor of safety} = \frac{1.2D + 1.6L}{D + L}\left(\frac{1}{\phi}\right) = \frac{1.2 + 1.6(L/D)}{1 + (L/D)}\left(\frac{1}{\phi}\right) \tag{1.3}$$

The factor of safety for the various values of ϕ and L/D ratios is shown below.

ϕ	0.9				0.8				0.7			
L/D	0	1	2	3	0	1	2	3	0	1	2	3
Factor of Safety	1.33	1.56	1.63	1.67	1.50	1.74	1.83	1.88	1.71	2.00	2.10	2.15

For members subjected to flexure (beams), with tension-controlled sections, $\phi = 0.9$, and the factor of safety ranges between 1.33 for $L/D = 0$ and 1.67 for $L/D = 3$. These values are less than those specified by the ACI Code 318–99 of 1.56 for $L/D = 0$ and 1.81 for $L/D = 3.0$ based on load factors of 1.4 for the dead load and 1.7 for the live load. This reduction ranges between 17 and 8% respectively.

For members subjected to axial forces (spiral columns), $\phi = 0.7$, and the factor of safety ranges between 1.71 for $L/D = 0$ and 2.15 for $L/D = 3$. The increase in the factor of safety in columns reflects the greater overall safety requirements of these critical building elements.

A general format of Eq. 1.2 may be written as follows [8]:

$$\phi R \geq v_0 \Sigma (v_i Q_i) \qquad (1.4)$$

where

R_n = nominal strength of the structural number
ϕ = undercapacity factor (<1.0)
ΣQ_i = sum of load effects
v_i = overload factor
v_0 = analysis factor (>1.0)

The subscript i indicates the load type, such as dead load, live load, and wind load. The analysis factor, v_0, is greater than 1.0 and is introduced to account for uncertainties in structural analysis. The overload factor, v_i, is introduced to account for several factors such as an increase in live load due to a change in the use of the structure and variations in erection procedures. The design concept is referred to as load and resistance factor design (LRFD) [8,9].

1.9 STRUCTURAL CONCRETE ELEMENTS

Structural concrete can be used for almost all buildings, whether single story or multistory. The concrete building may contain some or all of the following main structural elements, which are explained in detail in other chapters of the book:

- *Slabs* are horizontal plate elements in building floors and roofs. They may carry gravity loads as well as lateral loads. The depth of the slab is usually very small relative to its length or width (Chapters 9 and 17).
- *Beams* are long, horizontal or inclined members with limited width and depth. Their main function is to support loads from slabs (Chapters 3 and 4).
- *Columns* are critical members that support loads from beams or slabs. They may be subjected to axial loads or axial loads and moments (Chapters 10 and 11).
- *Frames* are structural members that consist of a combination of beams and columns or slabs, beams, and columns. They may be statically determinate or statically indeterminate frames (Chapter 16).
- *Footings* are pads or strips that support columns and spread their loads directly to the soil (Chapter 13).
- *Walls* are vertical plate elements resisting gravity as well as lateral loads as in the case of basement walls (Chapter 14).

1.10 STRUCTURAL CONCRETE DESIGN

The first step in the design of a building is the general planning carried out by the architect to determine the layout of each floor of the building to meet the owner's requirements. Once the architectural plans are approved, the structural engineer then determines the most adequate structural system to ensure the safety and stability of the building. Different structural options must be considered to determine the most economical solution based on the materials available and the soil condition. This result is normally achieved by

1. Idealizing the building into a structural model of load-bearing frames and elements
2. Estimating the different types of loads acting on the building
3. Performing the structural analysis using computer or manual calculations to determine the maximum moments, shear, torsional forces, axial loads, and other forces
4. Proportioning the different structural elements and calculating the reinforcement needed
5. Producing structural drawings and specifications with enough details to enable the contractor to construct the building properly

1.11 ACCURACY OF CALCULATIONS

In the design of concrete structures, exact calculations to determine the size of the concrete elements are not needed. Calculators and computers can give an answer to many figures after the decimal point. For a practical size of a beam, slab, or column, each dimension should be approximated to the nearest 1 or $\frac{1}{2}$ inch. Moreover, the steel bars available in the market are limited to specific diameters and areas, as shown in Table A.12 (Appendix A). The designer should choose a group of bars from the table with an area equal to or greater than the area obtained from calculations. Also, the design equations in this book based on the ACI Code are approximate. Therefore, for a practical and economical design, it is adequate to use four figures (or the full number with no fractions if it is greater than four figures) for the calculation of forces, stresses, moments, or dimensions such as length or width of section. For strains, use five or six figures because strains are very small quantities measured in a millionth of an inch (for example, a strain of 0.000358 in./in.). Stresses are obtained by multiplying the strains by the modulus of elasticity of the material, which has a high magnitude (for example, 29,000,000 lb/in.^{2}) for steel. Any figures less than five or six figures in strains will produce quite a change in stresses.

Examples

For forces, use 28.45 K, 2845 lb, 567.8 K (four figures).
For force/length, use 2.451 K/ft or 2451 lb/ft.
For length or width, use 14.63 in., 1.219 ft (or 1.22 ft).
For areas, use 7.537 in.^{2}, and for volumes, use 48.72 in.^{3}.
For strains, use 0.002078.

1.12 CONCRETE HIGH-RISE BUILDINGS

High-rise buildings are becoming the dominant feature of many U.S. cities; a great number of these buildings are designed and constructed in structural concrete.

Although at the beginning of the century the properties of concrete and joint behavior of steel and concrete were not fully understood, a 16-story building, the Ingalls Building, was constructed in Cincinnati in 1902 with a total height of 210 ft (64 m). In 1922, the Medical Arts Building, with a height of 230 ft (70 m), was constructed in Dallas, Texas. The design of concrete buildings was based on elastic theory concepts and a high factor of safety, resulting in large concrete sections in beams and columns. After extensive research, high-strength concrete and high-strength steel were allowed in the design of reinforced concrete members. Consequently, small concrete sections as well as savings in materials were achieved, and new concepts of structural design were possible.

To visualize how high concrete buildings can be built, some structural concrete skyscrapers are listed in Table 1.3. The CN Tower is the world's tallest free-standing concrete structure.

The reader should realize that most concrete buildings are relatively low and range from one to five stories. Skyscrapers and high-rise buildings constitute less than 10% of all concrete buildings.

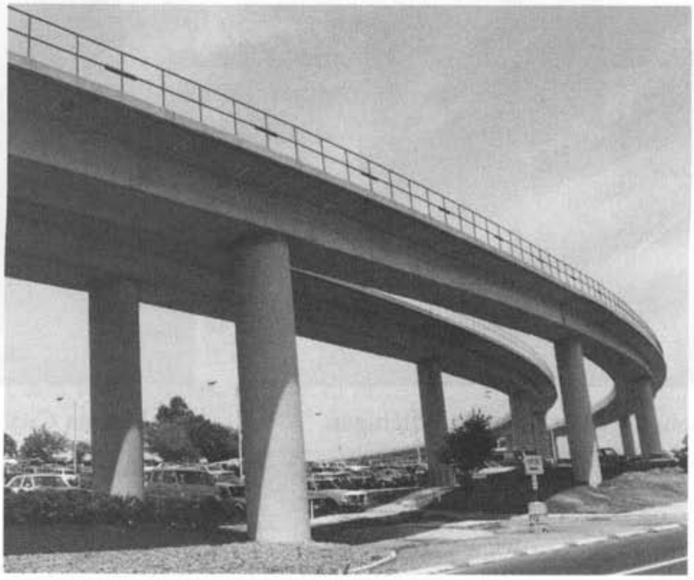
Photos of some different concrete buildings and structures are shown.

Renaissance Center, Detroit, Michigan.

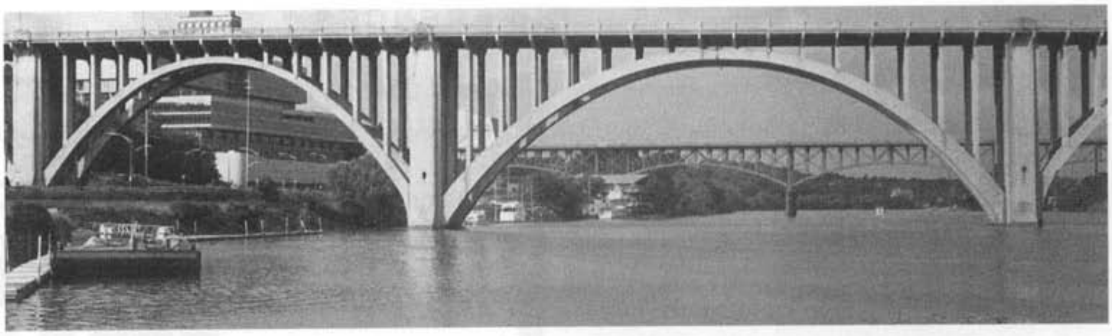
Marina City Towers, Chicago, Illinois.

City Center, Minneapolis, Minnesota.

CN Tower, Toronto, Canada
(height 1465 ft, or 447 m).



Concrete bridge for the city transit system, Washington, DC.



Concrete bridge, Knoxville, Tennessee.

Reinforced concrete grain silo using the slip form system, Brookings, South Dakota.

Table 1.3 Examples of Reinforced Concrete Skyscrapers

Year	Structure	Location	Stories	Height, ft (m)
1965	Lake Point Tower	Chicago	70	645 (197)
1969	One Shell Plaza	Houston	52	714 (218)
1975	Peachtree Center Plaza Hotel	Atlanta	71	723 (220)
1976	Water Tower Place	Chicago	74	859 (262)
1976	CN Tower	Toronto	—	1465 (447)
1977	Renaissance Center Westin Hotel	Detroit	73	740 (226)
1983	City Center	Minneapolis	40	528 (158)

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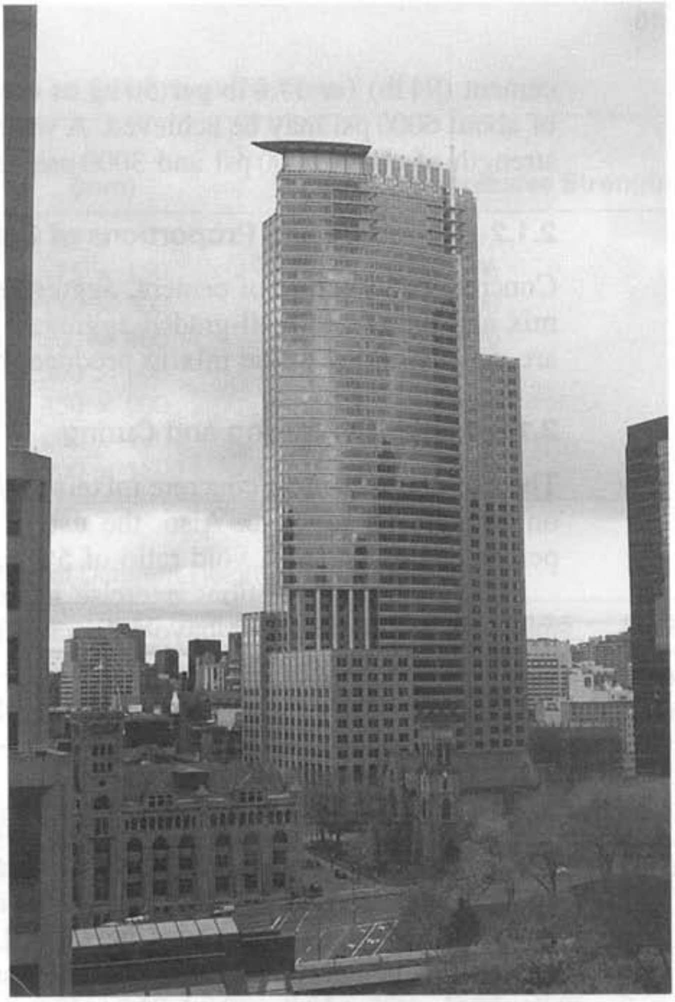
CHAPTER 2

PROPERTIES OF REINFORCED CONCRETE



IBM Building, Montreal, Canada (the highest concrete building in Montreal, with 50 stories).

2.1 FACTORS AFFECTING THE STRENGTH OF CONCRETE

In general, concrete consists of coarse and fine aggregate, cement, water, and—in many cases—different type of admixture. The materials are mixed together until a cement paste is developed, filling most of the voids in the aggregates and producing a uniform dense concrete. The plastic concrete is then placed in a mold and left to set, harden, and develop adequate strength. For the design of concrete mixtures, as well as composition and properties of concrete materials, the reader is referred to Refs. 1–6.

The strength of concrete depends upon many factors and may vary within wide limits with the same production method. The main factors that affect the strength of concrete are described next.

2.1.1 Water–Cement Ratio

The water–cement ratio is one of the most important factors affecting the strength of concrete. For complete hydration of a given amount of cement, a water–cement ratio (by weight) equal to 0.25 is needed. A water–cement ratio of about 0.35 or higher is needed for the concrete to be reasonably workable without additives. This ratio corresponds to 4 gal of water per sack of cement (94 lb) (or 17.8 lb per 50 kg of cement). Based on this cement ratio, a concrete strength of about 6000 psi may be achieved. A water–cement ratio of 0.5 and 0.7 may produce a concrete strength of about 5000 psi and 3000 psi, respectively.

2.1.2 Properties and Proportions of Concrete Constituents

Concrete is a mixture of cement, aggregate, and water. An increase in the cement content in the mix and the use of well-graded aggregate increase the strength of concrete. Special admixtures are usually added to the mix to produce the desired quality and strength of concrete.

2.1.3 Method of Mixing and Curing

The use of mechanical concrete mixers and the proper time of mixing both have favorable effects on strength of concrete. Also, the use of vibrators produces dense concrete with a minimum percentage of voids. A void ratio of 5% may reduce the concrete strength by about 30%.

The curing conditions exercise an important influence on the strength of concrete. Both moisture and temperature have a direct effect on the hydration of cement. The longer the period of moist storage, the greater the strength. If the curing temperature is higher than the initial temperature of casting, the resulting 28-day strength of concrete is reached earlier than 28 days.

2.1.4 Age of the Concrete

The strength of concrete increases appreciably with age, and hydration of cement continues for months. In practice, the strength of concrete is determined from cylinders or cubes tested at the age of 7 days and 28 days. As a practical assumption, concrete at 28 days is 1.5 times as strong as at 7 days: The range varies between 1.3 and 1.7. The British code of practice [2] accepts concrete if the strength at 7 days is not less than two-thirds of the required 28-day strength. For a normal portland cement, the increase of strength with time, relative to 28-day strength, may be assumed as follows:

Age	7 days	14 days	28 days	3 months	6 months	1 year	2 years	5 years
Strength ratio	0.67	0.86	1.0	1.17	1.23	1.27	1.31	1.35

2.1.5 Loading Conditions

The compressive strength of concrete is estimated by testing a cylinder or cube to failure in a few minutes. Under sustained loads for years, the ultimate strength of concrete is reduced by about 30%. Under 1-day sustained loading, concrete may lose about 10% of its compressive strength. Sustained loads and creep effect as well as dynamic and impact effect, if they occur on the structure, should be considered in the design of reinforced concrete members.

2.1.6 Shape and Dimensions of the Tested Specimen

The common sizes of concrete specimens used to predict the compressive strength are either 6-by 12-in. (150- by 300-mm) cylinders or 6-in. (150-mm) cubes. When a given concrete is tested in compression by means of cylinders of like shape but of different sizes, the larger specimens give lower strength indexes. Table 2.1 [4] gives the relative strength for various sizes of cylinders as

Table 2.1 Effect of Size of Compression Specimen on Strength of Concrete

Size of cylinder (in.)	Size of cylinder (mm)	Relative Compressive Strength
2 × 4	50 × 100	1.09
3 × 6	75 × 150	1.06
6 × 12	150 × 300	1.00
8 × 16	200 × 400	0.96
12 × 24	300 × 600	0.91
18 × 36	450 × 900	0.86
24 × 48	600 × 1200	0.84
36 × 72	900 × 1800	0.82

Table 2.2 Strength Correction Factor for Cylinders of Different Height–Diameter Ratios

Ratio	2.0	1.75	1.50	1.25	1.10	1.00	0.75	0.50
Strength correction factor	1.00	0.98	0.96	0.93	0.90	0.87	0.70	0.50
Strength relative to standard cylinder	1.00	1.02	1.04	1.06	1.11	1.18	1.43	2.00

Table 2.3 Relative Strength of Cylinder versus Cube [6]

Compressive Strength											
	(psi)	1000	2200	2900	3500	3800	4900	5300	5900	6400	7300
	(N/mm^2)	7.0	15.5	20.0	24.5	27.0	24.5	37.0	41.5	45.0	51.5
Strength Ratio of Cylinder to Cube		0.77	0.76	0.81	0.87	0.91	0.93	0.94	0.95	0.96	0.96

a percentage of the strength of the standard cylinder; the heights of all cylinders are twice the diameters.

Sometimes concrete cylinders of nonstandard shape are tested. The greater the ratio of specimen height to diameter, the lower the strength indicated by the compression test. To compute the equivalent strength of the standard shape, the results must be multiplied by a correction factor. Approximate values of the correction factor are given in Table 2.2, extracted from ASTM C 42/C 42M-03. The relative strengths of a cylinder and a cube for different compressive strengths are shown in Table 2.3.

2.2 COMPRESSIVE STRENGTH

In designing structural members, it is assumed that the concrete resists compressive stresses and not tensile stresses; therefore, compressive strength is the criterion of quality concrete. The other concrete stresses can be taken as a percentage of the compressive strength, which can be easily and accurately determined from tests. Specimens used to determine compressive strength may be cylindrical, cubical, or prismatic.

Test specimens in the form of a 6-in. (150 mm) or 8-in. (200 mm) cube are used in Great Britain, Germany, and other parts of Europe.

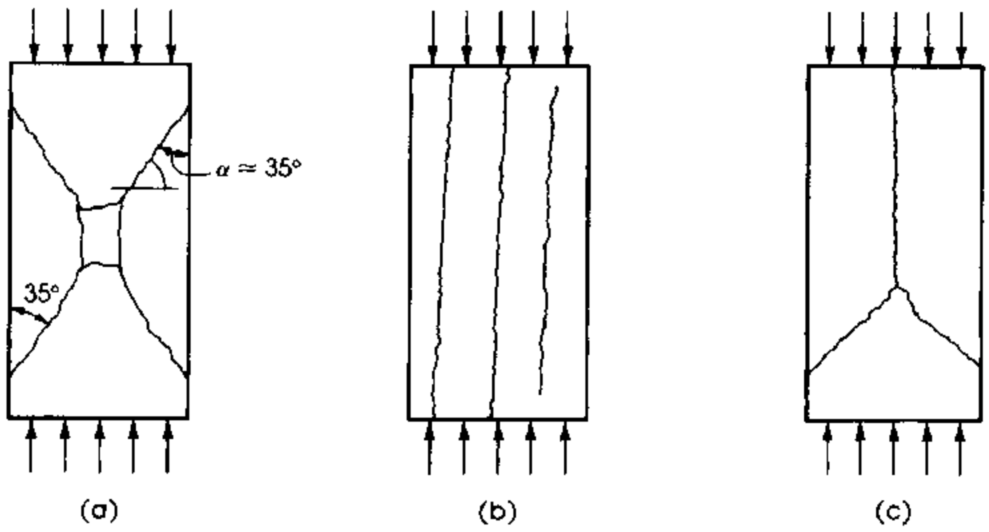


Figure 2.1 Modes of failure of standard concrete cylinders.

Prism specimens are used in France, Russia, and other countries and are usually 70 by 70 by 350 mm or 100 by 100 by 500 mm. They are cast with their longer sides horizontal and are tested, like cubes, in a position normal to the position of cast.

Before testing, the specimens are moist-cured and then tested at the age of 28 days by gradually applying a static load until rupture occurs. The rupture of the concrete specimen may be caused by the applied tensile stress (failure in cohesion), the applied shearing stress (sliding failure), the compressive stress (crushing failure), or combinations of these stresses.

The failure of the concrete specimen can be in one of three modes [5], as shown in Fig. 2.1. First, under axial compression, the specimen may fail in shear, as in Fig. 2.1*a*. Resistance to failure is due to both cohesion and internal friction.

The second type of failure (Fig. 2.1*b*) results in the separation of the specimen into columnar pieces by what is known as splitting, or columnar, fracture. This failure occurs when the strength of concrete is high, and lateral expansion at the end bearing surfaces is relatively unrestrained.

The third type of failure (Fig. 2.1*c*) is seen when a combination of shear and splitting failure occurs.

2.3 STRESS-STRAIN CURVES OF CONCRETE

The performance of a reinforced concrete member under load depends, to a great extent, on the stress-strain relationship of concrete and steel and on the type of stress applied to the member. Stress-strain curves for concrete are obtained by testing a concrete cylinder to rupture at the age of 28 days and recording the strains at different load increments.

Figure 2.2 shows typical stress-strain curves for concretes of different strengths. All curves consist of an initial relatively straight elastic portion, reaching maximum stress at a strain of about 0.002; then rupture occurs at a strain of about 0.003. Concrete having a compressive strength between 3000 and 6000 psi (21 and 42 N/mm^2) may be adopted. High-strength concrete with a compressive strength greater than 6000 psi (6000–15,000 psi) is becoming an important building material for the design of concrete structures.

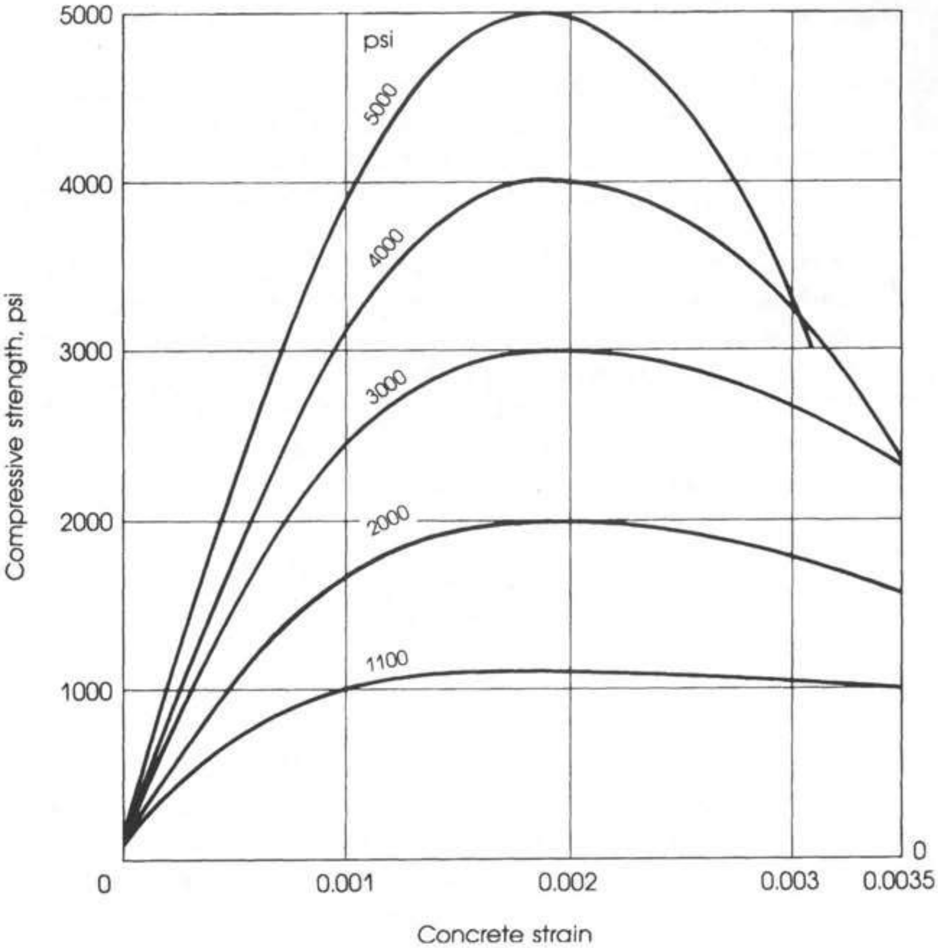
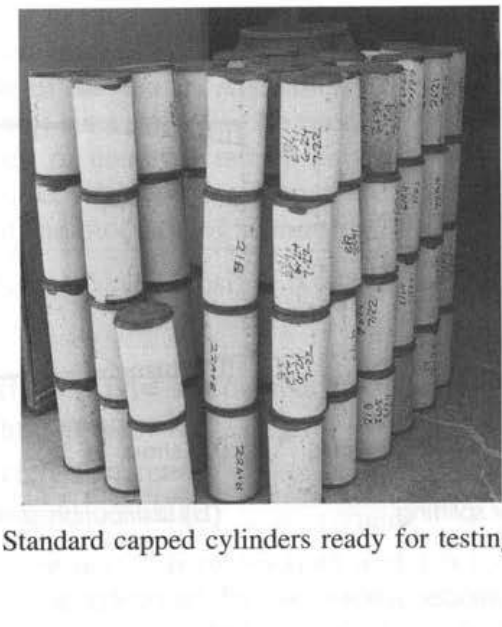


Figure 2.2 Typical stress–strain curves of concrete.



Standard capped cylinders ready for testing.

2.4 TENSILE STRENGTH OF CONCRETE

Concrete is a brittle material, and it cannot resist the high tensile stresses that are important when considering cracking, shear, and torsional problems. The low tensile capacity can be attributed to the high stress concentrations in concrete under load, so that a very high stress is reached in some portions of the specimen, causing microscopic cracks, while the other parts of the specimen are subjected to low stress.

Direct tension tests are not reliable for predicting the tensile strength of concrete, due to minor misalignment and stress concentrations in the gripping devices. An indirect tension test in the form of splitting a 6- by 12-in. (150- by 300-mm) cylinder was suggested by the Brazilian Fernando Carneiro. The test is usually called the *splitting test*. In this test, the concrete cylinder is placed with its axis horizontal in a compression testing machine. The load is applied uniformly along two opposite lines on the surface of the cylinder through two plywood pads, as shown in Fig. 2.3. Considering an element on the vertical diameter and at a distance y from the top fibers, the element is subjected to a compressive stress

$$f_c = \frac{2P}{\pi LD}\left(\frac{D^2}{y(D-y)} - 1\right) \tag{2.1}$$

and a tensile stress

$$f'_{sp} = \frac{2P}{\pi LD} \tag{2.2}$$

where P is the compressive load on the cylinder and D and L are the diameter and length of the cylinder. For a 6- by 12-in. (150- by 300-mm) cylinder and at a distance $y = D/2$, the compression strength is $f_c = 0.0265P$, and the tensile strength is $f'_{sp} = 0.0088P = f_c/3$.

The splitting strength of f'_{sp} can be related to the compressive strength of concrete in that it varies between six and seven times $\sqrt{f'_c}$ for normal concrete and between four and five times $\sqrt{f'_c}$ for lightweight concrete. The direct tensile stress, f'_t, can also be estimated from the

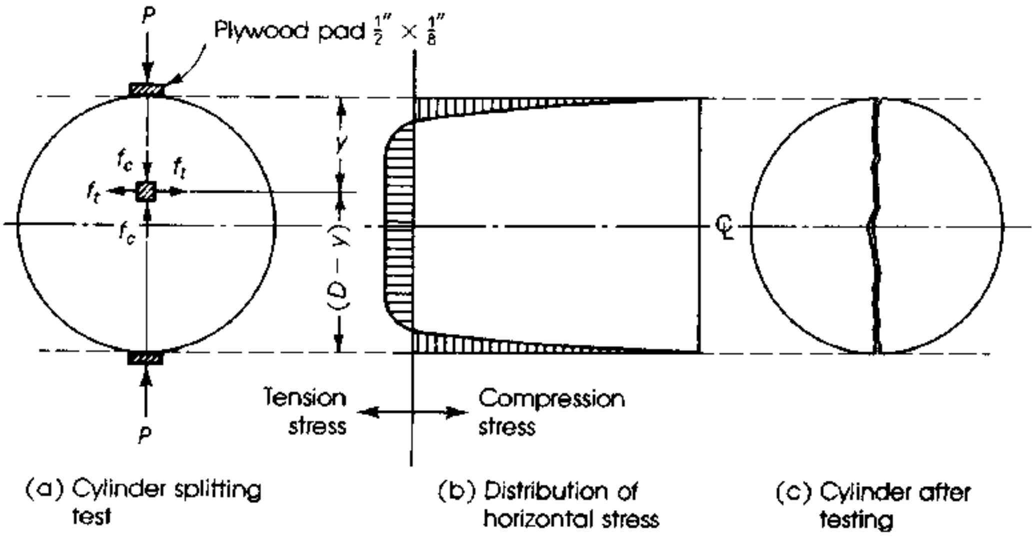
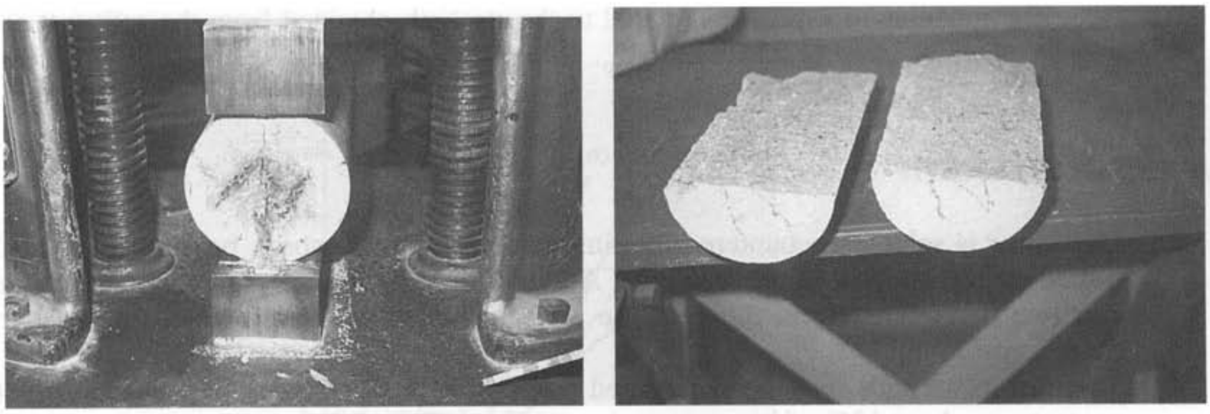


Figure 2.3 Cylinder splitting test [6]: (*a*) configuration of test, (*b*) distribution of horizontal stress, and (*c*) cylinder after testing.



Concrete cylinder splitting test.

split test: Its value varies between 0.5 f'_{sp} and 0.7 f'_{sp}. The smaller of these values applies to higher-strength concrete. The splitting strength, f'_{sp}, can be estimated as 10% of the compressive strength up to $f'_c = 6000$ psi (42 N/mm^2). For higher values of compressive strength, f'_{sp} can be taken as 9% of f'_c.

In general, the tensile strength of concrete ranges from 7% to 11% of its compressive strength, with an average of 10%. The lower the compressive strength, the higher the relative tensile strength.

2.5 FLEXURAL STRENGTH (MODULUS OF RUPTURE) OF CONCRETE

Experiments on concrete beams have shown that ultimate tensile strength in bending is greater than the tensile stress obtained by direct or splitting tests. Flexural strength is expressed in terms of the modulus of rupture of concrete (f_r), which is the maximum tensile stress in concrete in bending. The modulus of rupture can be calculated from the flexural formula used for elastic materials, $f_r = Mc/I$, by testing a plain concrete beam. The beam, 6 by 6 by 28 in. (150 by 150 by 700 mm), is supported on a 24-in. (600-mm) span and loaded to rupture by two loads, 4 in. (100 mm) on either side of the center. A smaller beam of 4 by 4 by 20 in. (100 by 100 by 500 mm) on a 16-in. (400-mm) span may also be used.

The modulus of rupture of concrete ranges between 11% and 23% of the compressive strength. A value of 15% can be assumed for strengths of about 4000 psi (28 N/mm^2). The ACI Code prescribes the value of the modulus of rupture as

$$f_r = 7.5\lambda\sqrt{f'_c}\ \text{(psi)} = 0.62\lambda\sqrt{f'_c}\ (\text{N/mm}^2) \tag{2.3}$$

where

λ is a modification factor for type of concrete (ACI 8.6.1)

$\lambda = 1.0$ Normal-weight concrete

$\lambda = 0.85$ Sand-lightweight concrete

$\lambda = 0.75$ for all-lightweight concrete

Linear interpolation shall be permitted between 0.85 and 1.0 on the basis of volumetric fractions, for concrete containing normal-weight fine aggregate and a blend of lightweight and normal-weight coarse aggregate.

The modulus of rupture as related to the strength obtained from the split test on cylinders may be taken as $f_r = (1.25 \text{ to } 1.50) f'_{sp}$.

2.6 SHEAR STRENGTH

Pure shear is seldom encountered in reinforced concrete members, because it is usually accompanied by the action of normal forces. An element subjected to pure shear breaks transversely into two parts. Therefore, the concrete element must be strong enough to resist the applied shear forces.

Shear strength may be considered as 20% to 30% greater than the tensile strength of concrete, or about 12% of its compressive strength. The ACI Code allows a nominal shear stress of $2\lambda\sqrt{f'_c}$ psi ($0.17\lambda\ \sqrt{f'_c}$ N/mm^2) on plain concrete sections. For more information, refer to Chapter 8.

2.7 MODULUS OF ELASTICITY OF CONCRETE

One of the most important elastic properties of concrete is its modulus of elasticity, which can be obtained from a compressive test on concrete cylinders. The modulus of elasticity, E_c, can be defined as the change of stress with respect to strain in the elastic range:

$$E_c = \frac{\text{unit stress}}{\text{unit strain}} \tag{2.4}$$

The modulus of elasticity is a measure of stiffness, or the resistance of the material to deformation. In concrete, as in any elastoplastic material, the stress is not proportional to the strain, and the stress–strain relationship is a curved line. The actual stress–strain curve of concrete can be obtained by measuring the strains under increments of loading on a standard cylinder.

The *initial tangent modulus* (Fig. 2.4) is represented by the slope of the tangent to the curve at the origin under elastic deformation. This modulus is of limited value and cannot be determined with accuracy. Geometrically, the tangent modulus of elasticity of concrete, E_c, is the slope of the tangent to the stress–strain curve at a given stress. Under long-time action of load and due to the development of plastic deformation, the stress-to-total-strain ratio becomes a variable nonlinear quantity.

For practical applications, the *secant modulus* can be used. The secant modulus is represented by the slope of a line drawn from the origin to a specific point of stress (B) on the stress–strain curve (Fig. 2.4). Point B is normally located at $f'_c/2$.

The ACI Code section 8.5.1 gives a simple formula for calculating the modulus of elasticity of normal and lightweight concrete considering the secant modulus at a level of stress, f_c, equal to half the ultimate concrete strength, f'_c,

$$E_c = 33w^{1.5}\sqrt{f'_c}\ \text{psi}(w \text{ in pcf}) = 0.043\, w^{1.5}\sqrt{f'_c}\ \text{N/mm}^2 \tag{2.5}$$

where w = unit weight of concrete (between 90 and 160 lb/ft^3 (pcf) or 1400 to 2600 kg/m^3) and f'_c = ultimate strength of a standard concrete cylinder. For normal-weight concrete, w is approximately 145 pcf (2320 kg/m^3); thus,

$$E_c = 57{,}600\sqrt{f'_c}\ \text{psi} = 4780\sqrt{f'_c}\ \text{MPa} \tag{2.6}$$

$$E_c = \frac{df_c}{d\varepsilon_c}$$

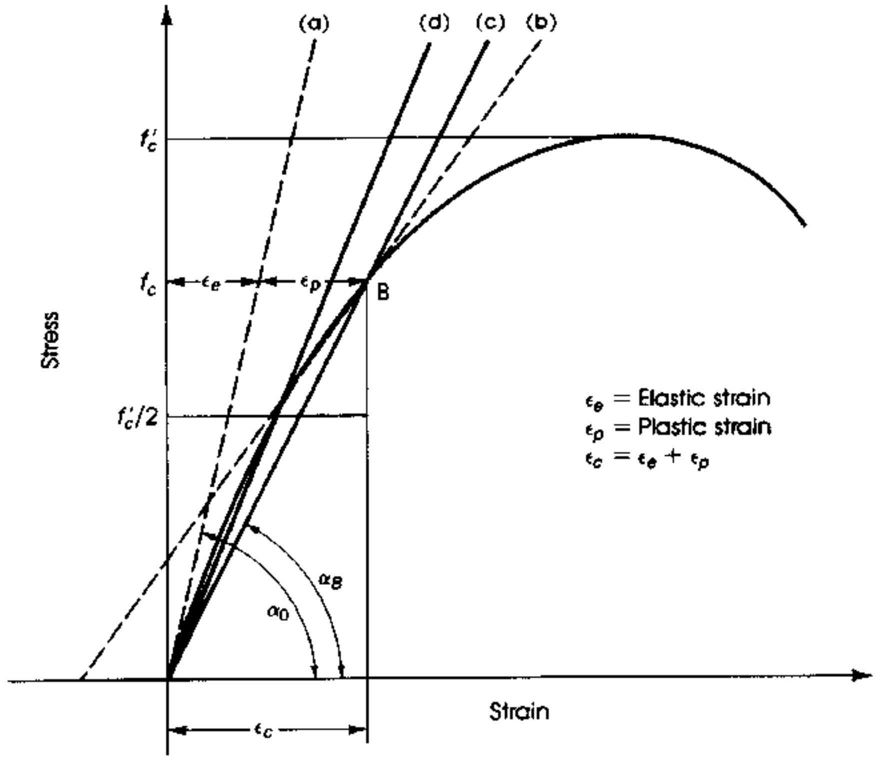


Figure 2.4 Stress–strain curve and modulus of elasticity of concrete. Lines a-d represent (*a*) initial tangent modulus, (*b*) tangent modulus at a stress, f_c, (*c*) secant modulus at a stress, f_c, and (*d*) secant modulus at a stress $f_c'/2$.

The ACI Code allows the use of $E_c = 57{,}000\sqrt{f_c'}$ (psi) $= 4700\sqrt{f_c'}$ MPa. The module of elasticity, E_c, for different values of f_c' are shown in Table A.10.

2.8 POISSON'S RATIO

Poisson's ratio, μ, is the ratio of the transverse to the longitudinal strains under axial stress within the elastic range. This ratio varies between 0.15 and 0.20 for both normal and lightweight concrete. Poisson's ratio is used in structural analysis of flat slabs, tunnels, tanks, arch dams, and other statically indeterminate structures. For isotropic elastic materials, Poisson's ratio is equal to 0.25. An average value of 0.18 can be used for concrete.

2.9 SHEAR MODULUS

The modulus of elasticity of concrete in shear ranges from about 0.4 to 0.6 of the corresponding modulus in compression. From the theory of elasticity, the shear modulus is taken as follows:

$$G_c = \frac{E_c}{2(1+\mu)} \tag{2.7}$$

where μ = Poisson's ratio of concrete. If μ is taken equal to $1/6$, then $G_c = 0.43E_c = 24{,}500\sqrt{f_c'}$.

Test on a standard concrete cylinder to determine the modulus of elasticity of concrete.

2.10 MODULAR RATIO

The modular ratio, n, is the ratio of the modulus of elasticity of steel to the modulus of elasticity of concrete: $n = E_s/E_c$.

Because the modulus of elasticity of steel is considered constant and is equal to 29×10^6 psi and $E_c = 33w^{1.5}\sqrt{f'_c}$,

$$n = \frac{29 \times 10^6}{33w^{1.5}\sqrt{f'_c}} \tag{2.8}$$

For normal-weight concrete, $E_c = 57{,}400\sqrt{f'_c}$; hence, n can be taken as

$$n = \frac{500}{\sqrt{f'_c}}(f'_c \text{ in psi}) = \frac{42}{\sqrt{f'_c}}(f'_c \text{ in N/mm}^2) \tag{2.9}$$

The significance and the use of the modular ratio are explained in Chapter 6.

2.11 VOLUME CHANGES OF CONCRETE

Concrete undergoes volume changes during hardening. If it loses moisture by evaporation, it shrinks, but if the concrete hardens in water, it expands. The causes of the volume changes in concrete can be attributed to changes in moisture content, chemical reaction of the cement with water, variation in temperature, and applied loads.

2.11.1 Shrinkage

The change in the volume of drying concrete is not equal to the volume of water removed [6]. The evaporation of free water causes little or no shrinkage. As concrete continues to dry, water evaporates and the volume of the restrained cement paste changes, causing concrete to shrink, probably due to the capillary tension that develops in the water remaining in concrete. Emptying of the capillaries causes a loss of water without shrinkage, but once the absorbed water is removed, shrinkage occurs.

Many factors influence the shrinkage of concrete caused by the variations in moisture conditions [5]:

1. *Cement and water content*. The more cement or water content in the concrete mix, the greater the shrinkage.
2. *Composition and fineness of cement*. High-early-strength and low-heat cements show more shrinkage than normal portland cement. The finer the cement, the greater the expansion under moist conditions.
3. *Type, amount, and gradation of aggregate*. The smaller the size of aggregate particles, the greater the shrinkage. The greater the aggregate content, the smaller the shrinkage [7].
4. *Ambient conditions, moisture, and temperature*. Concrete specimens subjected to moist conditions undergo an expansion of 200 to 300 $\times 10^{-6}$, but if they are left to dry in air, they shrink. High temperature speeds the evaporation of water and, consequently, increases shrinkage.
5. *Admixtures*. Admixtures that increase the water requirement of concrete increase the shrinkage value.
6. *Size and shape of specimen*. As shrinkage takes place in a reinforced concrete member, tension stresses develop in the concrete, and equal compressive stresses develop in the steel. These stresses are added to those developed by the loading action. Therefore, cracks may develop in concrete when a high percentage of steel is used. Proper distribution of reinforcement, by producing better distribution of tensile stresses in concrete, can reduce differential internal stresses.

The values of final shrinkage for ordinary concrete vary between 200 and 700 $\times 10^{-6}$. For normal-weight concrete, a value of 300 $\times 10^{-6}$ may be used. The British Code [8] gives a value of 500 $\times 10^{-6}$, which represents an unrestrained shrinkage of 1.5 mm in a 3 m length of thin, plain concrete sections. If the member is restrained, a tensile stress of about 10 N/mm^2 (1400 psi) arises. If concrete is kept moist for a certain period after setting, shrinkage is reduced; therefore, it is important to cure the concrete for a period of no fewer than 7 days.

Exposure of concrete to wind increases the shrinkage rate on the upwind side. Shrinkage causes an increase in the deflection of structural members, which in turn increases with time. Symmetrical reinforcement in the concrete section may prevent curvature and deflection due to shrinkage.

Generally, concrete shrinks at a high rate during the initial period of hardening, but at later stages the rate diminishes gradually. It can be said that 15% to 30% of the shrinkage value occurs in 2 weeks, 40% to 80% occurs in 1 month, and 70% to 85% occurs in 1 year.

2.11.2 Expansion Due to Rise in Temperature

Concrete expands with increasing temperature and contracts with decreasing temperature. The coefficient of thermal expansion of concrete varies between 4 and 7 $\times 10^{-6}$ per degree Fahrenheit.

An average value of 5.5×10^{-6} per degree Fahrenheit (12×10^{-6} per degree Celsius) can be used for ordinary concrete. The British code [8] suggests a value of 10^{-5} per degree Celsius. This value represents a change of length of 10 mm in a 30-m member subjected to a change in temperature of 33°C. If the member is restrained and unreinforced, a stress of about 7 N/mm^2 (1000 psi) may develop.

In long reinforced concrete structures, expansion joints must be provided at lengths of 100 to 200 ft (30 to 60 m). The width of the expansion joint is about 1 in. (25 mm). Concrete is not a good conductor of heat, whereas steel is a good one. The ability of concrete to carry load is not much affected by temperature.

2.12 CREEP

Concrete is an elastoplastic material, and beginning with small stresses, plastic strains develop in addition to elastic ones. Under sustained load, plastic deformation continues to develop over a period that may last for years. Such deformation increases at a high rate during the first 4 months after application of the load. This slow plastic deformation under constant stress is called *creep*.

Figure 2.5 shows a concrete cylinder that is loaded. The instantaneous deformation is ε_1, which is equal to the stress divided by the modulus of elasticity. If the same stress is kept for a period of time, an additional strain, ε_2, due to creep effect, can be recorded. If load is then released, the elastic strain, ε_1, will be recovered, in addition to some creep strain. The final permanent plastic strain, ε_3, will be left, as shown in Fig. 2.5. In this case, $\varepsilon_3 = (1 - \alpha)\varepsilon_2$, where α is the ratio of the recovered creep strain to the total creep strain. The ratio α ranges between 0.1 and 0.2. The magnitude of creep recovery varies with the previous creep and depends appreciably upon the period of the sustained load. Creep recovery rate will be less if the loading period is increased, probably due to the hardening of concrete while in a deformed condition.

The ultimate magnitude of creep varies between 0.2×10^{-6} and 2×10^{-6} per unit stress (lb/in.^{2}) per unit length. A value of 1×10^{-6} can be used in practice. The ratio of creep strain to elastic strain may be as high as 4.

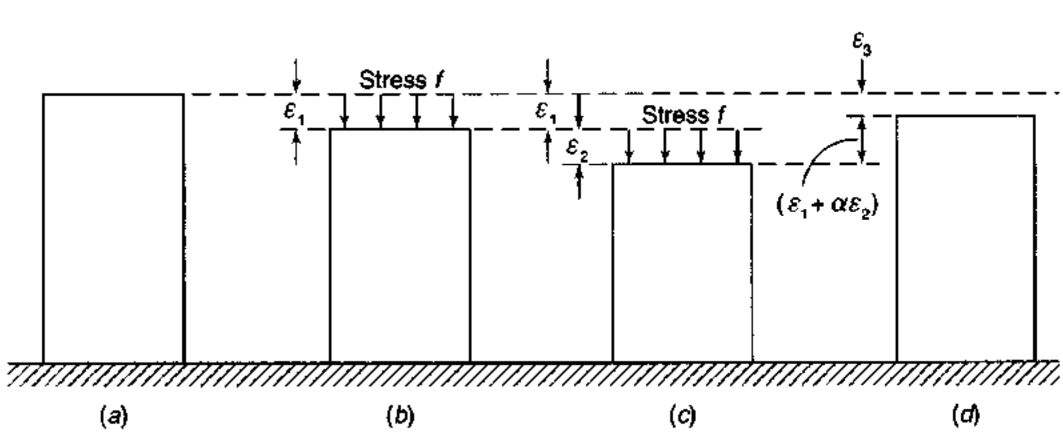


Figure 2.5 Deformation in a loaded concrete cylinder: (*a*) specimen unloaded, (*b*) elastic deformation, (*c*) elastic plus creep deformation, (*d*) permanent deformation after release of load.

Creep takes place in the hardened cement matrix around the strong aggregate. It may be attributed to slippage along planes within the crystal lattice, internal stresses caused by changes in the crystal lattice, and gradual loss of water from the cement gel in the concrete.

The different factors that affect the creep of concrete can be summarized as follows [9]:

1. *Level of stress*. Creep increases with an increase of stress in specimens made from concrete of the same strength and with the same duration of load.
2. *Duration of loading*. Creep increases with the loading period. About 80% of the creep occurs within the first 4 months; 90% occurs after about 2 years.
3. *Strength and age of concrete*. Creep tends to be smaller if concrete is loaded at a late age. Also, creep of 2000 psi (14 N/mm^2)–strength concrete is about 1.41×10^{-6}, whereas that of 4000 psi (28 N/mm^2)–strength concrete is about 0.8×10^{-6} per unit stress and length of time.
4. *Ambient conditions*. Creep is reduced with an increase in the humidity of the ambient air.
5. *Rate of loading*. Creep increases with an increase in the rate of loading when followed by prolonged loading.
6. *Percentage and distribution of steel reinforcement in a reinforced concrete member*. Creep tends to be smaller for higher proportion or better distribution of steel.
7. *Size of the concrete mass*. Creep decreases with an increase in the size of the tested specimen.
8. *Type, fineness, and content of cement*. The amount of cement greatly affects the final creep of concrete, as cement creeps about 15 times as much as concrete.
9. *Water–cement ratio*. Creep increases with an increase in the water–cement ratio.
10. *Type and grading of aggregate*. Well-graded aggregate will produce dense concrete and consequently a reduction in creep.
11. *Type of curing*. High-temperature steam curing of concrete, as well as the proper use of a plasticizer, will reduce the amount of creep.

Creep develops not only in compression, but also in tension, bending, and torsion.

The ratio of the rate of creep in tension to that in compression will be greater than 1 in the first 2 weeks, but this ratio decreases over longer periods [5].

Creep in concrete under compression has been tested by many investigators. Troxell, Raphale, and Davis [10] measured creep strains periodically for up to 20 years and estimated that of the total creep after 20 years, 18% to 35% occurred in 2 weeks, 30% to 70% occurred in 3 months, and 64% to 83% occurred in 1 year.

For normal concrete loaded after 28 days, $C_r = 0.13\sqrt[3]{t}$, where C_r = creep strain per unit stress per unit length. Creep augments the deflection of reinforced concrete beams appreciably with time. In the design of reinforced concrete members, long-term deflection may be critical and has to be considered in proper design. Extensive deformation may influence the stability of the structure.

Sustained loads affect the strength as well as the deformation of concrete. A reduction of up to 30% of the strength of unreinforced concrete may be expected when concrete is subjected to a concentric sustained load for 1 year.

The fatigue strength of concrete is much smaller than its static strength. Repeated loading and unloading cycles in compression lead to a gradual accumulation of plastic deformations. If concrete in compression is subjected to about 2 million cycles, its fatigue limit is about 50% to

60% of the static compression strength. In beams, the fatigue limit of concrete is about 55% of its static strength [11].

2.13 MODELS FOR PREDICTING THE SHRINKAGE AND CREEP OF CONCRETE

2.13.1 The ACI 209 Model

The American Concrete Institute recommend the ACI 209 model [12]. Branson and Christianson [13] first developed this model in 1970. The ACI 209 model was used for many years in the design of concrete structures. The model is simple to use but limited in its accuracy.

Shrinkage calculation. Calculation of shrinkage using the ACI 209 model can be performed if the following parameters and conditions are known: curing method (moist-cured or steam-cured concrete), relative humidity, H, type of cement, specimen shape, ultimate shrinkage strain, ε_{shu}, age of concrete after casting, t, age of the concrete drying commenced, usually taken as the age at the end of moist curing, t_c.

The shrinkage strain is defined as follows:

$$\varepsilon_s(t) = \frac{(t - t_c)}{b + (t - t_c)} K_{ss} K_{sh} \varepsilon_{shu} \tag{2.10}$$

where

t = Age of concrete after casting (days)

t_c = Age of the concrete drying commenced (days)

b = Constant in determining shrinkage strain, depends on curing method according to Table 2.4

K_{ss} = Shape and size correction factor for shrinkage according to the Eq. 2.11

K_{sh} = Relative humidity correction factor for shrinkage according to Eq. 2.12

ε_{shu} (ultimate shrinkage strain) 780×10^{-6} (mm/mm) (for both moist- and steam-cured concrete)

Shape and size correction factor for shrinkage should be calculated as follows:

$$K_{ss} = 1.14 - 0.0035\left(\frac{V}{S}\right) \tag{2.11}$$

where

V = volume of the specimen (mm^3)

S = surface of the specimen (mm^2)

Relative humidity correction factor for shrinkage is

$$K_{sh} = \begin{cases} 1.40 - 0.01H & \text{for } 40\% \le H \le 80\% \\ 3.00 - 0.03H & \text{for } 81\% \le H \le 100\% \end{cases} \tag{2.12}$$

Table 2.4 Values of Constant b as a Function of Curing Method

Moist-Cured Concrete	Steam-Cured Concrete
$b = 35$	$b = 55$

where

H = Relative humidity (in %)

Creep calculation. The total-load dependent strain at time t, ε_{ic} (t, t_0) of a concrete member uniaxially loaded at time t_0 with a constant stress σ may be calculated as follows:

$$\varepsilon_{ic}(t, t_0) = \varepsilon_i(t_0) + \varepsilon_c(t, t_0) \tag{2.13}$$

where

$\varepsilon_i(t_0)$ = The initial elastic strain at loading

$\varepsilon_c(t, t_0)$ = The creep strain at time $t \geq t_0$.

$$\varepsilon_i(t_0) = \frac{\sigma}{E_{\mathrm{cmt}_0}} \tag{2.14}$$

$$\varepsilon_c(t, t_0) = \frac{\sigma}{E_{\mathrm{cmt}_0}} C_c(t) \tag{2.15}$$

where

E_{cmt_0} = Modulus of elasticity at age of loading (MPa) as given in Eq. 2.17

$C_c(t)$ = Creep coefficient at time t, as given in Eq. 2.19

Usually, the total-load dependent strain is presented with compliance function, also called creep function, $J(t, t_0)$, which represent the total-load dependent strain at time t produced by a unit constant stress that has been acting since time t_0.

$$J(t, t_0) = \frac{1 + C_c(t)}{E_{\mathrm{cmt}_0}} \tag{2.16}$$

$$E_{\mathrm{cmt}_0} = 0.043(\gamma)^{3/2}\sqrt{f_c'(t_0)} \tag{2.17}$$

where

γ = Concrete unit weight (kg/m^3)

$f_c'(t_0)$ = Mean concrete compressive strength at age of loading (MPa)

$$f_c'(t_0) = f_{\mathrm{cm}_{28}} \frac{t_0}{b + ct_0} \tag{2.18}$$

where

$f_{\mathrm{cm}_{28}}$ = Average 28-day concrete compressive strength (MPa)

b and $c$are constants according to Table 2.5:

Table 2.5 Constants b and c as a Function of Cement Type and Curing Method

Type of Cement	**Moist-Cured Concrete**		**Steam-Cured Concrete**	
I	$b = 4$	$c = 0.85$	$b = 1$	$c = 0.95$
III	$b = 2.30$	$c = 0.92$	$b = 0.70$	$c = 0.98$

Table 2.6 Correction Factors

Curing Method	t_0 (days)	H	K_{ca}	K_{ch}	K_{cs}
Moist Cured	≥ 1 day	$\geq 40\%$	N/A	N/A	N/A
	≥ 7 days	$\geq 40\%$	$1.25(t_0)^{-0.118}$	$1.27 - 0.0067H$	$1.14 - 0.0035(V/S)$
Stem Cured	≥ 1 day	$\geq 40\%$	$1.13(t_0)^{-0.095}$	$1.27 - 0.0067H$	$1.14 - 0.0035(V/S)$
	≥ 7 days	$\geq 40\%$	N/A	N/A	N/A

Creep coefficient, $C_c(t)$, can be determined as follows:

$$C_c(t) = \frac{t^{0.60}}{10 + t^{0.60}} C_{cu} K_{ch} K_{ca} K_{cs} \tag{2.19}$$

where

C_{cu} = Ultimate creep coefficient = 2.35
K_{ch} = Relative humidity correction factor for creep determined from Table 2.6
K_{ca} = Age at loading correction factor determined from Table 2.6
K_{cs} = Shape and size correction factor for creep determined from Table 2.6

2.13.2 The B3 Model

The model was developed by Bazant and Baweja [14].

Shrinkage calculation. Required parameters for calculation of shrinkage strain using the B3 model are concrete mean compressive strength at 28 days, curing conditions, cement type, relative humidity, water content in concrete, and specimen shape.

The shrinkage strain can be estimated using the following equation:

$$\varepsilon_s(t) = (\varepsilon_{shu})(K_h)S(t) \tag{2.20}$$

where

ε_{shu} = Ultimate shrinkage strain according to Eq. 2.21
K_h = Humidity function for shrinkage according to Table 2.9
$S(t)$ = Time function for shrinkage according to Eq. 2.22

Ultimate shrinkage strain can be calculated using the following equation:

$$\varepsilon_{shu} = -\alpha_1\alpha_2[0.019(w)^{2.1}(f_{cm_{28}})^{-0.28} + 270] \times 10^{-6} \tag{2.21}$$

where

α_1 = Type of cement correction factor according to Table 2.7
α_2 = Curing condition correction factor according to the Table 2.8
w = Water content (kg/m^3)
$f_{cm_{28}}$ = Mean compressive concrete strength at 28 days (MPa)

Type of cement correction factor α_1 can be determined using Table 2.7.
Curing condition correction factor α_2 can be determined using Table 2.8.
Humidity function for shrinkage, K_h, should be determined according to Table 2.9.

Table 2.7 Correction Factor α_1 as a Function of Cement Type

Type of Cement	α_1
I	1.00
II	0.85
III	1.10

Table 2.8 Correction Factor α_2 as a Function of Type of Curing

Type of Curing	α_2
Steam cured	0.75
Water cured or $H = 100\%$	1.00
Sealed during curing	1.20

Table 2.9 Humidity Function for Shrinkage, K_h

Humidity	K_h
$H \leq 98\%$	$1 - (H/100)^3$
$H = 100\%$	-0.2
$98\% \leq H \leq 100\%$	Linear interpolation

where

H is relative humidity (%)

Time function for shrinkage, $S(t)$, should be calculated according to the following equation:

$$S(t) = \tanh\sqrt{\frac{t - t_c}{T_{sh}}} \tag{2.22}$$

where

t = Age of concrete after casting (days)
t_c = Age of the concrete drying commenced (days)
T_{sh} = Shrinkage half-time (days) according to the Eq. 2.23

$$T_{sh} = 0.085(t_c)^{-0.08}(f_{cm_{28}})^{-0.25}[2K_s(V/S)]^2 \tag{2.23}$$

where K_s = Cross-section shape correction factor according to Table 2.10
K_s can be assumed to be 1 if type of member is not defined.

Creep calculation. The creep function, also called creep compliance, $J(t, t_0)$ is given by Eq. 2.24:

$$J(t, t_0) = q_1 + C_0(t, t_0) + C_d(t, t_0, t_c) \tag{2.24}$$

Table 2.10 Correction Factor, K_s as a Function of Cross-Section Shape

Cross-Section Shape	K_s
Infinite slab	1.00
Infinite cylinder	1.15
Infinite square prism	1.25
Sphere	1.30
Cube	1.55

where

q_1 = The instantaneous strain, given in Eq. 2.25

$C_0(t, t_0)$ = The compliance function for basic creep composed of three terms, an aging viscoelastic term, a nonaging viscoelastic term and an aging flow term given in Eq. 2.27

$C_d(t, t_0, t_c)$ = The compliance function for drying creep, given in Eq. 2.35

$$q_1 = \frac{0.6}{E_{\mathrm{cm}_{28}}} \tag{2.25}$$

where

$E_{\mathrm{cm}_{28}}$ = Modulus of elasticity of concrete at 28 days as given in the following equation:

$$E_{\mathrm{cm}_{28}} = 4735\sqrt{f_{\mathrm{cm}_{28}}} \tag{2.26}$$

The compliance function for basic creep, $C_0(t, t_0)$, should be calculated as follows:

$$C_0(t, t_0) = q_2 Q(t, t_0) + q_3 \ln[1 + (t - t_0)^{0.1}] + q_4 \ln\left(\frac{t}{t_0}\right) \tag{2.27}$$

where

q_2 = Aging viscoelastic compliance parameter

$Q(t, t_0)$ = The binomial integral

q_3 = Nonaging viscoelastic compliance parameter

q_4 = Flow compliance parameter

t_0 = Age of concrete at loading (days)

$$q_2 = 185.4(c)^{0.5}(f_{\mathrm{cm}_{28}})^{-0.9} \times 10^{-6} \tag{2.28}$$

where c is the cement content (kg/m^3).

$$Q(t, t_0) = Q_f(t_0)\left[1 + \frac{Q_f(t_0)^{r(t_0)}}{Z(t, t_0)^{r(t_0)}}\right]^{-1/r(t_0)} \tag{2.29}$$

where

$$Q_f(t_0) = \frac{1}{0.086(t_0)^{2/9} + 1.21(t_0)^{4/9}} \tag{2.30}$$

$$Z(t, t_0) = \frac{\ln[1 + (t - t_0)^{0.1}]}{\sqrt{t_0}} \tag{2.31}$$

$$r(t_0) = 1.7(t_0)^{0.12} + 8 \tag{2.32}$$

$$q_3 = 0.29 q_2 \left(\frac{w}{c}\right)^4 \tag{2.33}$$

$$q_4 = 20.3 \left(\frac{a}{c}\right)^{-0.7} \times 10^{-6} \tag{2.34}$$

The compliance function for drying creep, $C_d(t, t_0, t_c)$, should be calculated as follows:

$$C_d(t, t_0, t_c) = q_5 \sqrt{\exp[-8H(t)] - \exp[-8H(t_0)]} \tag{2.35}$$

where

q_5 = Drying creep compliance parameter that can be calculated from the following equation:

$$q_5 = \frac{0.757 |\varepsilon_{\text{shu}} \times 10^6|^{-0.6}}{f_{\text{cm}_{28}}} \tag{2.36}$$

where

ε_{shu} = Ultimate shrinkage strain, given by Eq. 2.21

$H(t)$ and $H(t_0)$ are spatial averages of pore relative humidity.

$$H(t) = 1 - \left[\left(1 - \frac{H}{100}\right) S(t)\right] \tag{2.37}$$

$$H(t_0) = 1 - \left[\left(1 - \frac{H}{100}\right) S(t_0)\right] \tag{2.38}$$

$S(t)$ is given by Eq. 2.22 and

$$S(t_0) = \tanh\sqrt{\frac{t_0 - t_c}{T_{\text{sh}}}} \tag{2.39}$$

T_{sh} is given by Eq. 2.23.

2.13.3 The GL 2000 Model

The GL 2000 Model was developed by Gardner et. al and is described in Ref. 15.

Shrinkage calculation. Parameters required for calculation of shrinkage strain using the GL 2000 model are mean 28-day concrete compressive strength, $f_{\text{cm}_{28}}$, relative humidity, H, age of concrete at the beginning of shrinkage, t_c, type of cement, and specimen shape.

The shrinkage strain can be calculated using the following equation:

$$\varepsilon_s(t) = \varepsilon_{\text{shu}} \beta(h) \beta(t) \tag{2.40}$$

where

ε_{shu} = Ultimate shrinkage strain according to Eq. 2.41
$\beta(h)$ = Correction term for effect of humidity according to Eq. 2.42
$\beta(t)$ = Correction term for effect of time according to Eq. 2.43

Ultimate shrinkage strain should be calculated from the following equation:

$$\varepsilon_{shu} = (900)K\left(\frac{30}{f_{cm_{28}}}\right)^{1/2} \times 10^{-6} \tag{2.41}$$

where

K = Shrinkage constant, which depends on cement type as shown in Table 2.11
$f_{cm_{28}}$ = Mean 28-day concrete compressive strength (MPa)

Shrinkage constant K can be determined from Table 2.11.

Correction term for effect of humidity, $\beta(h)$, should be calculated as shown:

$$\beta(h) = 1 - 1.18\left(\frac{H}{100}\right)^{4} \tag{2.42}$$

where

H = Relative humidity (%)

Correction term for effect of time, $\beta(t)$, should be determined as follows:

$$\beta(t) = \left(\frac{t - t_c}{t - t_c + 0.12(V/S)^2}\right)^{1/2} \tag{2.43}$$

where

t = Age of concrete after casting (days)
t_c = Age of concrete at the beginning of shrinkage (days)
V/S = Volume-to-surface area ratio (mm)

Creep calculation. The creep compliance is composed of two parts: the elastic strain and the creep strain according to the following equation:

$$J(t, t_0) = \frac{1}{E_{cmt_0}} + \frac{\phi(t, t_0)}{E_{cm_{28}}} \tag{2.44}$$

Table 2.11 Shrinkage Constant, K, as a Function of Cement Type

Type of Cement	K
I	1.00
II	0.75
III	1.15

where

E_{cmt_0} = Modulus of elasticity of concrete at loading (MPa)
$E_{cm_{28}}$ = Modulus of elasticity of concrete at 28 days (MPa)
$\phi(t, t_0)$ = Creep coefficient

$$E_{cmt_0} = 3500 + 4300\sqrt{f_{cmt_0}} \tag{2.45}$$

where f_{cmt_0} = Concrete mean compressive strength at loading (MPa), which can be determined as follows:

$$f_{cmt_0} = f_{cm28}\frac{t^{3/4}}{a + bt^{3/4}}$$

Coefficients a and b are related to the cement type as shown in Table 2.12.

$$E_{cm_{28}} = 3500 + 4300\sqrt{f_{cm_{28}}} \tag{2.46}$$

Creep coefficient, $\phi(t, t_0)$, can be calculated as shown:

$$\phi(t, t_0) = \Phi(t_c)2\left(\frac{(t-t_0)^{0.3}}{(t-t_0)^{0.3}+14}\right) + \left(\frac{7}{t_0}\right)^{0.5}\left(\frac{t-t_0}{t-t_0+7}\right)^{0.5}$$
$$+ 2.5(1 - 1.086h^2)\left(\frac{t-t_0}{t-t_0+0.12(V/S)^2}\right)^{0.5} \tag{2.47}$$

$$\text{If} \quad t_0 = t_c \quad \text{then } \Phi(t_c) = 1 \tag{2.48}$$

$$\text{If} \quad t_0 > t_c \quad \text{then } \Phi(t_c) = \left[1 - \left(\frac{t_0 - t_c}{t_0 - t_c + 0.12(V/S)^2}\right)^{0.5}\right]^{0.5} \tag{2.49}$$

$h = H/100$ (H = Relative humidity(%))

2.13.4 The CEB 90 Model

The CEB 90 Model was developed by Muller and Hillsdorf [16].

Shrinkage calculation. Parameters required for calculation of shrinkage strain using the CEB 90 model are mean 28-day concrete compressive strength, $f_{cm_{28}}$, relative humidity, H, age of concrete at the beginning of shrinkage, t_c, type of cement, and specimen shape.

The strain due to shrinkage may be calculated from the following equation:

$$\varepsilon_s(t, t_c) = \varepsilon_{cs_0}\beta_s(t, t_c) \tag{2.50}$$

Table 2.12 Coefficient a and b as a Function of Cement Type

Cement Type	a	b
I	2.8	0.77
II	3.4	0.72
III	1.0	0.92

where

ε_{cs_0} = Notional shrinkage coefficient according to Eq. 2.51
$\beta_s(t, t_c)$ = Coefficient describing development of shrinkage with time according to Eq. 2.54

Notional shrinkage coefficient is

$$\varepsilon_{cs_0} = \varepsilon_s(f_{cm_{28}})\beta_{RH} \tag{2.51}$$

where

$\varepsilon_s(f_{cm_{28}})$ = Concrete strength factor on shrinkage according to Eq. 2.52
β_{RH} = Relative humidity factor on notional shrinkage coefficient according to Table 2.13

Concrete strength factor on shrinkage, $\varepsilon_s(f_{cm_{28}})$, can be calculated as follows:

$$\varepsilon_s(f_{cm_{28}}) = \left[160 + 10(\beta_{sc})\left(9 - \frac{f_{cm_{28}}}{10}\right)\right] \times 10^{-6} \tag{2.52}$$

where

β_{sc} = Coefficient that depends on type of cement according to Table 2.14.
$f_{cm_{28}}$ = Mean 28-day concrete compressive strength (MPa)

Coefficient β_{sc} dependent on humidity, β_{RH}, should be determined according to Table 2.14, where

$$\beta_{arh} = 1 - \left(\frac{H}{100}\right)^3 \tag{2.53}$$

The development of shrinkage with time is given by

$$\beta_s(t - t_c) = \sqrt{\frac{(t - t_c)}{0.56(h_e/4)^2 + (t - t_c)}} \tag{2.54}$$

Table 2.13 Determination of Coefficient β_{RH}

Humidity	β_{RH}
$40\% \le H < 99\%$	$-1.55 \times \beta_{arh}$
$H \ge 99\%$	0.25

Table 2.14 Coefficient β_{sc}

Type of Cement	European Type	American Type	β_{sc}
Slow hardening	SL	II	4
Normal/rapid hardening	R	I	5
Rapid hardening, high strength	RS	III	8

where

t = Age of concrete (days)
t_c = Age of concrete at the beginning of shrinkage (days)
h_e = Effective thickness to account for volume/surface ratio (mm)

Effective thickness, h_e, can be determined as follows:

$$h_e = \frac{2A_c}{u} \tag{2.55}$$

where

A_c = Cross-section of the structural member (mm^2)
u = Perimeter of the structural member in contact with the atmosphere (mm)

Creep calculation. Creep compliance represents the total stress dependent strain per unit stress. It can be calculated as

$$J(t, t_0) = \frac{1}{E_{\text{cmt}_0}} + \frac{\phi(t, t_0)}{E_{\text{cm}_{28}}} \tag{2.56}$$

where

E_{cmt_0} = Modulus of elasticity at age of loading (MPa)
$E_{\text{cm}_{28}}$ = Modulus of elasticity at 28 days (MPa)
$\phi(t, t_0)$ = Creep coefficient

$$E_{\text{cmt}_0} = E_{\text{cm}_{28}} \exp\left[0.5S\left(1 - \sqrt{\frac{28}{t}}\right)\right] \tag{2.57}$$

S is the coefficient that depends on cement type and can be determined from Table 2.15.

$$E_{\text{cm}_{28}} = 21500\sqrt[3]{\frac{f_{\text{cm}_{28}}}{10}} \tag{2.58}$$

Creep coefficient, $\phi(t, t_0)$, can be evaluated from the given equation:

$$\phi(t, t_0) = \phi_0 \beta_c(t, t_0) \tag{2.59}$$

Table 2.15 Coefficient S as a Function of Cement Type

Cement Type	European Type	U.S. Type	S
Slow hardening	SL	II	0.38
Normal/rapid hardening	R	I	0.25
Rapid hardening high strength	RS	III	0.20

where

ϕ_0 = Notional creep coefficient

$\beta_c(t, t_0)$ = Equation describing development of creep with time after loading

$$\phi_0 = \phi_{\mathrm{RH}}\beta(f_{\mathrm{cm}_{28}})\beta(t_0) \tag{2.60}$$

where ϕ_{RH} = Relative humidity factor on the notional creep coefficient, which is given by

$$\phi_{\mathrm{RH}} = 1 + \frac{1 - H/100}{0.16\sqrt[3]{h_e/4}} \tag{2.61}$$

$\beta(f_{\mathrm{cm}_{28}})$ = Concrete strength factor on the notional creep coefficient, which is given by

$$\beta(f_{\mathrm{cm}_{28}}) = \frac{5.3}{\sqrt{f_{\mathrm{cm}_{28}}/10}} \tag{2.62}$$

$\beta\ (t_0)$ = Age of concrete at loading factor on the notional creep coefficient, which is given by

$$\beta(t_0) = \frac{1}{0.1 + t_0^{0.2}} \tag{2.63}$$

An equation describing development of creep with time after loading, $\beta_c(t, t_0)$, can be calculated using the following equation:

$$\beta_c(t, t_0) = \left(\frac{t - t_0}{\beta_H + t - t_0}\right)^{0.3} \tag{2.64}$$

$$\beta_H = 1.5h_e[1 + (0.012H)^{18}] + 250 \leq 1500 \text{ days} \tag{2.65}$$

2.13.5 The CEB 90-99 Model

The CEB 90-99 is a modification of the CEB 90 and is described in Ref. 17.

Shrinkage calculation. In this new model, total shrinkage contains of autogenous and drying shrinkage component. In high-performance concrete, autogenous shrinkage is significant and needs to be considered in prediction of shrinkage. This new approach was necessary so that shrinkage of normal as well as high-performance concrete can be predicted with sufficient accuracy [1].

Total shrinkage strain can be calculated using the following equation:

$$\varepsilon_s(t, t_c) = \varepsilon_{\mathrm{as}}(t) + \varepsilon_{\mathrm{ds}}(t, t_c) \tag{2.66}$$

where

$\varepsilon_{\mathrm{as}}(t)$ = Autogenous shrinkage at time t

$\varepsilon_{\mathrm{ds}}(t, t_c)$ = Drying shrinkage at time t

Autogenous shrinkage, $\varepsilon_{\mathrm{as}}(t)$, should be calculated as follows:

$$\varepsilon_{\mathrm{as}}(t) = \varepsilon_{\mathrm{as}_0}(f_{\mathrm{cm}_{28}})\beta_{\mathrm{as}}(t) \tag{2.67}$$

where

$\varepsilon_{as_0}(f_{cm_{28}})$ = National autogenous shrinkage coefficient according to Eq. 2.68
$\beta_{as}(t)$ = Function to describe the time-development of autogenous shrinkage, from Eq. 2.69

National autogenous shrinkage coefficient, $\varepsilon_{cas_0}(f_{cm})$, can be calculated as follows:

$$\varepsilon_{as_0}(f_{cm_{28}}) = -\alpha_{as}\left(\frac{f_{cm_{28}}/10}{6 + f_{cm_{28}}/10}\right)^{2.5} \times 10^{-6} \tag{2.68}$$

where

α_{as} = Coefficient that depends on type of cement
= 800 for slowly hardening cements
= 700 for normal or rapidly hardening cements
= 600 for rapidly hardening high-strength cements
$f_{cm_{28}}$ = Mean compressive strength of concrete at an age of 28 days (MPa)

Function $\beta_{as}(t)$ should be calculated using the following equation:

$$\beta_{as}(t) = 1 - \exp[-0.2(t)^{0.5}] \tag{2.69}$$

where t = Age of concrete (days)

Drying shrinkage, $\varepsilon_{ds}(t, t_c)$, can be estimated by the following equation:

$$\varepsilon_{ds}(t, t_c) = \varepsilon_{ds_0}(f_{cm_{28}})\beta_{RH}(H)\beta_{ds}(t - t_c) \tag{2.70}$$

where

$\varepsilon_{ds_0}(f_{cm_{28}})$ = Notional drying shrinkage coefficient according to Eq. 2.71
$\beta_{RH}(H)$ = Coefficient to take into account the effect of relative humidity on drying shrinkage according to Eq. 2.72
$\beta_{ds}(t - t_c)$ = Function to describe the time development of drying shrinkage according to Eq. 2.74

Notional drying shrinkage coefficient, $\varepsilon_{ds_0}(f_{cm_{28}})$, may be calculated from the following equation:

$$\varepsilon_{ds_0}(f_{cm_{28}}) = [(220 + 110\alpha_{ds_1})\exp(-\alpha_{ds_2} f_{cm_{28}}/10)] \times 10^{-6} \tag{2.71}$$

where

α_{ds_1} = Coefficient that depends on type of cement
= 3 for slowly hardening cements
= 4 for normal or rapidly hardening cements
= 6 for rapidly hardening high-strength cements
α_{ds_2} = Coefficient that depends on type of cement
= 0.13 for slowly hardening cements
= 0.11 for normal or rapidly hardening cements
= 0.12 for rapidly hardening high-strength cements

Coefficient $\beta_{RH}(H)$ should be calculated as follows:

$$\beta_{RH} = \begin{cases} -1.55\left[1-\left(\frac{H}{100}\right)^3\right] & \text{for } 40\% \le H < 99\% \times \beta_{s_1} \\ 0.25 & \text{for } H \ge 99\% \times \beta_{s_1} \end{cases} \tag{2.72}$$

where

H = Ambient relative humidity (%)

β_{s_1} = Coefficient to take into account the self-desiccation in high-performance concrete.

It can be determined as follows:

$$\beta_{s_1} = \left(\frac{35}{f_{cm_{28}}}\right)^{0.1} \le 1.0 \tag{2.73}$$

Function $\beta_{ds}(t - t_c)$ may be estimated as follows:

$$\beta_{ds}(t-t_c) = \left(\frac{(t-t_c)}{0.56(h_e/4)^2 + (t-t_c)}\right)^{0.5} \tag{2.74}$$

where

t_c = Concrete age at the beginning of drying (days)

$h_e = \frac{2A_c}{u}$ = notional size of member (mm), where A_c is the cross-section (mm^2) and u is the perimeter of the member in contact with the atmosphere (mm)

Creep calculation. Total stress-dependent strain per unit stress, also called creep compliance or creep function can be determined as follows:

$$J(t, t_0) = \frac{1}{E_{cmt_0}} + \frac{\phi(t, t_0)}{E_{cm_{28}}} \tag{2.75}$$

where

E_{cmt_0} = Modulus of elasticity at age of loading (MPa)

$E_{cm_{28}}$ = Modulus of elasticity at day 28 (MPa)

$\phi(t, t_0)$ = Creep coefficient

$$E_{cmt_0} = E_{cm_{28}} \exp\left[0.5S\left(1 - \sqrt{\left(\frac{28}{t_0}\right)}\right)\right] \tag{2.76}$$

S is the coefficient that depends on cement type and compressive strength and can be determined from Table 2.16.

$$E_{cm_{28}} = 21500\sqrt[3]{\frac{f_{cm_{28}}}{10}} \tag{2.77}$$

Creep coefficient, $\phi(t, t_0)$, can be evaluated from the given equation:

$$\phi(t, t_0) = \phi_0 \beta_c(t, t_0) \tag{2.78}$$

Table 2.16 Coefficient S as a Function of Cement Type and Compressive Strength

$f_{cm_{28}}$ (MPa)	Type of Cement	S
	Rapidly hardening high strength	0.20
≤60	Normal and rapidly hardening	0.25
	Slow hardening	0.38
>60	All types	0.20

where

ϕ_0 = Notional creep coefficient

$\beta_c(t, t_0)$ = Equation describing development of creep with time after loading

$$\phi_0 = \phi_{RH}\beta(f_{cm_{28}})\beta(t_0) \tag{2.79}$$

ϕ_{RH} = Relative humidity factor on the notional creep coefficient

$$\phi_{RH} = \left[1 + \frac{1 - H/100}{0.16\sqrt[3]{h_e/4}}\alpha_1\right]\alpha_2 \tag{2.80}$$

where

$$\alpha_1 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.7} \tag{2.81}$$

$$\alpha_2 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.2} \tag{2.82}$$

$\beta(f_{cm_{28}})$ = Concrete strength factor on the notional creep coefficient,

$$\beta(f_{cm_{28}}) = \frac{5.3}{\sqrt{f_{cm_{28}}/10}} \tag{2.83}$$

$\beta(t_0)$ = Age of concrete at loading factor on the notional creep coefficient

$$\beta(t_0) = \frac{1}{0.1 + t_0^{0.2}} \tag{2.84}$$

where

$$t_0 = t_{0,T}\left[\frac{9}{2 + t_{0,T}^{1.2}} + 1\right]^{\alpha} \geq 0.5 \text{ days} \tag{2.85}$$

t_0 = Age of concrete at loading (days)

$t_{0,T}$ = Age of concrete at loading adjusted according to the concrete temperature; for $T = 20\,^{\circ}\mathrm{C}$, $t_{0,T}$ corresponds to t_0

α = Coefficient that depends on type of cement

= −1 for slowly hardening cement

= 0 for normal or rapidly hardening cement

= 1 for rapidly hardening high-strength cement

An equation describing development of creep with time after loading, $\beta_c(t, t_0)$, can be calculated using the following equation:

$$\beta_c(t, t_0) = \left(\frac{t - t_0}{\beta_H + t - t_0}\right)^{0.3} \tag{2.86}$$

$$\beta_H = 1.5h_e[1 + (0.012H)^{18}] + 250\alpha_3 \leq 1500\alpha_3 \tag{2.87}$$

$$\alpha_3 = \left[\frac{35}{f_{\mathrm{cm}_{28}}}\right]^{0.5} \tag{2.88}$$

2.13.6 The AASHTO Model

Shrinkage calculation. Parameters required for calculation of shrinkage strain using the AASHTO model are: curing method (moist-cured or steam-cured concrete), 28-day concrete compressive strength, $f_{\mathrm{cm}_{28}}$, relative humidity, H, drying time of concrete, t, type of cement, and specimen shape.

The strain due to shrinkage may be calculated from the following equation:

- For moist-cured concrete:

$$\varepsilon_{sh} = -k_s k_h \left(\frac{t}{35.0 + t}\right) 0.51 \times 10^{-3} \tag{2.89}$$

- For steam-cured concrete:

$$\varepsilon_{sh} = -k_s k_h \left(\frac{t}{55.0 + t}\right) 0.56 \times 10^{-3} \tag{2.90}$$

where

t = drying time (day)
k_s = size factor for shrinkage specified in Eq. 2.91
k_h = humidity factor for shrinkage specified in Eq. 2.92

Size factor for shrinkage should be calculated as follows:

$$k_s = \left[\frac{\dfrac{t}{26e^{0.36(V/S)} + t}}{\dfrac{t}{45 + t}}\right]\left[\frac{1064 - 94(V/S)}{923}\right] \tag{2.91}$$

where

V = Volume of the specimen (in.^{3})
S = Surface of the specimen (in.^{2})

Humidity factor for shrinkage is:

$$k_h = \frac{140 - H}{70} \quad \text{for } H < 80\%$$
$$k_h = \frac{3(100 - H)}{70} \quad \text{for } H \geq 80\% \tag{2.92}$$

where

H = Relative humidity (%)

Creep calculation. The creep compliance represents the total stress dependent strain per unit stress. It can be calculated as:

$$J(t, t_0) = \frac{1}{E_c} + \frac{\psi(t, t_0)}{E_c} \tag{2.93}$$

where

$\Psi(t, t_0)$ = Creep coefficient as given in Eq. 2.94

E_c = Modulus of elasticity at 28 days (ksi) as given in Eq. 2.97

The creep coefficient may be calculated from the following equation:

$$\psi(t, t_0) = 3.5k_c k_f \left(1.58 - \frac{H}{120}\right) t_0^{-0.118} \frac{(t - t_0)^{0.6}}{10.0 + (t - t_0)^{0.6}} \tag{2.94}$$

where

t = Maturity of concrete (day)

t_0 = Age of concrete when load is initially applied (day)

H = Relative humidity (%)

k_f = Factor for the effect of concrete strength as given in Eq. 2.95

k_c = Factor for the effect of the volume-to-surface ratio of the component as given in Eq. 2.96

The factor for the effect of concrete strength should be calculated as follows:

$$k_f = \frac{1}{0.67 + \dfrac{f_{cm_{28}}}{9}} \tag{2.95}$$

where

$f_{cm_{28}}$ = Specified concrete compressive strength at 28 days (Ksi)

The factor for the effect of the volume-to-surface ratio of the component should be calculated as follows:

$$k_c = \left[\frac{\dfrac{t}{26e^{0.36(V/S)} + t}}{\dfrac{t}{45 + t}}\right]\left[\frac{1.80 + 1.77e^{-0.54(V/S)}}{2.587}\right] \tag{2.96}$$

where

V = Volume of the specimen ($in.^3$)

S = Surface of the specimen ($in.^2$)

The modulus of elasticity at 28 days should be calculated as follows:

$$E_c = 33000\omega_c^{1.5}\sqrt{f_c'} \tag{2.97}$$

where

ω_c = Concrete unit weight (Kcf)

f_c' = Specified concrete compressive strength at 28 days (Ksi)

Example 2.1

Calculate shrinkage strain and creep compliance for the concrete specimen given below. Use the ACI 209 model.

Given factors:

Humidity = 75%
$h_e = 2V/S = 2A_c/u = 76\,\text{mm}$
$f_{\text{cm}_{28}} = 45.2$ MPa
$w = 207.92\,\text{kg/m}^3$
$w/c = 0.46$
$a/c = 3.73$
$t = 35$ days
$t_0 = 28$ days
$t_c = 8$ days
$\gamma = 2405\,\text{kg/m}^3$
Cement type III
Moist-cured concrete

Solution

Shrinkage calculation

$$\varepsilon_s(t) = \frac{(t - t_c}{b + (t - t_c)} K_{\text{ss}} K_{\text{sh}} \varepsilon_{\text{shu}}$$

$$\varepsilon_{\text{shu}} = 780 \times 10^{-6}\ \text{mm/mm}$$

According to Table 2.4, $b = 35$

$$V/S = 38\ \text{mm}$$

$$K_{\text{ss}} = 1.14 - 0.0035\left(\frac{V}{S}\right) = 1.14 - 0.0035(38) = 1.007$$

For $H = 75\%$,

$$K_{\text{sh}} = 1.40 - 0.01H = 1.40 - 0.01(75) = 0.65$$

$$\varepsilon_s(t) = \frac{(t - t_c)}{b + (t - t_c)} K_{\text{ss}} K_{\text{sh}} \varepsilon_{\text{shu}}$$

$$= \frac{(35 - 8)}{35 + (35 - 8)}(1.007)(0.65)(780 \times 10^{-6}) = 222.3 \times 10^{-6}\ \text{mm/mm}$$

Creep calculation

$$J(t, t_0) = \frac{1 + C_c(t)}{E_{\text{cmt}_0}} \qquad \text{(Eq. 2.7)}$$

Determination of E_{cmt_0}

$$b = 2.30,\ c = 0.92\ \text{(Table 2.5)}$$

$$f_c'(t_0) = f_{\text{cm}_{28}} \frac{t_0}{b + ct_0} = 45.2 \frac{28}{2.3 + 0.92 \times 28} = 45.1\ \text{MPa}$$

$$E_{\text{cmt}_0} = 0.043(\gamma)^{3/2}\sqrt{f_c'(t_0)} = 0.043(2405)^{3/2}\sqrt{45.1} = 34058.8\ \text{MPa}$$

Determination of $C_c(t)$

$$C_{cu} = 2.35$$

$$K_{ch} = 1.27 - 0.0067(H) = 1.27 - 0.0067(75) = 0.767$$

$$K_{ca} = 1.25(t_0)^{-0.118} = 1.25(28)^{-0.118} = 0.844$$

$$K_{cs} = 1.14 - 0.0035(V/S) = 1.14 - 0.0035(38) = 1.007$$

$$C_c(t) = \frac{t^{0.60}}{10 + t^{0.60}} C_{cu} K_{ch} K_{ca} K_{cs} = \frac{35^{0.60}}{10 + 35^{0.60}} 2.35 \times 0.767 \times 0.844 \times 1.00 = 0.702$$

$$J(t, t_0) = \frac{1 + C_c(t)}{E_{cmt_0}} = \frac{1 + 0.702}{34058.8} = 49.9 \times 10^{-6} \frac{1}{\text{MPa}}$$

Example 2.2

Using the B3 model, calculate shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

$$\varepsilon_s(t) = (\varepsilon_{shu})(K_h)S(t)$$

Determination of ε_{shu}

$$\alpha_1 = 1.10 \text{ (Table 2.7)}$$

$$\alpha_2 = 1.0 \text{ (Table 2.8)}$$

$$\varepsilon_{shu} = \alpha_1 \alpha_2 [0.019(w)^{2.1}(f_{cm_{28}})^{-0.28} + 270] \times 10^{-6}$$

$$= (1.10)(1.0)[0.019(207.92)^{2.1}(45.2)^{-0.28} + 270] \times 10^{-6} = 827 \times 10^{-6} \text{ mm/mm}$$

Determination of K_h

According to the Table 2.6, for $H = 75\%$

$$K_h = 1 - \left(\frac{H}{100}\right)^3 = 1 - \left(\frac{75}{100}\right)^3 = 0.578$$

Determination of $S(t)$

$K_s = 1.0$, since the type of member is not defined

$$T_{sh} = 0.085(t_c)^{-0.08}(f_{cm_{28}})^{-0.25}[2K_s(V/S)]^2$$

$$= 0.085(8)^{-0.08}(45.2)^{-0.25}[2(1.0)(38)]^2 = 160.3$$

$$S(t) = \tanh\sqrt{\frac{t - t_c}{T_{sh}}} = \tanh\sqrt{\frac{35 - 8}{160.3}} = 0.389$$

$$\varepsilon_s(t) = (\varepsilon_{shu})(K_h)S(t) = (827 \times 10^{-6})(0.578)(0.389) = 185.9 \times 10^{-6} \text{ mm/mm}$$

Creep calculation

$$J(t, t_0) = q_1 + C_0(t, t_0) + C_d(t, t_0, t_c) \tag{2.15}$$

Determination of q_1

$$E_{cm_{28}} = 4735\sqrt{f_{cm_{28}}} = 4735\sqrt{45.2} = 31833.9 \text{ MPa}$$

$$q_1 = \frac{0.6}{E_{cm_{28}}} = \frac{0.6}{31833.9} = 18.85 \times 10^{-6} \frac{1}{\text{MPa}}$$

Calculation of $C_0(t, t_0)$

$$c = \frac{w}{w/c} = \frac{207.92}{0.46} = 452 \text{ kg/m}^3$$

$$q_2 = 185.4(c)^{0.5}(f_{cm_{28}})^{-0.9} \times 10^{-6} = 185.4(452)^{0.5}(45.2)^{-0.9} \times 10^{-6}$$

$$= 127.6 \times 10^{-6}$$

$$Q_f(t_0) = \frac{1}{0.086(t_0)^{2/9} + 1.21(t_0)^{4/9}} = \frac{1}{0.086(28)^{2/9} + 1.21(28)^{4/9}} = 0.182$$

$$Z(t, t_0) = \frac{\ln[1 + (t - t_0)^{0.1}]}{\sqrt{t_0}} = \frac{\ln[1 + (35 - 28)^{0.1}]}{\sqrt{28}} = 0.150$$

$$r(t_0) = 1.7(t_0)^{0.12} + 8 = 1.7(28)^{0.12} + 8 = 10.54$$

$$Q(t, t_0) = Q_f(t_0)\left[1 + \frac{Q_f(t_0)^{r(t_0)}}{Z(t, t_0)^{r(t_0)}}\right]^{-1/r(t_0)} = 0.182\left[1 + \frac{0.182^{10.54}}{0.150^{10.54}}\right]^{-1/10.54} = 0.148$$

$$q_3 = 0.29q_2\left(\frac{w}{c}\right)^4 = 0.29(127.6 \times 10^{-6})(0.46)^4 = 1.66 \times 10^{-6}$$

$$q_4 = 20.3\left(\frac{a}{c}\right)^{-0.7} \times 10^{-6} = 20.3(3.73)^{-0.7} \times 10^{-6} = 8.08 \times 10^{-6}$$

$$C_0(t, t_0) = q_2 Q(t, t_0) + q_3 \ln[1 + (t - t_0)^{0.1}] + q_4 \ln\left(\frac{t}{t_0}\right)$$

$$= (127.6 \times 10^{-6})(0.148) + (1.66 \times 10^{-6})\ln[1 + (35 - 28)^{0.1}] + (8.08 \times 10^{-6})\ln\left(\frac{35}{28}\right)$$

$$= 22.01 \times 10^{-6}\frac{1}{\text{MPa}}$$

Calculation of $C_d(t, t_0, t_c)$:

$$q_5 = \frac{0.757|\varepsilon_{shu} \times 10^6|^{-0.6}}{f_{cm_{28}}} = \frac{0.757|827 \times 10^{-6} \times 10^6|^{-0.6}}{45.2} = 297.5 \times 10^{-6}$$

$$S(t) = 0.389$$

$$S(t_0) = \tanh\sqrt{\frac{t_0 - t_c}{T_{sh}}} = \tanh\sqrt{\frac{28 - 8}{160.3}} = 0.339$$

$$H(t) = 1 - \left[\left(1 - \frac{H}{100}\right)S(t)\right] = 1 - \left(1 - \frac{75}{100}\right)0.389] = 0.903$$

$$H(t_0) = 1 - \left[\left(1 - \frac{H}{100}\right)S(t_0)\right] = 1 - \left[\left(1 - \frac{75}{100}\right)0.339\right] = 0.915$$

$$C_d(t,t_0,t_c) = q_5\sqrt{\exp[-8H(t)] - \exp[-8H(t_0)]}$$

$$= (297\times10^{-6})\sqrt{\exp[-8\times0.903] - \exp[-8\times0.915]} = 2.43\times10^{-6}\frac{1}{\text{MPa}}$$

$$J(t,t_0) = q_1 + C_0(t,t_0) + C_d(t,t_0,t_c)$$

$$= (18.85\times10^{-6}) + (22.01\times10^{-6}) + (2.43\times10^{-6}) = 43.3\times10^{-6}\frac{1}{\text{MPa}}$$

Example 2.3

Using the GL 2000 model, calculate the shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

$$\varepsilon_s(t) = \varepsilon_{\text{shu}}\beta(h)\beta(t)$$

Calculation of ε_{shu}

$K = 1.15$ (Table 2.11)

$$\varepsilon_{\text{shu}} = (900)K\left(\frac{30}{f_{\text{cm}_{28}}}\right)^{1/2}\times10^{-6} = (900)(1.15)\left(\frac{30}{45.2}\right)^{1/2}\times10^{-6} = 843.2\times10^{-6}\text{ mm/mm}$$

Calculation of $\beta(h)$

$$\beta(h) = 1 - 1.18\left(\frac{H}{100}\right)^4 = 1 - 1.18\left(\frac{75}{100}\right)^4 = 0.627$$

Calculation of $\beta(t)$:

$$\beta(t) = \left(\frac{t - t_c}{t - t_c + 0.12(V/S)^2}\right)^{1/2} = \left(\frac{35-8}{35-8+0.12(38)^2}\right)^{1/2} = 0.367$$

$$\varepsilon_s(t) = \varepsilon_{\text{shu}}\beta(h)\beta(t) = (843.2\times10^{-6})(0.627)(0.367) = 194\times10^{-6}\text{ mm/mm}$$

Creep calculation

$$J(t,t_0) = \frac{1}{E_{\text{cmt}_0}} + \frac{\phi(t,t_0)}{E_{\text{cm}_{28}}}$$

Calculation of E_{cmt_0} and $E_{\text{cm}_{28}}$

$$t_0 = 28\text{ days} \Rightarrow E_{\text{cmt}_0} = E_{\text{cm}_{28}}$$

$$E_{\text{cm}_{28}} = 3500 + 4300\sqrt{f_{\text{cm}_{28}}} = 3500 + 4300\sqrt{45.2} = 32409.3\text{ MPa}$$

Calculation of $\phi(t,t_0)$

$t_0 = 28 > t_c = 8$ days

$$\Phi(t_c) = \left[1 - \left(\frac{t_0 - t_c}{t_0 - t_c + 0.12(V/S)^2}\right)^{0.5}\right]^{0.5} = \left[1 - \left(\frac{28-8}{28-8+0.12(38)^2}\right)^{0.5}\right]^{0.5} = 0.824$$

$$h = H/100 = 75/100 = 0.75$$

$$\phi(t, t_0) = \Phi(t_c)\left[2\left(\frac{(t-t_0)^{0.3}}{(t-t_0)^{0.3}+14}\right) + \left(\frac{7}{t_0}\right)^{0.5}\left(\frac{t-t_0}{t-t_0+7}\right)^{0.5}\right.$$
$$\left. + 2.5(1-1.086h^2)\left(\frac{t-t_0}{t-t_0+0.12(V/S)^2}\right)^{0.5}\right]$$
$$= 0.824\left[2\left(\frac{(35-28)^{0.3}}{(35-28)^{0.3}+14}\right) + \left(\frac{7}{28}\right)^{0.5}\left(\frac{35-28}{35-28+7}\right)^{0.5}\right.$$
$$\left. + 2.5(1-1.086(0.75)^2)\left(\frac{35-28}{35-28+0.12(38)^2}\right)^{0.5}\right] = 0.636$$

$$J(t, t_0) = \frac{1}{E_{cmt_0}} + \frac{\phi(t, t_0)}{E_{cm_{28}}} = \frac{1}{32409.3} + \frac{0.636}{32409.3} = 50.5 \times 10^{-6}\frac{1}{\text{MPa}}$$

Example 2.4

Using the CEB 90 model, calculate shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

$$\varepsilon_s(t, t_c) = (\varepsilon_{cs_0})\beta_s(t, t_c)$$

Calculation of ε_{cs_0}

$$\beta_{sc} = 8$$

$$\varepsilon_s(f_{cm_{28}}) = \left[160 + 10(\beta_{sc})\left(9 - \frac{f_{cm_{28}}}{10}\right)\right] \times 10^{-6}$$
$$= \left[160 + 10(8)\left(9 - \frac{45.2}{10}\right)\right] \times 10^{-6} = 518.4 \times 10^{-6} \text{ mm/mm}$$

For $H = 75\%$,

$$\beta_{RH} = -1.55\beta_{arh}$$

$$\beta_{arh} = 1 - (H/100)^3 = 1 - (75/100)^3 = 0.578$$

$$\beta_{RH} = -1.55\beta_{arh} = -1.55 \times 0.578 = -0.896$$

$$\varepsilon_{cs_0} = \varepsilon_s(f_{cm_{28}})(\beta_{RH}) = (518.4 \times 10^{-6})(-0.896) = -464.2 \times 10^{-6} \text{ mm/mm}$$

Calculation of $\beta_s(t - t_c)$

$$h_e = \frac{2A_c}{u} = 76 \text{ mm}$$

$$\beta_s(t-t_c) = \sqrt{\frac{(t-t_c)}{0.56(h_e/4)^2 + (t-t_c)}} = \sqrt{\frac{(35-8)}{0.56(76/4)^2 + (35-8)}} = 0.343$$

$$\varepsilon_s(t, t_c) = (\varepsilon_{cs_0})\beta_s(t-t_c) = (-464.2 \times 10^{-6})(0.343) = -159.3 \times 10^{-6} \text{ mm/mm}$$

Creep calculation

$$J(t,t_0)=\frac{1}{E_{\text{cmt}_0}}+\frac{\phi(t,t_0)}{E_{\text{cm}_{28}}}$$

Calculation of E_{cmt_0} and $E_{\text{cm}_{28}}$

$$t_0=28\text{ days}\Rightarrow E_{\text{cmt}_0}=E_{\text{cm}_{28}}$$

$$E_{\text{cm}_{28}}=21500\sqrt[3]{\frac{f_{\text{cm}_{28}}}{10}}=21500\sqrt[3]{\frac{45.2}{10}}=35548\text{ MPa}$$

Calculation of $\phi(t,t_0)$

$$\phi_{\text{RH}}=1+\frac{1-H/100}{0.16\sqrt[3]{h_e/4}}=1+\frac{1-75/100}{0.16\sqrt[3]{76/4}}=1.586$$

$$\beta(f_{\text{cm}_{28}})=\frac{5.3}{\sqrt{f_{\text{cm}_{28}}/10}}=\frac{5.3}{\sqrt{45.2/10}}=2.49$$

$$\beta(t_0)=\frac{1}{0.1+t_0^{0.02}}=\frac{1}{0.1+28^{0.2}}=0.488$$

$$\phi_0=\phi_{\text{RH}}\beta(f_{\text{cm}_{28}})\beta(t_0)=(1.586)(2.49)(0.488)=1.927$$

$$\beta_H=1.5h_e[1+(0.012H)^{18}]+250=1.5(76)[1+(0.012\times 75)^{18}]+250$$

$$=379\le 1500\text{ days}$$

$$\beta_c(t,t_0)=\left(\frac{t-t_0}{\beta_H+t-t_0}\right)^{0.3}=\left(\frac{35-28}{379+35-28}\right)^{0.3}=0.3$$

$$\phi(t,t_0)=\phi_0\beta_c(t,t_0)=1.927\times 0.3=0.578$$

$$J(t,t_0)=\frac{1}{E_{\text{cmt}_0}}+\frac{\phi(t,t_0)}{E_{\text{cm}_{28}}}=\frac{1}{35548}+\frac{0.578}{35548}=44.4\times 10^{-6}\frac{1}{\text{MPa}}$$

Example 2.5

Using the new CEB 90–99 model to calculate shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

$$\varepsilon_s(t,t_c)=\varepsilon_{\text{as}}(t)+\varepsilon_{\text{ds}}(t,t_c)$$

Calculation of $\varepsilon_{\text{as}}(t)$

$$\alpha_{\text{as}}=600\text{ for rapidly hardening high-strength cements}$$

$$\varepsilon_{\text{as}_0}(f_{\text{cm}_{28}})=-\alpha_{\text{as}}\left(\frac{f_{\text{cm}_{28}}/10}{6+f_{\text{cm}_{28}}/10}\right)^{2.5}\times 10^{-6}$$

$$=-600\left(\frac{45.2/10}{6+45.2/10}\right)^{2.5}\times 10^{-6}=-72.6\times 10^{-6}\text{ mm/mm}$$

$$\beta_{\text{as}}(t)=1-\exp(-0.2(t)^{0.5})=1-\exp(-0.2(35)^{0.5})=0.694$$

$$\varepsilon_{\text{as}}(t)=\varepsilon_{\text{as}_0}(f_{\text{cm}_{28}})\beta_{\text{as}}(t)=(-72.6\times 10^{-6})(0.694)=-50.4\times 10^{-6}\text{ mm/mm}$$

Calculation of $\varepsilon_{ds}\ (t, t_c)$

$$\alpha_{ds_1} = 6 \text{ for rapidly hardening high-strength cements}$$

$$\alpha_{ds_2} = 0.12 \text{ for rapidly hardening high-strength cements}$$

$$\varepsilon_{ds_0}(f_{cm_{28}}) = [(220 + 110\alpha_{ds_1})\exp(-\alpha_{ds_2} f_{cm_{28}}/10)] \times 10^{-6}$$

$$= [(220 + 110 \times 6)\exp(-0.12 \times 45.2/10)] \times 10^{-6} = 511.6 \times 10^{-6} \text{ mm/mm}$$

$$\beta_{s1} = \left(\frac{35}{f_{cm_{28}}}\right)^{0.1} = \left(\frac{35}{45.2}\right)^{0.1} = 0.97 \leq 1.0$$

For 40% < H = 75% < 99% (0.97) = 96.5%,

$$\beta_{RH} = -1.55\left[1 - \left(\frac{H}{100}\right)^3\right] = -1.55\left[1 - \left(\frac{75}{100}\right)^3\right] = -0.896$$

$$\beta_{ds}(t - t_c) = \left(\frac{(t - t_c)}{0.56(h/4)^2 + (t - t_c)}\right)^{0.5}$$

$$= \left(\frac{(35 - 8)}{0.56(76/100)^2 + (35 - 8)}\right)^{0.5} = 0.343$$

$$\varepsilon_{ds}(t, t_c) = \varepsilon_{ds_0}(f_{cm_{28}})\beta_{RH}(H)\beta_{ds}(t - t_c)$$

$$= (511.6 \times 10^{-6})(-0.896)(0.343) = -157.2 \times 10^{-6} \text{ mm/mm}$$

$$\varepsilon_s(t, t_c) = \varepsilon_{as}(t) + \varepsilon_{ds}(t, t_c) = (-50.4 \times 10^{-6}) + (-157.2 \times 10^{-6}) = -207.6 \times 10^{-6} \text{ mm/mm}$$

Creep calculation

$$J(t, t_0) = \frac{1}{E_{cmt_0}} + \frac{\phi(t, t_0)}{E_{cm_{28}}}$$

Calculation of E_{cmt_0} and $E_{cm_{28}}$

$$t_0 = 28 \text{ days} \Rightarrow E_{cmt_0} = E_{cm_{28}}$$

$$E_{cm_{28}} = 21{,}500\sqrt[3]{\frac{f_{cm_{28}}}{10}} = 21{,}500\sqrt[3]{\frac{45.2}{10}} = 35{,}548 \text{ MPa}$$

Calculation of $\phi(t, t_0)$

$$\alpha_1 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.7} = \left[\frac{35}{45.2}\right]^{0.7} = 0.836$$

$$\alpha_2 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.2} = \left[\frac{35}{45.2}\right]^{0.2} = 0.950$$

$$\phi_{RH} = \left[1 + \frac{1 - H/100}{0.16\sqrt[3]{h_e/4}}\alpha_1\right]\alpha_2 = \left[1 + \frac{1 - 75/100}{0.16\sqrt[3]{76/4}}0.836\right]0.950 = 1.419$$

$$\beta(f_{cm_{28}}) = \frac{5.3}{\sqrt{f_{cm_{28}}/10}} = \frac{5.3}{\sqrt{45.2/10}} = 2.49$$

$$t_0 = t_{0,T}\left[\frac{9}{2 + t_{0,T}^{1.2}} + 1\right]^{\alpha} = 28\left[\frac{9}{2 + 28^{1.2}} + 1\right] = 32.5 \geq 0.5 \text{ days}$$

$$\beta(t_0) = \frac{1}{0.1 + t_0^{0.2}} = \frac{1}{0.1 + 32.5^{0.2}} = 0.475$$

$$\phi_0 = \phi_{RH}\beta(f_{cm_{28}})\beta(t_0) = 1.415 \times 2.49 \times 0.475 = 1.674$$

$$\alpha_3 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.5} = \left[\frac{35}{45.2}\right]^{0.5} = 0.880$$

$$\beta_H = 1.5h_e[1 + (0.012H)^{18}] + 250\alpha_3$$

$$= 1.5 \times 76 \times [1 + (0.012 \times 75)^{18}] + 250 \times 0.88 = 351 \leq 1500 \times 0.880 = 1320$$

$$\beta_c(t, t_0) = \left(\frac{t - t_0}{\beta_H + t - t_0}\right)^{0.3} = \left(\frac{35 - 28}{351 + 35 - 28}\right)^{0.3} = 0.307$$

$$\phi(t, t_0) = \phi_0\beta_c(t, t_0) = 1.674 \times 0.307 = 0.514$$

$$J(t, t_0) = \frac{1}{E_{cmt_0}} + \frac{\phi(t, t_0)}{E_{cm_{28}}} = \frac{1}{35{,}548} + \frac{0.514}{35{,}548} = 42.6 \times 10^{-6}\frac{1}{\text{MPa}}$$

Example 2.6

Using the AASHTO model, calculate shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

For moist-cured concrete, ε_{sh} should be taken as:

$$\varepsilon_{sh} = -K_s K_h \left(\frac{t}{35.0 + t}\right) 0.51 \times 10^{-3}$$

Determination of K_s:

$$\text{V/S} = 38 \text{ mm} = 1.5 \text{ in.}$$

$$K_s = \left[\frac{\dfrac{t}{26e^{0.36(V/S)} + t}}{\dfrac{t}{45 + t}}\right]\left[\frac{1064 - 94(V/S)}{923}\right]$$

$$K_s = \left[\frac{\dfrac{t}{26e^{0.36(1.5)} + t}}{\dfrac{t}{45 + t}}\right]\left[\frac{1064 - 94(1.5)}{923}\right] = 1$$

Determination of K_h:

For $H = 75\%$,

$$K_h = \left[\frac{140 - H}{70}\right] = \frac{140 - 75}{70} = 0.93$$

Calculation of ε_{sh}:

$$\varepsilon_{sh} = -1 \times 0.93 \times \left(\frac{35}{35.0 + 35}\right) 0.51 \times 10^{-3} = -237.15 \times 10^{-6} \text{ in/in.}$$

Creep calculation
The creep coefficient should be taken as:

$$\psi(t, t_0) = 3.5K_cK_f\left(1.58 - \frac{H}{120}\right)t_0^{-0.118}\frac{(t-t_0)^{0.6}}{10.0 + (t-t_0)^{0.6}}$$

Determination of k_c:

$$k_c = \left[\frac{\dfrac{t}{26e^{0.36(V/S)}+t}}{\dfrac{t}{45+t}}\right]\left[\frac{1.80 - 1.77e^{-0.54(V/S)}}{2.587}\right]$$

$$k_c = \left[\frac{\dfrac{t}{26e^{0.36(1.5)}+t}}{\dfrac{t}{45+t}}\right]\left[\frac{1.80 + 1.77e^{-0.54\times 1.5}}{2.587}\right] = 1$$

Determination of k_f:

$$f_{cm_{28}} = 45.2 \text{ MPa} = 6.55 \text{ Ksi}$$

$$k_f = \left[\frac{1}{0.67 + \dfrac{f_{cm28}}{9}}\right] = \frac{1}{0.67 + \dfrac{6.55}{9}} = 0.715$$

Calculation of $\psi\ (t, t_0)$:

$$\psi(t, t_0) = 3.5 \times 1 \times 0.715\left(1.58 - \frac{75}{120}\right) \times 28^{-0.118} \times \frac{(35-28)^{0.6}}{10.0 + (35-28)^{0.6}}$$

$$\psi(t, t_0) = 0.3923$$

Determination of E_c:

$$\omega_c = 2405 \text{ Kg/m}^3 = 0.15 \text{ Kcf}$$

$$E_c = 33000\omega_c^{1.5}\sqrt{f_c'}$$

$$E_c = 33000 \times 0.15^{1.5}\sqrt{6.55} = 4906.5 \text{ Ksi}$$

Calculation of $J(t, t_0)$:

$$J(t, t_0) = \frac{1}{E_c} + \frac{\psi(t, t_0)}{E_c}$$

$$J(t, t_0) = \frac{1}{4906.5} + \frac{0.3923}{4906.5} = 284 \times 10^{-6}\frac{1}{\text{Ksi}}$$

$$J(t, t_0) = 284 \times 10^{-6}\frac{1}{\text{Ksi}} = 41.2 \times 10^{-6}\frac{1}{\text{MPa}}$$

2.14 UNIT WEIGHT OF CONCRETE

The unit weight, w, of hardened normal concrete ordinarily used in buildings and similar structures depends on the concrete mix, maximum size and grading of aggregates, water–cement ratio, and strength of concrete. The following values of the unit weight of concrete may be used:

1. Unit weight of plain concrete using maximum aggregate size of $\frac{3}{4}$ in. (20 mm) varies between 145 and 150 lb/ft^3 (2320 to 2400 kg/m^3). For concrete of strength less than 4000 psi (280 kg/cm^2), a value of 145 lb/ft^3 (2320 kg/m^3) can be used, whereas for higher-strength concretes, w can be assumed to be equal to 150 lb/ft^3 (2400 kg/m^3).
2. Unit weight of plain concrete of maximum aggregate size of 4 to 6 in. (100 to 150 mm) varies between 150 and 160 lb/ft^3 (2400 to 2560 kg/m^3). An average value of 155 lb/ft^3 may be used.
3. Unit weight of reinforced concrete, using about 0.7% to 1.5% of steel in the concrete section, may be taken as 150 lb/ft^3 (2400 kg/m^3). For higher percentages of steel, the unit weight, w, can be assumed to be 155 lb/ft^3 (2500 kg/m^3).
4. Unit weight of lightweight concrete used for fireproofing, masonry, or insulation purposes varies between 20 and 90 lb/ft^3 (320 and 1440 kg/m^3). Concrete of upper values of 90 pcf or greater may be used for load-bearing concrete members.

The unit weight of heavy concrete varies between 200 and 270 lb/ft^3 (3200 and 4300 kg/m^3). Heavy concrete made with natural barite aggregate of $1\frac{1}{2}$ in. maximum size (38 mm) weighs about 225 lb/ft^3 (3600 kg/m^3). Iron ore sand and steel-punchings aggregate produce a unit weight of 270 lb/ft^3 (4320 kg/m^3). [18].

2.15 FIRE RESISTANCE

Fire resistance of a material is its ability to resist fire for a certain time without serious loss of strength, distortion, or collapse [19]. In the case of concrete, fire resistance depends on the thickness, type of construction, type and size of aggregates, and cement content. It is important to consider the effect of fire on tall buildings more than on low or single-story buildings, because occupants need more time to escape.

Reinforced concrete is a much better fire-resistant material than steel. Steelwork heats rapidly, and its strength drops appreciably in a short time. Concrete itself has low thermal conductivity. The effect of temperatures below 250°C is small on concrete, but definite loss is expected at higher temperatures.

2.16 HIGH-PERFORMANCE CONCRETE

High-performance concrete may be assumed to imply that the concrete exhibits combined properties of strength, toughness, energy absorption, durability, stiffness, and a relatively higher ductility than normal concrete. This improvement in concrete quality may be achieved by using a new generation of additives and superplasticizers, which improves the workability of concrete and, consequently, its strength. Also, the use of active microfillers such as silica fume, fly ash, and polymer improves the strength, porosity, and durability of concrete. The addition of different

Casting and finishing precast concrete wall panels.

types of fiber to the concrete mix enhances many of its properties, including ductility, strength, toughness, and many other properties.

Because it is difficult to set a limit to measure high-performance concrete, one approach is to define a lower-bound limit based on the shape of its stress-strain response in tension [20]. If the stress-strain relationship curve shows a quasi strain-hardening behavior—or, in other words, a postcracking strength larger than the cracking strength with an elastic-plastic behavior—then high performance is achieved [20]. In this behavior, multicracking stage is reached with high energy-absorption capacity. Substantial progress has been made recently in understanding the behavior and practical application of high-performance concrete.

2.17 LIGHTWEIGHT CONCRETE

Lightweight concrete is a concrete that has been made lighter than conventional normal-weight concrete and, consequently, it has a relatively lower density. Basically, reducing the density requires the inclusion of air in the concrete composition. This, however, can be achieved in four distinct ways:

1. By omitting the finer sizes from the aggregate grading, thereby creating what is called *no-fines* concrete. It is a mixture of cement, water, and coarse aggregate only ($\frac{3}{4}-\frac{3}{8}$), mixed to produce concrete with many uniformly distributed voids.
2. By replacing the gravel or crushed rock aggregate with a hollow cellular or porous aggregate, which includes air in the mix. This type is called *lightweight aggregate concrete*. Lightweight aggregate may be natural, such as pumice, pozzolans, and volcanic slags; artificial (from industrial by-products), such as furnace clinker and foamed slag; or industrially produced, such as perlite, vermiculite, expanded clay, shale, and slate.
3. By creating gas bubbles in a cement slurry, which, when it sets, leaves a spongelike structure. This type is called *aerated concrete*.
4. By forming air cells in the slurry by chemical reaction or by vigorous mixing of the slurry with a preformed stable foam, which is produced by using special foam concentrate in a high-speed mixer. This type is called *cellular concrete*.

Structural lightweight concrete has a unit weight that ranges from 90 to 115 lb/ft^3, compared with 145 lb/ft^3 for normal-weight concrete. It is used in the design of floor slabs in buildings and other structural members where high-strength concrete is not required. Structural lightweight concrete can be produced with a compressive strength of 2500 to 5000 psi for practical applications.

2.18 FIBROUS CONCRETE

Fibrous concrete is made primarily of concrete constituents and discrete reinforcing fibers. The brittle nature of concrete and its low flexural tensile strength are major reasons for the growing interest in the performance of fibers in concrete technology. Various types of fibers—mainly steel, glass, and organic polymers—have been used in fibrous concrete. Generally, the length and diameter of the fibers do not exceed 3 in. (75 mm) and 0.04 in. (1 mm), respectively. The addition of fibers to concrete improves its mechanical properties, such as ductility, toughness, shear, flexural strength, impact resistance, and crack control. A convenient numerical parameter describing a fiber is its aspect ratio, which is the fiber length divided by an equivalent fiber diameter. Typical aspect ratios range from about 30 to 150, with the most common ratio being about 100. More details on fibrous concrete are given in [21].

2.19 STEEL REINFORCEMENT

Reinforcement, usually in the form of steel bars, is placed in the concrete member, mainly in the tension zone, to resist the tensile forces resulting from external load on the member. Reinforcement is also used to increase the member's compression resistance. Steel costs more than concrete, but it has a yield strength about 10 times the compressive strength of concrete. The function and behavior of both steel and concrete in a reinforced concrete member are discussed in Chapter 3.

Longitudinal bars taking either tensile or compression forces in a concrete member are called *main reinforcement*. Additional reinforcement in slabs, in a direction perpendicular to the main reinforcement, is called *secondary*, or *distribution, reinforcement*. In reinforced concrete beams, another type of steel reinforcement is used, transverse to the direction of the main steel and bent in a box or U shape. These are called *stirrups*. Similar reinforcements are used in columns, where they are called *ties*. Refer to Figure 8.8 and Figure 10.3.

2.19.1 Types of Steel Reinforcement

Different types of steel reinforcement are used in various reinforced concrete members. These types can be classified as follows:

Round bars. Round bars are used most widely for reinforced concrete. Round bars are available in a large range of diameters, from $\frac{1}{4}$ (6 mm) to $1\frac{3}{8}$ (36 mm), plus two special types, $1\frac{3}{4}$ (45 mm) and $2\frac{1}{4}$ (57 mm). Round bars, depending on their surfaces, are either plain or deformed bars. Plain bars are used mainly for secondary reinforcement or in stirrups and ties. Deformed bars have projections or deformations on the surface for the purpose of improving the bond with concrete and reducing the width of cracks opening in the tension zone.

The diameter of a plain bar can be measured easily, but for a deformed bar, a nominal diameter is used that is the diameter of a circular surface with the same area as the section of the deformed bar. Requirements of surface projections on bars are specified by ASTM Specification A 305, or A 615. The bar sizes are designated by numbers 3 through 11, corresponding to the diameter in one-eighths of an inch. For instance, a no. 7 bar has a nominal diameter of $\frac{7}{8}$ in. and a no. 4 bar has a nominal diameter of $\frac{1}{2}$ in. The two largest sizes are designated no. 14 and no. 18, respectively. American standard bar marks are shown on the steel reinforcement to indicate the initial of the producing mill, the bar size, and the type of steel (Fig. 2.6). The grade of the reinforcement is indicated on the bars by either the continuous-line system or the number system. In the first system, one longitudinal line is added to the bar, in addition to the main ribs, to indicate the high-strength grade of 60 ksi (420 N/mm^2), according to ASTM Specification A 617. If only the main ribs are shown on the bar, without any additional lines, the steel is of the ordinary grade according to ASTM A 615 for the structural grade ($f_y = 40$ ksi, or 280 N/mm^2). In the number system, the yield strength of the high-strength grades is marked clearly on every bar. For ordinary grades, no strength marks are indicated. The two types are shown in Fig. 2.6.

Welded fabrics and mats. Welded fabrics and mats consist of a series of longitudinal and transverse cold-drawn steel wires, generally at right angles and welded together at all points of intersection. Steel reinforcement may be built up into three-dimensional cages before being placed in the forms.

Prestressed concrete wires and strands. Prestressed concrete wires and strands use special high-strength steel (see Chapter 20). High-tensile steel wires of diameters 0.192 in. (5 mm) and 0.276 in. (7 mm) are used to form the prestressing cables by winding six steel wires around a seventh wire of slightly larger diameter. The ultimate strength of prestressed strands is 250 ksi or 270 ksi.

2.19.2 Grades and Strength

Different grades of steel are used in reinforced concrete. Limitations on the minimum yield strength, ultimate strength, and elongation are explained in ASTM specifications for reinforcing steel bars (Table 2.17). The properties and grades of metric reinforcing steel are shown in Tables 2.18 and 2.19.

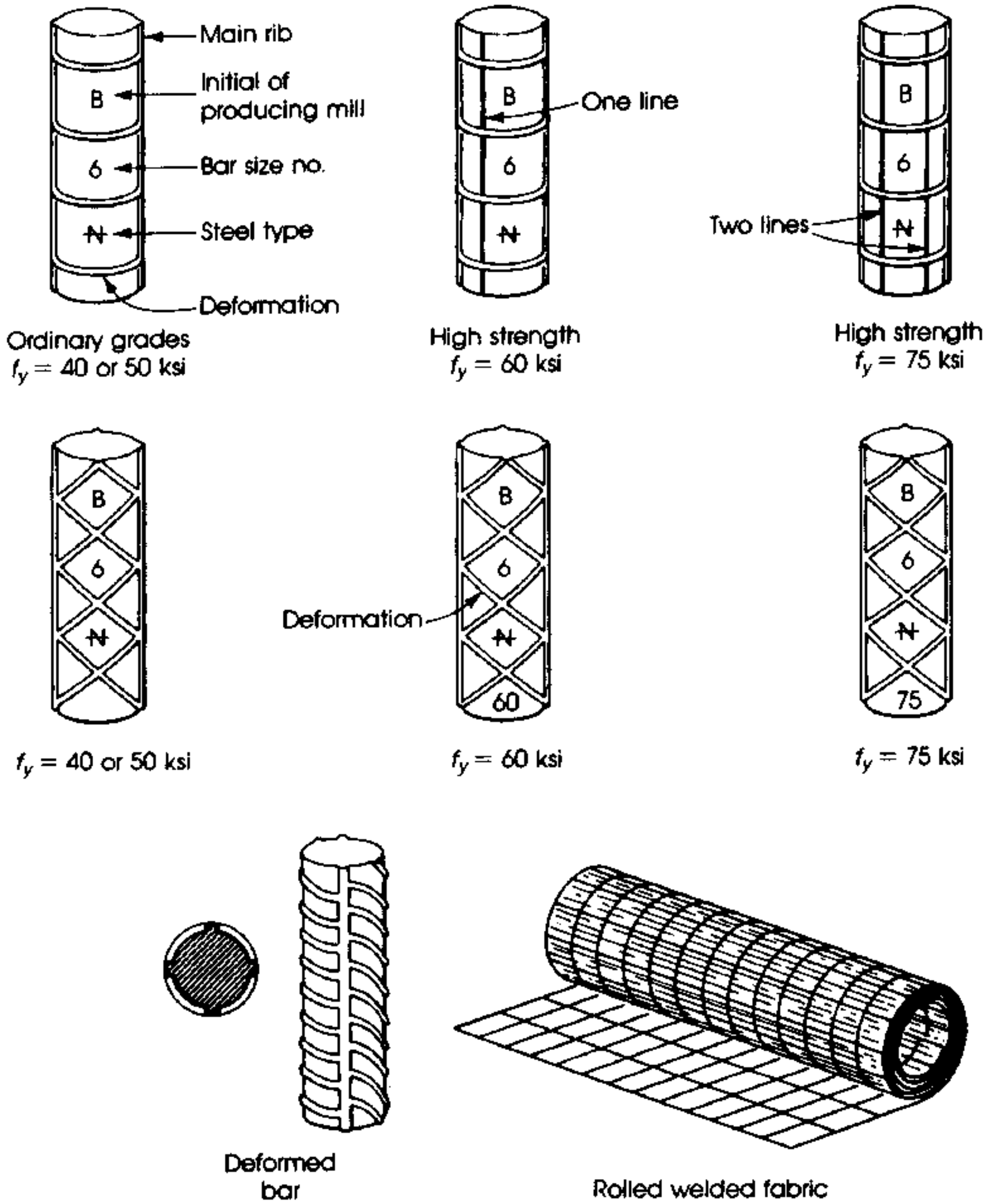


Figure 2.6 Some types of deformed bars and American standard bar marks.

2.19.3 Stress–Strain Curves

The most important factor affecting the mechanical properties and stress–strain curve of the steel is its chemical composition. The introduction of carbon and alloying additives in steel increases its strength but reduces its ductility. Commercial steel rarely contains more than 1.2% carbon; the proportion of carbon used in structural steels varies between 0.2% and 0.3%.

Two other properties are of interest in the design of reinforced concrete structures; the first is the modulus of elasticity, E_s. It has been shown that the modulus of elasticity is constant for all types of steel. The ACI Code has adopted a value of $E_s = 29 \times 10^6$ psi (2.0×10^5 MPa).

Table 2.17 Grade of ASTM Reinforcing Steel Bars

	Minimum Yield Strength f_y		Ultimate Strength f_{su}	
Steel	**ksi**	**MPa**	**ksi**	**MPa**
Billet steel				
Grade 40	40	276	70	483
60	60	414	90	621
75	75	518	100	690
Rail steel				
Grade 50	50	345	80	551
60	60	414	90	621
Deformed wire				
Reinforcing	75	518	85	586
Fabric	70	483	80	551
Cold-drawn wire				
Reinforcing	70	483	80	551
Fabric	65	448	75	518
Fabric	56	386	70	483

Table 2.18 ASTM 615 M (Metric) for Reinforcing Steel Bars

Bar No.	Diameter (mm)	Area (mm^2)	Weight (kg/m)
10 M	11.3	100	0.785
15 M	16.0	200	1.570
20 M	19.5	300	2.355
25 M	25.2	500	3.925
30 M	29.9	700	5.495
35 M	35.7	1000	7.850
45 M	43.7	1500	11.770
55 M	56.4	2500	19.600

Table 2.19 ASTM Metric Specifications

		Grade	
ASTM	**Bar size no.**	**MPa**	**ksi**
A615 M	10, 15, 20	300	43.5
Billet steel	10–55	400	58.0
	35, 45, 55	500	72.5
A616 M	10–35	350	50.75
Rail steel	10–35	400	58.0
A617 M	10–35	300	43.5
Axle steel	10–35	400	58.0
A706	10–55	400	58.0
Low alloy			

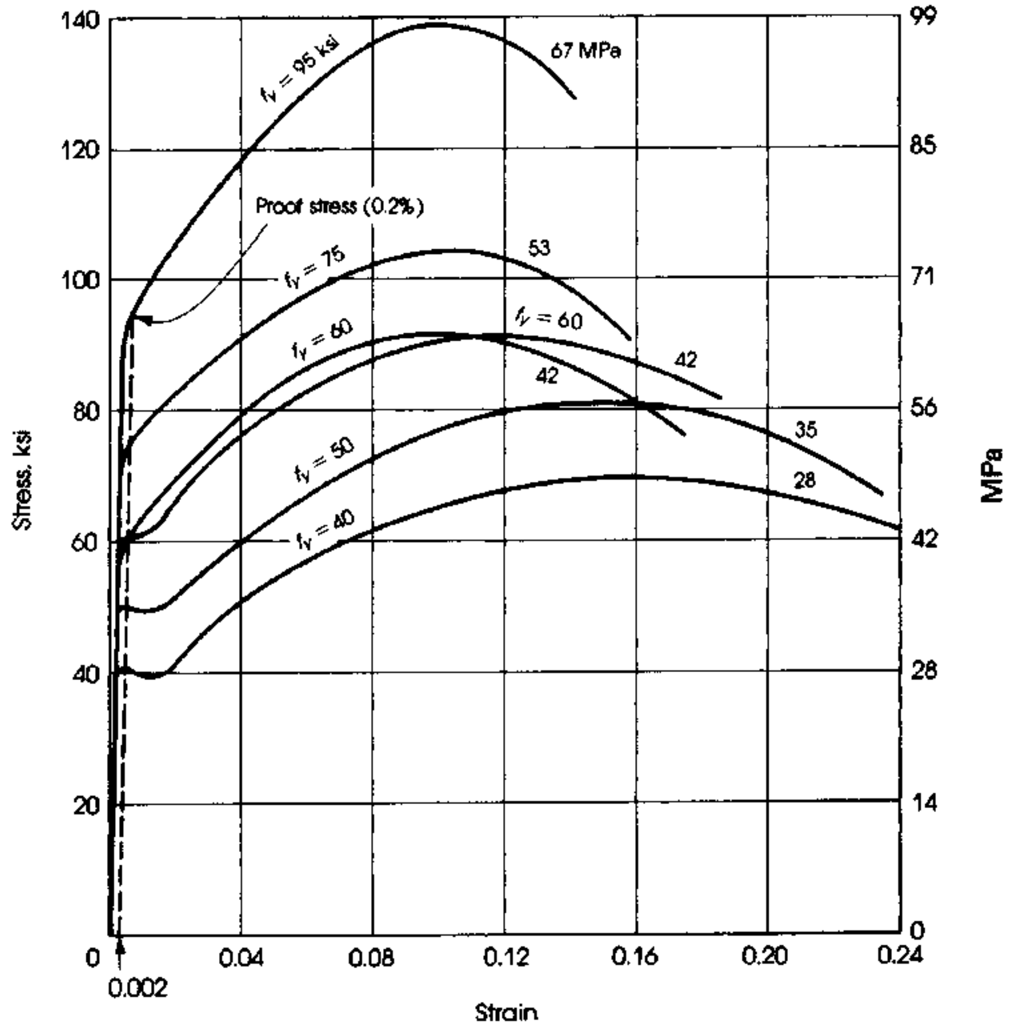


Figure 2.7 Typical stress-strain curves for some reinforcing steel bars of different grades. Note that 60-ksi steel may or may not show a definite yield point.

The modulus of elasticity is the slope of the stress-strain curve in the elastic range up to the proportional limit; E_s = stress/strain. Second is the yield strength, f_y. Typical stress—strain curves for some steel bars are shown in Fig. 2.7. In high-tensile steel, a definite yield point may not show on the stress-strain curve. In this case, ultimate strength is reached gradually under an increase of stress (Fig. 2.7). The yield strength or proof stress is considered the stress that leaves a residual strain of 0.2% on the release of load, or a total strain of 0.5% to 0.6% under load.

SUMMARY

Section 2.1

The main factors that affect the strength of concrete are the water—cement ratio, properties and proportions of materials, age of concrete, loading conditions, and shape of tested specimen.

$$f'_c \text{ (cylinder)} = 0.85 f'_c \text{ (cube)} = 1.10 f'_c \text{ (prism)}$$

Sections 2.2–2.6

1. The usual specimen used to determine the compressive strength of concrete at 28 days is a 6- by 12-in. (150- by 300-mm) cylinder. Compressive strength between 3000 and 6000 psi is usually specified for reinforced concrete structures. Maximum stress, f_c', is reached at an estimated strain of 0.002, whereas rupture occurs at a strain of about 0.003.
2. Tensile strength of concrete is measured indirectly by a splitting test performed on a standard cylinder using formula $f_{sp}' = 2P/\pi LD$. Tensile strength of concrete is approximately $0.1f_c'$.
3. Flexural strength (modulus of rupture, f_r) of concrete is calculated by testing a 6- by 6- by 28-in. plain concrete beam, $f_r = 7.5\lambda\sqrt{f_c'}$ (psi), where λ is a modification factor related to unit weight of concrete.
4. Nominal shear stress is $2\lambda\sqrt{f_c'}$ (psi).

Sections 2.7–2.9

The modulus of elasticity of concrete, E_c for unit weight w between 90 and 160 pcf, is $E_c = 33w^{1.5}\sqrt{f_c'}$ (psi) $= 0.043w^{1.5}\sqrt{f_c'}$ MPa.

For normal-weight concrete, $w = 145$ pcf.

$$E_c = 57{,}600\sqrt{f_c'} \qquad \text{or} \qquad E_c = 57{,}000\sqrt{f_c'} = 4700\sqrt{f_c'} \text{ MPa}$$

The shear modulus of concrete is $G_c = E_c/2(1 + \mu) = 0.43\ E_c$ for a Poisson's ratio $\mu = \frac{1}{6}$. Poisson's ratio, μ, varies between 0.15 and 0.20, with an average value of 0.18.

Section 2.10

Modular ratio is $n = E_s/E_c = 500/\sqrt{f_c'}$, where f_c' is in psi.

Section 2.11

1. Values of shrinkage for normal concrete fall between 200×10^{-6} and 700×10^{-6}. An average value of 300×10^{-6} may be used.
2. The coefficient of expansion of concrete falls between 4×10^{-6} and $7 \times 10^{-6}/^\circ\text{F}$.

Section 2.12–2.13

The ultimate magnitude of creep varies between 0.2×10^{-6} and 2×10^{-6} per unit stress per unit length. An average value of 1×10^{-6} may be adopted in practical problems. Of the ultimate (20-year) creep, 18% to 35% occurs in 2 weeks, 30% to 70% occurs in 3 months, and 64% to 83% occurs in 1 year.

Section 2.14

The unit weight of normal concrete is 145 pcf for plain concrete and 150 pcf for reinforced concrete.

Section 2.15

Reinforced concrete is a much better fire-resistant material than steel. Concrete itself has a low thermal conductivity. An increase in concrete cover in structural members such as walls, columns, beams, and floor slabs will increase the fire resistance of these members.

Sections 2.16–2.18

1. High-performance concrete implies that concrete exhibits properties of strength, toughness, energy absorption, durability, stiffness, and ductility higher than normal concrete.
2. Concrete is made lighter than normal-weight concrete by inclusion of air in the concrete composition. Types of lightweight concrete are no-fines concrete, lightweight aggregate concrete, aerated concrete, and cellular concrete.
3. Fibrous concrete is made of concrete constituents and discrete reinforcing fibers such as steel, glass, and organic polymers.

Section 2.19

The grade of steel mainly used is grade 60 ($f_y = 60$ ksi). The modulus of elasticity of steel is $E_s = 29 \times 10^6$ psi (2×10^{-5} MPa).

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PROBLEMS

2.1 Explain the modulus of elasticity of concrete in compression and the shear modulus.

2.2 Determine the modulus of elasticity of concrete by the ACI formula for a concrete cylinder that has a unit weight of 120 pcf (1920 kg/m^3) and a compressive strength of 3000 psi (21 MPa).

2.3 Estimate the modulus of elasticity and the shear modulus of a concrete specimen with a dry density of 150 pcf (2400 kg/m^3) and compressive strength of 4500 psi (31 MPa) using Poisson's ratio, $\mu = 0.18$.

2.4 What is meant by the modular ratio and Poisson's ratio? Give approximate values for concrete.

2.5 What factors influence the shrinkage of concrete?

2.6 What factors influence the creep of concrete?

2.7 What are the types and grades of the steel reinforcement used in reinforced concrete?

2.8 On the stress-strain diagram of a steel bar, show and explain the following: proportional limit, yield stress, ultimate stress, yield strain, and modulus of elasticity.

2.9 Calculate the modulus of elasticity of concrete, E_c, for the following types of concrete:

$$E_c = 33W^{1.5}\sqrt{f'_c}\ \text{(ft)},$$

$$E_c = 0.043W^{1.5}\sqrt{f'_c}\ \text{(SI)}$$

Density	**Strength f'_c**
160 pcf	5000 psi
145 pcf	4000 psi
125 pcf	2500 psi
2400 kg/m^3	35 MPa
2300 kg/m^3	30 MPa
2100 kg/m^3	25 MPa

2.10 Determine the modular ratio, n, and the modulus of rupture for each case of Problem 2.9. Tabulate your results.

$$f_r = 7.5\lambda\sqrt{f'_c}\ \text{(psi)} \quad f_r = 0.62\lambda\sqrt{f'_c}\ \text{(MPa)}$$

2.11 A standard normal 6 × 12-in. concrete cylinder was tested to failure, and the following loads and strains were recorded.

Load, kips	Strain x 10^{-4}	Load, kips	Strain x 10^{-4}
0.0	0.0	72	10.0
12	1.2	84	13.6
24	2.0	96	18.0
36	3.2	108	30.0
48	5.2	95	39.0
60	7.2	82	42.0

a. Draw the stress–strain diagram of concrete and determine the maximum stress and corresponding strain.

b. Determine the initial modulus and secant modulus.

c. Calculate the modulus of elasticity of concrete using the ACI formula for normal-weight concrete and compare results.

$$E_c = 57{,}000\sqrt{f'_c}\ \text{psi}$$

$$E_c = 4730\sqrt{f'_c}\ \text{MPa}$$

CHAPTER 3

FLEXURAL ANALYSIS OF REINFORCED CONCRETE BEAMS

Apartment building, Fort Lauderdale, Florida.

3.1 INTRODUCTION

The analysis and design of a structural member may be regarded as the process of selecting the proper materials and determining the member dimensions such that the design strength is equal or greater than the required strength. The required strength is determined by multiplying the actual applied loads, the dead load, the assumed live load, and other loads, such as wind, seismic, earth pressure, fluid pressure, snow, and rain loads, by load factors. These loads develop external forces such as bending moments, shear, torsion, or axial forces depending on how these loads are applied to the structure.

In proportioning reinforced concrete structural members, three main items can be investigated:

1. The safety of the structure, which is maintained by providing adequate internal design strength.
2. Deflection of the structural member under service loads. The maximum value of deflection must be limited and is usually specified as a factor of the span, to preserve the appearance of the structure.
3. Control of cracking conditions under service loads. Visible cracks spoil the appearance of the structure and also permit humidity to penetrate the concrete, causing corrosion of steel and consequently weakening the reinforced concrete member. The ACI Code implicitly limits crack widths to 0.016 in. (0.40 mm) for interior members and 0.013 in. (0.33 mm) for exterior members. Control of cracking is achieved by adopting and limiting the spacing of the tension bars (see Chapter 6).

It is worth mentioning that the strength design approach was first permitted in the United States in 1956 and in Britain in 1957. The latest ACI Code emphasizes the strength concept based on specified strain limits on steel and concrete that develop tension-controlled, compression-controlled, or transition conditions.

3.2 ASSUMPTIONS

Reinforced concrete sections are heterogeneous (nonhomogeneous), because they are made of two different materials, concrete and steel. Therefore, proportioning structural members by ultimate-strength design is based on the following assumptions:

1. Strain in concrete is the same as in reinforcing bars at the same level, provided that the bond between the steel and concrete is adequate.
2. Strain in concrete is linearly proportional to the distance from the neutral axis.
3. The modulus of elasticity of all grades of steel is taken as $E_s = 29 \times 10^6$ lb/in.^{2} (200,000 MPa or N/mm^2). The stress in the elastic range is equal to the strain multiplied by E_s.
4. Plane cross-sections continue to be plane after bending.
5. Tensile strength of concrete is neglected because (1) concrete's tensile strength is about 10% of its compressive strength, (2) cracked concrete is assumed to be not effective, and (3) before cracking, the entire concrete section is effective in resisting the external moment.
6. The method of elastic analysis, assuming an ideal behavior at all levels of stress, is not valid. At high stresses, nonelastic behavior is assumed, which is in close agreement with the actual behavior of concrete and steel.
7. At failure the maximum strain at the extreme compression fibers is assumed equal to 0.003 by the ACI Code provision.
8. For design strength, the shape of the compressive concrete stress distribution may be assumed to be rectangular, parabolic, or trapezoidal. In this text, a rectangular shape will be assumed (ACI Code, Section 10.2).

3.3 BEHAVIOR OF A SIMPLY SUPPORTED REINFORCED CONCRETE BEAM LOADED TO FAILURE

Concrete being weakest in tension, a concrete beam under an assumed working load will definitely crack at the tension side, and the beam will collapse if tensile reinforcement is not provided. Concrete cracks occur at a loading stage when its maximum tensile stress reaches the modulus of rupture of concrete. Therefore, steel bars are used to increase the moment capacity of the beam; the steel bars resist the tensile force, and the concrete resists the compressive force.

To study the behavior of a reinforced concrete beam under increasing load, let us examine how two beams were tested to failure. Details of the beams are shown in Fig. 3.1. Both beams had a section of 4.5 in. by 8 in. (110 mm by 200 mm), reinforced only on the tension side by two no. 5 bars. They were made of the same concrete mix. Beam 1 had no stirrups, whereas beam 2 was provided with no. 3 stirrups spaced at 3 in. The loading system and testing procedure were the same for both beams. To determine the compressive strength of the concrete and its modulus of elasticity, E_c, a standard concrete cylinder was tested, and strain was measured at different load increments. The following observations were noted at different distinguishable stages of loading.

Stage 1. At zero external load, each beam carried its own weight in addition to that of the loading system, which consisted of an I-beam and some plates. Both beams behaved similarly at this stage. At any section, the entire concrete section, in addition to the steel reinforcement, resisted

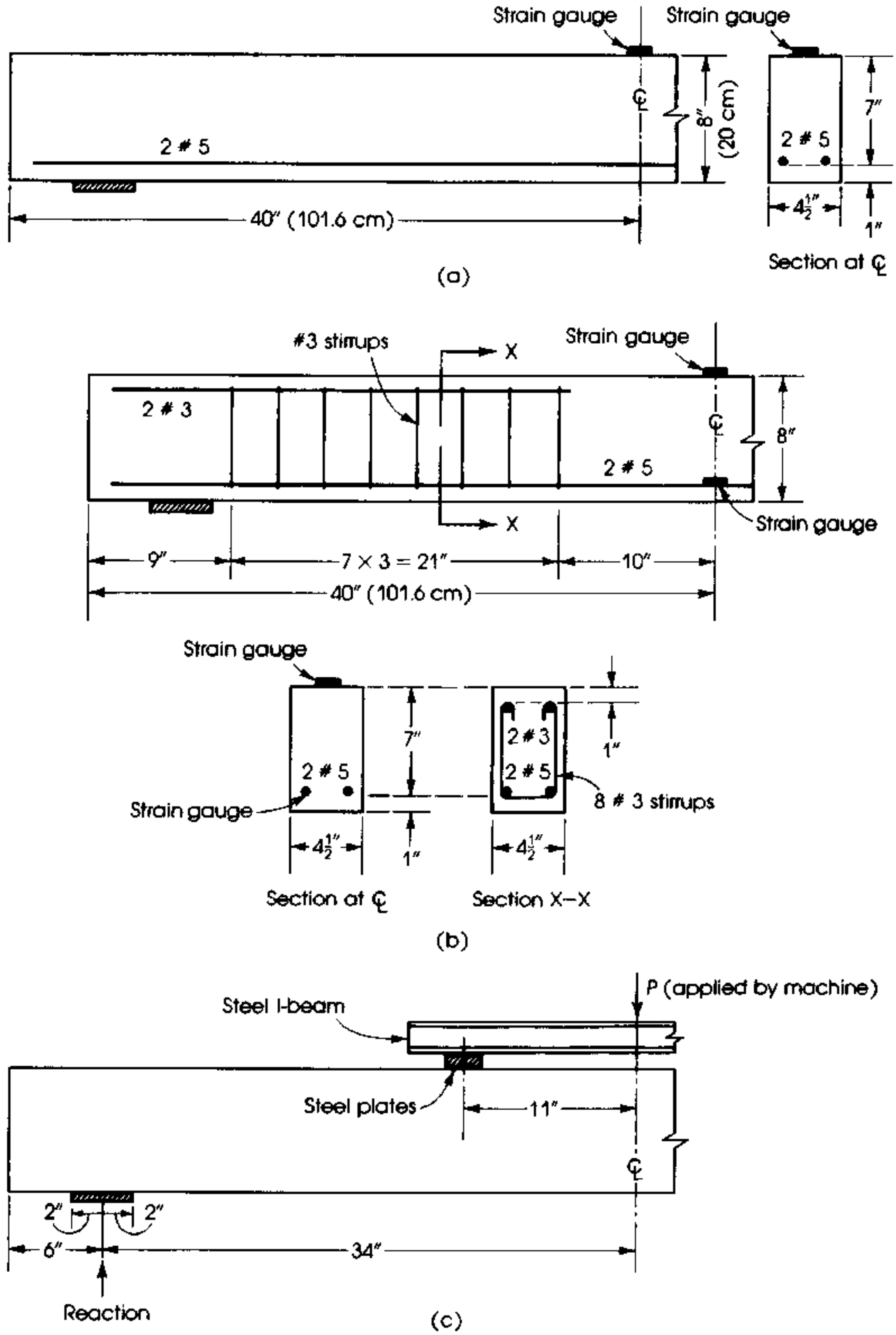
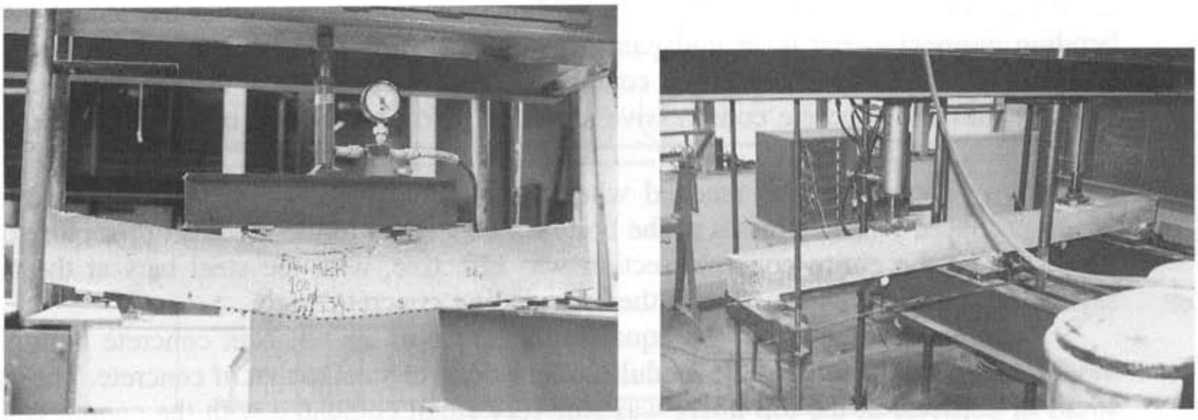
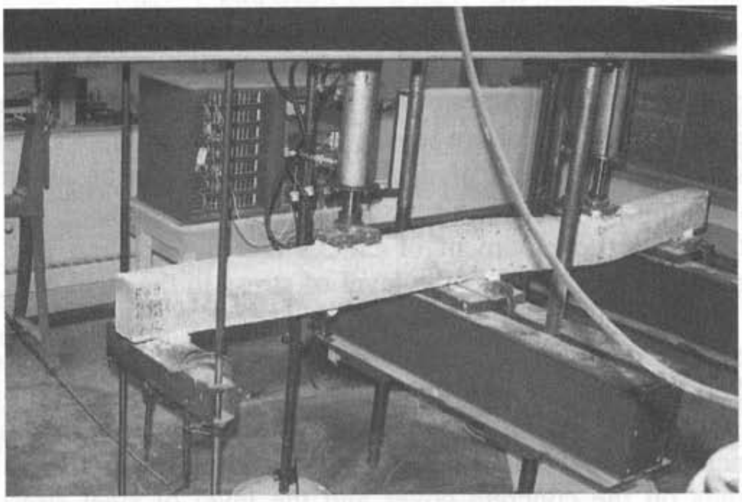


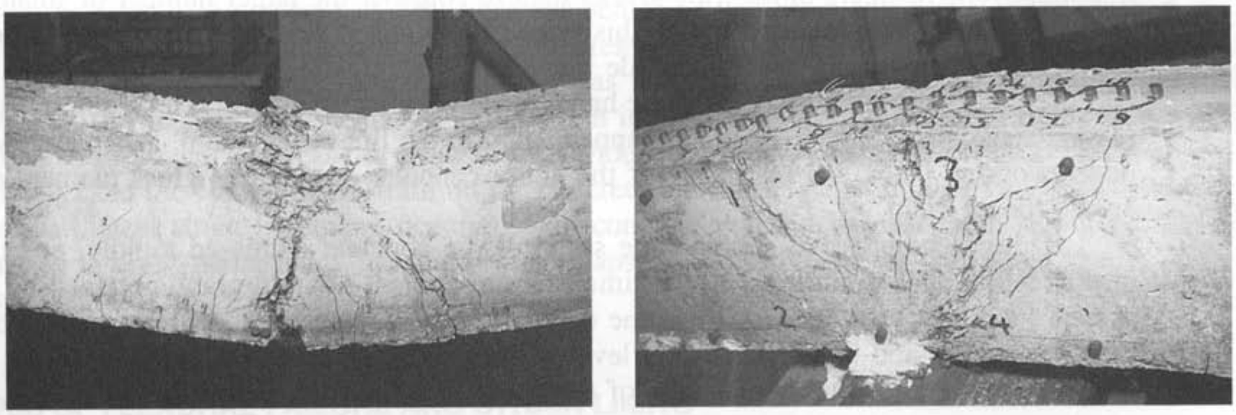
Figure 3.1 Details of tested beams: (*a*) beam 1, (*b*) beam 2, and (*c*) loading system. All beams are symmetrical about the centerline.



Test on a simply supported beam and a two-span continuous beam loaded to failure.



Two-span continuous reinforced concrete beam loaded to failure.



Failure conditions at the positive- and negative-moment sections in a continuous reinforced concrete beam.

the bending moment and shearing forces. Maximum stress occurred at the section of maximum bending moment—that is, at midspan. Maximum tension stress at the bottom fibers was much less than the modulus of rupture of concrete. Compressive stress at the top fibers was much less than the ultimate concrete compressive stress, f'_c. No cracks were observed at this stage.

Stage 2. This stage was reached when the external load, P, was increased from 0 to P_1, which produced tensile stresses at the bottom fibers equal to the modulus of rupture of concrete. At this stage the entire concrete section was effective, with the steel bars at the tension side sustaining a strain equal to that of the surrounding concrete.

Stress in the steel bars was equal to the stress in the adjacent concrete multiplied by the modular ratio, n, the ratio of the modulus of elasticity of steel to that of concrete. The compressive stress of concrete at the top fibers was still very small compared with the compressive strength, f'_c. The behavior of beams was elastic within this stage of loading.

Stage 3. When the load was increased beyond P_1, tensile stresses in concrete at the tension zone increased until they were greater than the modulus of rupture, f_r, and cracks developed. The neutral axis shifted upward, and cracks extended close to the level of the shifted neutral axis. Concrete in the tension zone lost its tensile strength, and the steel bars started to work effectively and to resist the entire tensile force. Between cracks, the concrete bottom fibers had tensile stresses, but they were of negligible value. It can be assumed that concrete below the neutral axis did not participate in resisting external moments.

In general, the development of cracks and the spacing and maximum width of cracks depend on many factors, such as the level of stress in the steel bars, distribution of steel bars in the section, concrete cover, and grade of steel used.

At this stage, the deflection of the beams increased clearly, because the moment of inertia of the cracked section was less than that of the uncracked section. Cracks started about the midspan of the beam, but other parts along the length of the beam did not crack. When load was again increased, new cracks developed, extending toward the supports. The spacing of these cracks depends on the concrete cover and the level of steel stress. The width of cracks also increased. One or two of the central cracks were most affected by the load, and their crack widths increased appreciably, whereas the other crack widths increased much less. It is more important to investigate those wide cracks than to consider the larger number of small cracks.

If the load were released within this stage of loading, it would be observed that permanent fine cracks of no significant magnitude were left. On reloading, cracks would open quickly, because the tensile strength of concrete had already been lost. Therefore, it can be stated that the second stage, once passed, does not happen again in the life of the beam. When cracks develop under working loads, the resistance of the entire concrete section and gross moment of inertia are no longer valid.

At high compressive stresses, the strain of the concrete increased rapidly, and the stress of concrete at any strain level was estimated from a stress—strain graph obtained by testing a standard cylinder to failure for the same concrete. As for the steel, the stresses were still below the yield stress, and the stress at any level of strain was obtained by multiplying the strain of steel, ε_s, by the modulus of elasticity of steel, E_s.

Stage 4. In beam 1, at a load value of 9500 lb (42.75 kN), shear stress at a distance of about the depth of the beam from the support increased and caused diagonal cracks at approximately 45° from horizontal in the direction of principal stresses resulting from the combined action of bending moment and shearing force. The diagonal crack extended downward to the level of

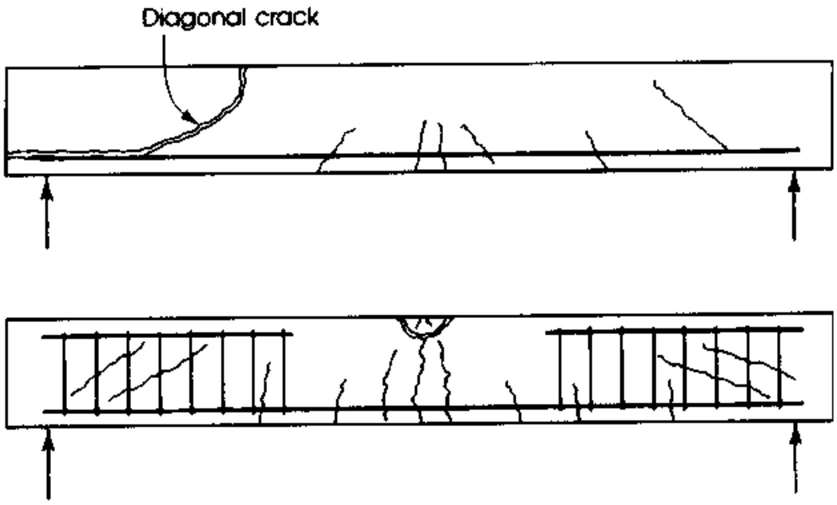


Figure 3.2 Shape of beam 1 at shear failure (*top*) and beam 2 at bending moment failure (*bottom*).

the steel bars and then extended horizontally at that level toward the support. When the crack, which had been widening gradually, reached the end of the beam, a concrete piece broke off and failure occurred suddenly (Fig. 3.2). The failure load was 13,600 lb (61.2 kN). Stresses in concrete and steel at the midspan section did not reach their failure stresses. (The shear behavior of beams is discussed in Chapter 8.)

In beam 2, at a load of 11,000 lb (49.5 kN), a diagonal crack developed similar to that of beam 1; then other parallel diagonal cracks appeared, and the stirrups started to take an effective part in resisting the principal stresses. Cracks did not extend along the horizontal main steel bars, as in beam 1. On increasing the load, diagonal cracks on the other end of the beam developed at a load of 13,250 lb (59.6 kN). Failure did not occur at this stage because of the presence of stirrups.

Stage 5. When the load on beam 2 was further increased, strains increased rapidly until the maximum carrying capacity of the beam was reached at ultimate load, $P_u = 16{,}200$ lb (72.9 kN).

In beam 2, the amount of steel reinforcement used was relatively small. When reached, the yield strain can be considered equal to yield stress divided by the modulus of elasticity of steel, $\varepsilon_y = f_y/E_s$; the strain in the concrete, ε_c, was less than the strain at maximum compressive stress, f'_c. The steel bars yielded, and the strain in steel increased to about 12 times that of the yield strain without increase in load. Cracks widened sharply, deflection of the beam increased greatly, and the compressive strain on the concrete increased. After another very small increase of load, steel strain hardening occurred, and concrete reached its maximum strain, ε'_c, and it started to crush under load; then the beam collapsed. Figure 3.2 shows the failure shapes of the two beams.

3.4 TYPES OF FLEXURAL FAILURE AND STRAIN LIMITS

3.4.1 Flexural Failure

Three types of flexural failure of a structural member can be expected depending on the percentage of steel used in the section.

1. Steel may reach its yield strength before the concrete reaches its maximum strength, Fig. 3.3a. In this case, the failure is due to the yielding of steel reaching a high strain equal to or greater than 0.005. The section contains a relatively small amount of steel and is called a tension-controlled section.
2. Steel may reach its yield strength at the same time as concrete reaches its ultimate strength, Fig. 3.3*b*. The section is called a balanced section.
3. Concrete may fail before the yield of steel, Fig. 3.3*c*, due to the presence of a high percentage of steel in the section. In this case, the concrete strength and its maximum strain of 0.003 are reached, but the steel stress is less than the yield strength, that is, f_s is less than f_y. The strain in the steel is equal to or less than 0.002. This section is called a compression-controlled section.

It can be assumed that concrete fails in compression when the concrete strain reaches 0.003. A range of 0.0025 to 0.004 has been obtained from tests and the ACI Code assumes a strain of 0.003.

In beams designed as tension-controlled sections, steel yields before the crushing of concrete. Cracks widen extensively, giving warning before the concrete crushes and the structure collapses. The ACI Code adopts this type of design. In beams designed as balanced or compression-controlled sections, the concrete fails suddenly, and the beam collapses immediately without warning. The ACI Code does not allow this type of design.

3.4.2 Strain Limits for Tension and Tension-Controlled Sections

The design provisions for both reinforced and prestressed concrete members are based on the concept of tension or compression-controlled sections, ACI Code, Section 10.3. Both are defined in terms of net tensile strain (NTS), (ε_t, in the extreme tension steel at nominal strength, exclusive of prestress strain. Moreover, two other conditions may develop: (1) the balanced strain condition and (2) the transition region condition. These four conditions are defined as follows:

1. Compression-controlled sections are those sections in which the net tensile strain, NTS, in the extreme tension steel at nominal strength is equal to or less than the compression-controlled strain limit at the time when concrete in compression reaches its assumed strain limit of 0.003, ($\varepsilon_c = 0.003$). For grade 60 steel, ($f_y = 60$ ksi), the compression-controlled strain limit may be taken as a net strain of 0.002, Fig. 3.4*a*. This case occurs mainly in columns subjected to axial forces and moments.
2. Tension-controlled sections are those sections in which the NTS, ε_t, is equal to or greater than 0.005 just as the concrete in the compression reaches its assumed strain limit of 0.003, Fig. 3.4*c*.
3. Sections in which the NTS in the extreme tension steel lies between the compression-controlled strain limit (0.002 for $f_y = 60$ ksi) and the tension-controlled strain limit of 0.005 constitute the transition region, Fig. 3.4*b*.
4. The balanced strain condition develops in the section when the tension steel, with the first yield, reaches a strain corresponding to its yield strength, f_y or $\varepsilon_s = f_y/E_s$, just as the maximum strain in concrete at the extreme compression fibers reaches 0.003, Fig. 3.5.

In addition to the above four conditions, Section 10.3.5 of the ACI Code indicates that the net tensile strain, ε_t, at nominal strength, within the transition region, shall not be less than 0.004 for reinforced concrete flexural members without or with an axial load less than 0.10 $f'_c A_g$, where A_g = gross area of the concrete section.

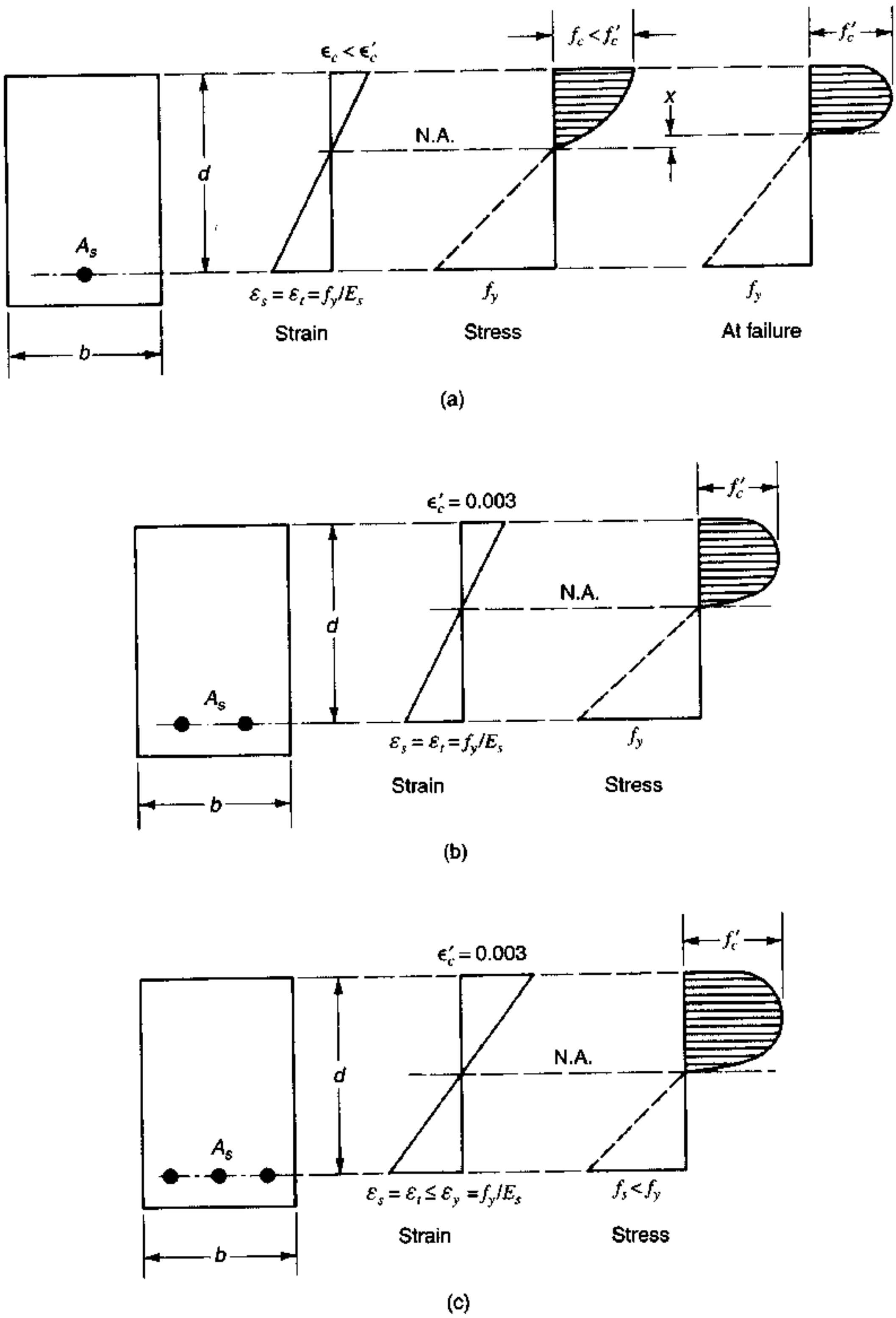


Figure 3.3 Stress and strain diagrams for (*a*) tension-controlled, (*b*) balanced, and (*c*) compression-controlled sections.

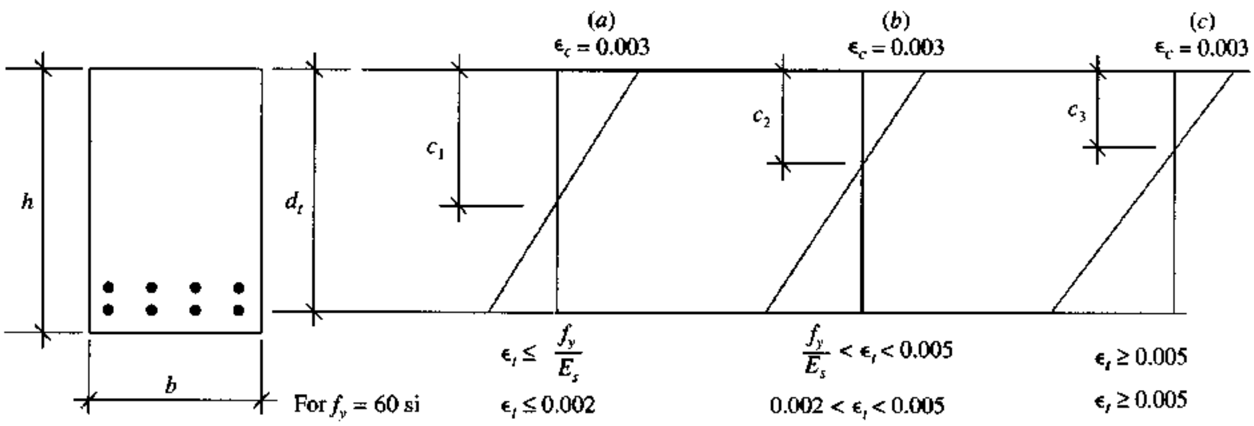


Figure 3.4 Strain limit distribution, $c_1 > c_2 > c_3$: (*a*) compression-controlled section, (*b*) transition region, and (*c*) tension-controlled section.

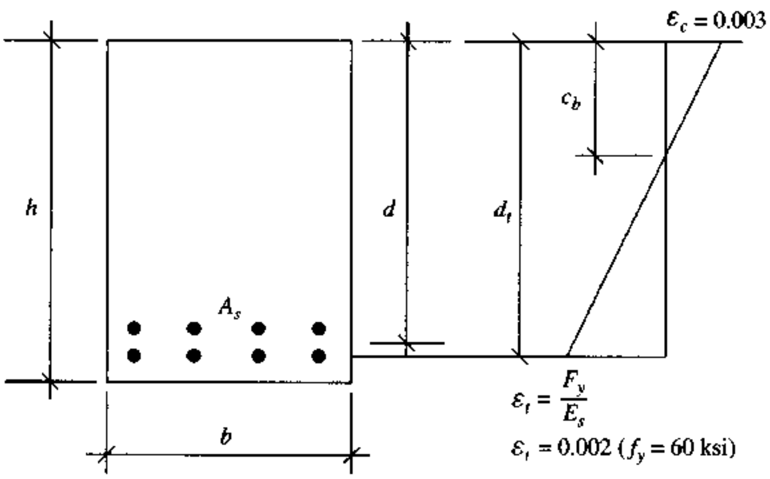


Figure 3.5 Balanced strain section (occurs at first yield or at distance d_t).

Note that d_t in Fig. 3.4, is the distance from the extreme concrete compression fiber to the extreme tension steel, while the effective depth, d, equals the distance from the extreme concrete compression fiber to the centroid of the tension reinforcement, Fig. 3.5. These cases are summarized in Table 3.1.

Table 3.1 Strain Limits of Fig. 3.4

Section Condition	Concrete Strain	Steel Strain	Notes (f_y = 60 ksi)
Compression-controlled	0.003	$\varepsilon_t \leq f_y/E_s$	$\varepsilon_t \leq 0.002$
Tension-controlled	0.003	$\varepsilon_t \geq 0.005$	$\varepsilon_t \geq 0.005$
Transition region	0.003	$f_y/E_s < \varepsilon_t < 0.005$	$0.002 < \varepsilon_t < 0.005$
Balanced strain	0.003	$\varepsilon_s = f_y/E_s$	$\varepsilon_s = 0.002$
Transition region (flexure)	0.003	$0.004 \leq \varepsilon_t < 0.005$	$0.004 \leq \varepsilon_t < 0.005$

3.5 LOAD FACTORS

The types of loads and the safety provisions were explained earlier in Sections 1.7 and 1.8.

For the design of structural members, the factored design load is obtained by multiplying the dead load by a load factor and the specified live load by another load factor. The magnitude of the load factor must be adequate to limit the probability of sudden failure and to permit an economical structural design. The choice of a proper load factor or, in general, a proper factor of safety depends mainly on the importance of the structure (whether a courthouse or a warehouse), the degree of warning needed prior to collapse, the importance of each structural member (whether a beam or column), the expectation of overload, the accuracy of artisanry, and the accuracy of calculations.

Based on historical studies of various structures, experience, and the principles of probability, the ACI Code adopts a load factor of 1.2 for dead loads and 1.6 for live loads. The dead load factor is smaller, because the dead load can be computed with a greater degree of certainty than the live load. Moreover, the choice of factors reflects the degree of the economical design as well as the degree of safety and serviceability of the structure. It is also based on the fact that the performance of the structure under actual loads must be satisfactorily within specific limits.

If the required strength is denoted by U (ACI Code, Section 9.2), and those due to wind and seismic forces are W and E, respectively, according to the ACI Code, the required strength U, shall be the most critical of the following factors (based on the ASCE 7-05):

1. In the case of dead, live, and wind loads,

$$U = 1.4D \tag{3.1a}$$

$$U = 1.2D + 1.6L \tag{3.1b}$$

$$U = 1.2D + 1.0L + 1.6W \tag{3.1c}$$

$$U = 0.9D + 1.6W \tag{3.1d}$$

2. In the case of dead, live, and seismic (earthquake) forces, E,

$$U = 1.2D + 1.0(L + E) \tag{3.2a}$$

$$U = 0.9D + 1.0E \tag{3.2b}$$

3. When the earth pressure load, H, is included,

$$U = 1.2D + 1.6(L + H) \tag{3.3a}$$

$$U = 0.9D + 1.6(W + H) \tag{3.3b}$$

$$U = 0.9D + 1.0E + 1.6H \tag{3.3c}$$

4. When pressure loads from fluids, F, are included,

$$U = 1.4(D + F) \tag{3.4a}$$

$$U = 1.2(D + F) + 1.6(L + H) \tag{3.4b}$$

5. For load combination due to roof live load, L_r, rain load, R, snow load, S, the effect of temperature T (including the effect of creep, shrinkage, and differential settlement) in addition to the above loads,

$$U = 1.2(D + F + T) + 1.6(L + H) + 0.5(L_r \text{ or } S \text{ or } R) \quad (3.5a)$$

$$U = 1.2D + 1.6(L_r \text{ or } S \text{ or } R) + (1.0L \text{ or } 0.8W) \quad (3.5b)$$

$$U = 1.2D + 1.6W + 1.0L + 0.5(L_r \text{ or } S \text{ or } R) \quad (3.5c)$$

$$U = 1.2D + 1.0E + 1.0L + 0.2S \quad (3.5d)$$

It is to noted that

1. The load factor L in Eqs. 3.1c, 3.2a, and 3.5a, b, c, and d shall be permitted to be reduced to $0.5L$, except for garages, areas occupied as places of public assembly, and all areas where the live load, L, is greater than 100 pounds per square foot (psf).
2. When the wind load, W, is not reduced by a directionality factor, it is permitted to use $1.3W$ in place of $1.6W$ in Eqs. 3.1d and 3.3b.
3. If the service level of the seismic load E is used, $1.4E$ shall be used in place of $1.0E$ in Eqs. 3.2a and b and 3.3c.
4. If the structural action due to H counteracts that due to W or E, the load factor of H shall be set to 0.
5. In a flood zone area, the flood load or load combinations of ASCE shall be used.
6. Impact effects shall be included with the live load, L.

The ACI Code does not specify a value for impact, but AASHTO specifications give a simple factor for impact, I, as a percentage of the live load, L, as follows:

$$I = 50/(125 + S) \leq 30\% \quad (3.6)$$

where I = percentage of impact, S = part of the span loaded, and live load including impact $= L(1 + I)$.

When a better estimation is known from experiments or experience, the adjusted value shall be used.

The above equations indicate that the dead load factor is 1.2, whereas the live load factor is 1.6. These values are less than those specified by the 1999 ACI Code of 1.4 for the dead load and 1.7 for the live load. The new factors are based on the ASCE specifications ASCE 7-05.

For applied concentrated dead and live loads, P_D, P_L, the factored concentrated load $P_U = 1.2P_D + 1.6P_L$; also $M_U = 1.2\ M_D + 1.6\ M_L$, where M_D and M_L are the service dead-load and live-load moments, respectively.

3.6 STRENGTH-REDUCTION FACTOR ϕ

The nominal strength of a section, say M_n, for flexural members, calculated in accordance with the requirements of the ACI Code provisions must be multiplied by the strength reduction factor, ϕ, which is always less than 1. The strength reduction factor has several purposes:

1. To allow for the probability of under-strength sections due to variations in dimensions, material properties, and inaccuracies in the design equations
2. To reflect the importance of the member in the structure
3. To reflect the degree of ductility and required reliability under the applied loads

The ACI Code, Section 9.3, specifies the following values to be used:

For tension-controlled sections,	$\phi = 0.90$
For compression-controlled section	
a. with spiral reinforcement,	$\phi = 0.75$
b. other reinforced members,	$\phi = 0.65$
For plain concrete,	$\phi = 0.60$
For shear and torsion,	$\phi = 0.75$
For bearing on concrete,	$\phi = 0.65$
For strut and tie models,	$\phi = 0.75$

A higher ϕ factor is used for tension-controlled sections than for compression-controlled sections, because the latter sections have less ductility and they are more sensitive to variations in concrete strength. Also, spirally reinforced compression members have a ϕ value of 0.75 compared to 0.65 for tied compression members; this variation reflects the greater ductility behavior of spirally reinforced concrete members under the applied loads. In the ACI Code provisions, the ϕ factor is based on the behavior of the cross-section at nominal strength, (P_n, M_n), defined in terms of the NTS, ε_t, in the extreme tensile strains, as given in Table 3.1. For tension-controlled members, $\phi = 0.9$. For compression-controlled members, $\phi = 0.75$ (with spiral reinforcement) and $\phi = 0.65$ for other members.

For the transition region, ϕ may be determined by linear interpolation between 0.65 (or 0.75) and 0.9. Figure 3.6*a* shows the variation of ϕ for grade 60 steel. The linear equations are as follows:

$$\phi = 0.75 + (\varepsilon_t - 0.002)(50) \quad \text{(for spiral members)} \tag{3.7}$$

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) \quad \text{(for other members)} \tag{3.8}$$

Alternatively, ϕ may be determined in the transition region, as a function of (c/d_t) for grade 60 steel as follows:

$$\phi = 0.75 + 0.15\left[\frac{1}{c/d_t} - \frac{5}{3}\right] \quad \text{(for spiral members)} \tag{3.9}$$

$$\phi = 0.65 + 0.25\left[\frac{1}{c/d_t} - \frac{5}{3}\right] \quad \text{(for other members)} \tag{3.10}$$

where c = the depth of the neutral axis at nominal strength (c_2 in Fig. 3.4). At the limit strain of 0.002 for grade 60 steel and from the triangles of Fig. 3.4*a*, $c/d_t = 0.003/(0.002 + 0.003) = 0.6$. Similarly, at a strain, $\varepsilon_t = 0.005$, $c/d_t = 0.003/(0.005 + 0.003) = 0.375$. Both values are shown in Fig. 3.6.

For reinforced concrete flexural members, the NTS, ε_t, should be equal to or greater than 0.004 (ACI Code, Section 10.3). In this case,

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.82 \tag{3.11}$$

Figure 3.6*b* shows the range of ϕ for flexural members. For grade 60 steel, the range varies between 0.9 for $\varepsilon_t \geq 0.005$ and 0.82 for $\varepsilon_t = 0.004$. Other values of ϕ can be obtained from Eq. 3.11 or by interpolation.

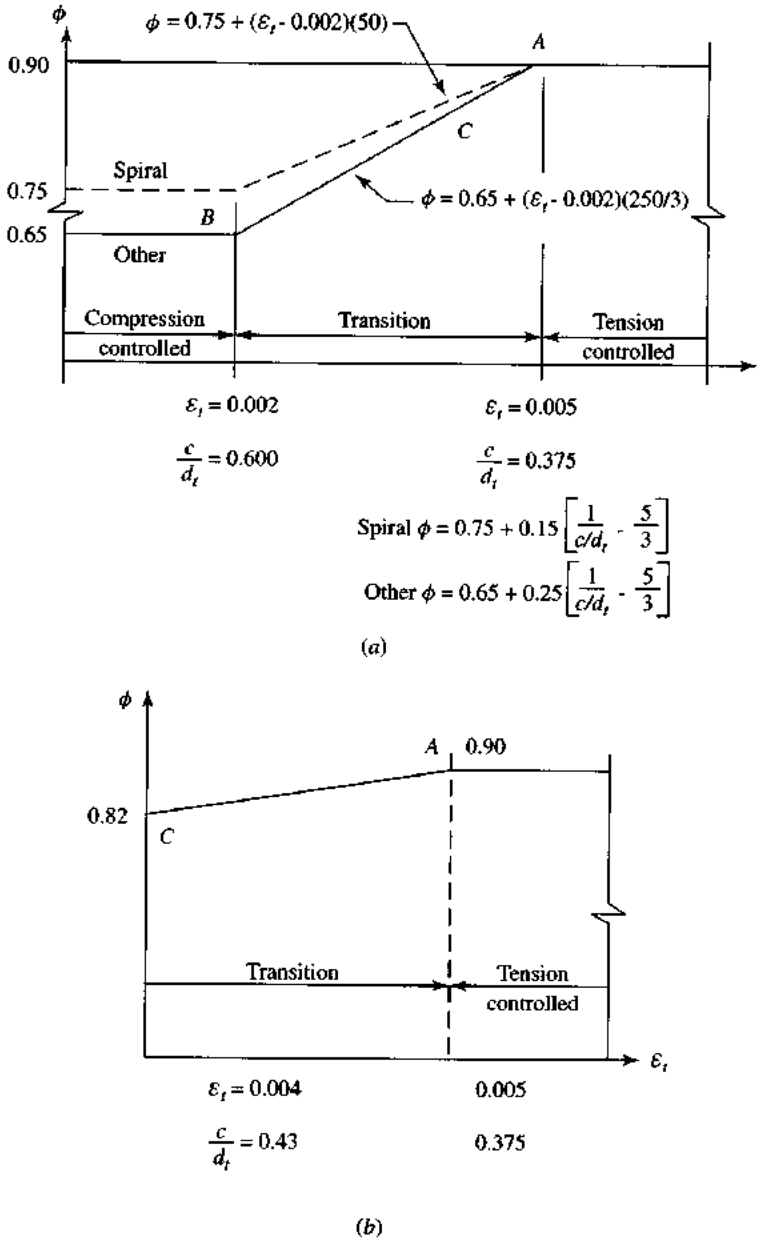


Figure 3.6 (a) Variation of ϕ, with the net tensile strain for grade 60 steel and for prestressed steel, [1]; (b) variation of ϕ and strain limit in flexural member with f_y = 60 ksi.

3.7 SIGNIFICANCE OF ANALYSIS AND DESIGN EXPRESSIONS

Two approaches for the investigations of a reinforced concrete member will be used in this book:

Analysis of a section implies that the dimensions and steel used in the section (in addition to concrete strength and steel yield strength) are given, and it is required to calculate the internal design moment capacity of the section so that it can be compared with the applied external required moment.

Design of a section implies that the external required moment is known from structural analysis, and it is required to compute the dimensions of an adequate concrete section and the amount of steel reinforcement. Concrete strength and yield strength of steel used are given.

3.8 EQUIVALENT COMPRESSIVE STRESS DISTRIBUTION

The distribution of compressive concrete stresses at failure may be assumed to be a rectangle, trapezoid, parabola, or any other shape that is in good agreement with test results.

When a beam is about to fail, the steel will yield first if the section is under-reinforced, and in this case the steel is equal to the yield stress. If the section is over-reinforced, concrete crushes first and the strain is assumed to be equal to 0.003, which agrees with many tests of beams and columns. A compressive force, C, develops in the compression zone and a tension force, T, develops in the tension zone at the level of the steel bars. The position of force T is known, because its line of application coincides with the center of gravity of the steel bars. The position of compressive force C is not known unless the compressive volume is known and its center of gravity is located. If that is done, the moment arm, which is the vertical distance between C and T, will consequently be known.

In Fig. 3.7, if concrete fails, $\varepsilon_c = 0.003$, and if steel yields, as in the case of a balanced section, $f_s = f_y$.

The compression force, C, is represented by the volume of the stress block, which has the nonuniform shape of stress over the rectangular hatched area of bc. This volume may be considered equal to $C = bc(\alpha_1 f_c')$, where $\alpha_1 f_c'$ is an assumed average stress of the nonuniform stress block.

The position of compression force C is at a distance z from the top fibers, which can be considered as a fraction of the distance c (the distance from the top fibers to the neutral axis), and z can be assumed to be equal to $\alpha_2 c$, where $\alpha_2 < 1$. The values of α_1 and α_2 have been estimated from many tests, and their values, as suggested by Mattock, Kriz, and Hognestad [3], are as follows:

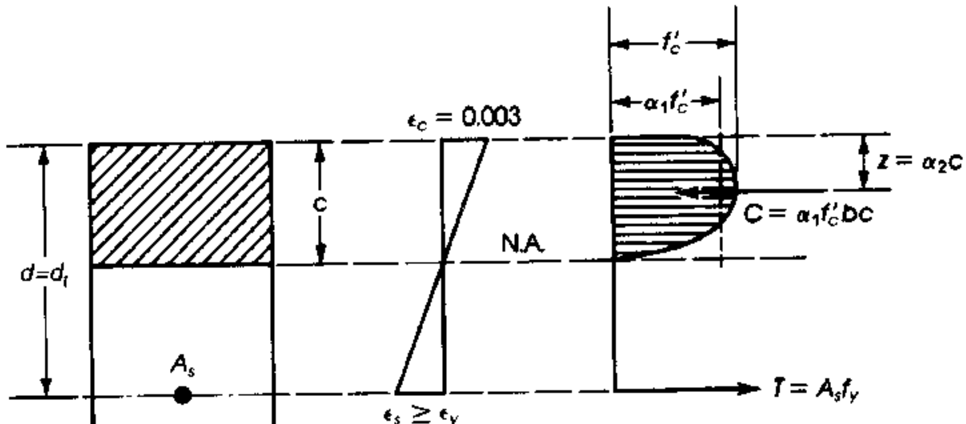


Figure 3.7 Ultimate forces in a rectangular section.

$\alpha_1 = 0.72$ for $f_c' \leq 4000$ psi (27.6 MPa); it decreases linearly by 0.04 for every 1000 psi (6.9 MPa) greater than 4000 psi

$\alpha_2 = 0.425$ for $f_c' < 4000$ psi (27.6 MPa); it decreases linearly by 0.025 for every 1000 psi greater than 4000 psi

The decrease in the value of α_1 and α_2 is related to the fact that high-strength concretes show more brittleness than low-strength concretes [2].

To derive a simple rational approach for calculations of the internal forces of a section, the ACI Code adopted an equivalent rectangular concrete stress distribution, which was first proposed by C. S. Whitney and checked by Mattock and others [3]. A concrete stress of 0.85 f_c' is assumed to be uniformly distributed over an equivalent compression zone bounded by the edges of the cross-section and a line parallel to the neutral axis at a distance $a = \beta_1 c$ from the fiber of maximum compressive strain, where c is the distance between the top of the compressive section and the neutral axis (Fig. 3.8). The fraction β_1 is 0.85 for concrete strengths $f_c' \leq 4000$ psi (27.6 MPa) and is reduced linearly at a rate of 0.05 for each 1000 psi (6.9 MPa) of stress greater than 4000 psi (Fig. 3.9), with a minimum value of 0.65.

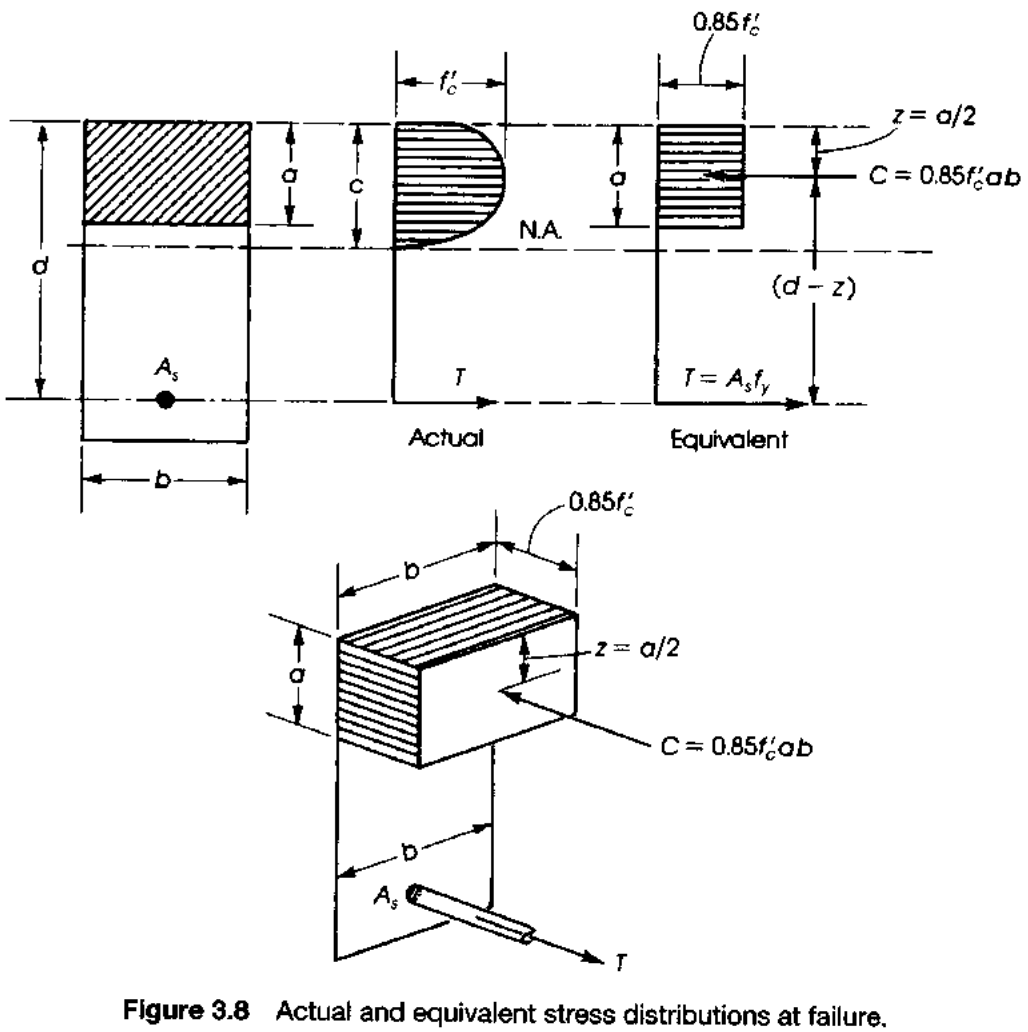


Figure 3.8 Actual and equivalent stress distributions at failure.

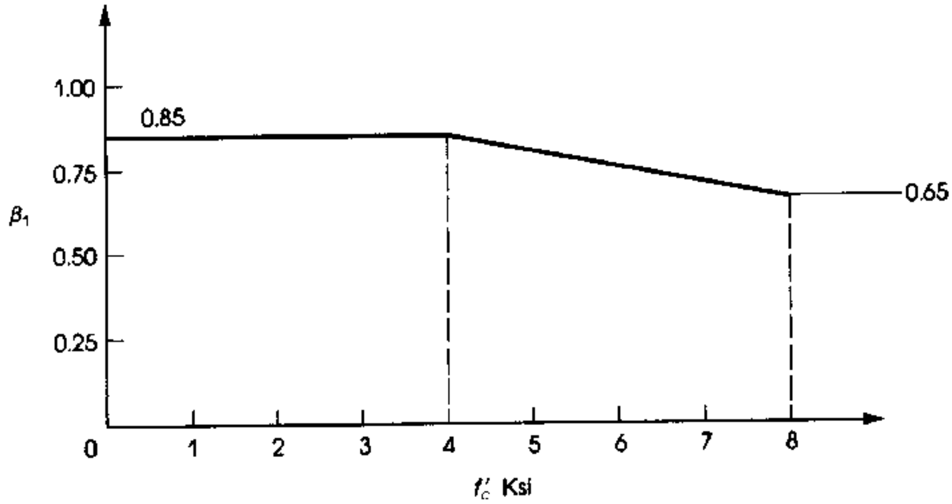


Figure 3.9 Values of β_1 for different compressive strengths of concrete, f'_c.

The preceding discussion applies in general to any section, and it is not confined to a rectangular shape. In the rectangular section, the area of the compressive zone is equal to ba, and every unit area is acted on by a uniform stress equal to $0.85f'_c$, giving a total stress volume equal to $0.85f'_c ab$, which corresponds to the compressive force, C. For any other shape, the force C is equal to the area of the compressive zone multiplied by a constant stress equal to $0.85f'_c$.

For example, in the section shown in Fig. 3.10, the force C is equal to the shaded area of the cross-section multiplied by $0.85f'_c$:

$$C = 0.85f'_c(6 \times 3 + 10 \times 2) = 32.3f'_c \text{ lb}$$

The position of the force C is at a distance z from the top fibers, at the position of the resultant force of all small-element forces of the section. As in the case when the stress is uniform and equals $0.85f'_c$, the resultant force C is located at the center of gravity of the compressive zone, which has a depth of a.

In this example, z is calculated by taking moments about the top fibers:

$$z = \frac{\left(6 \times 3 \times \frac{3}{2}\right) + 10 \times 2(1+3)}{6 \times 3 + 10 \times 2} = \frac{107}{38} = 2.82 \text{ in.}$$

3.9 SINGLY REINFORCED RECTANGULAR SECTION IN BENDING

We explained previously that a balanced condition is achieved when steel yields at the same time as the concrete fails, and that failure usually happens suddenly. This implies that the yield strain in the steel is reached ($\varepsilon_y = f_y/E_s$) and that the concrete has reached its maximum strain of 0.003. The percentage of reinforcement used to produce a balanced condition is called the *balanced steel ratio*, ρ_b. This value is equal to the area of steel, A_s, divided by the effective cross-section, bd:

$$\rho_b = \frac{A_s(\text{balanced})}{bd}$$

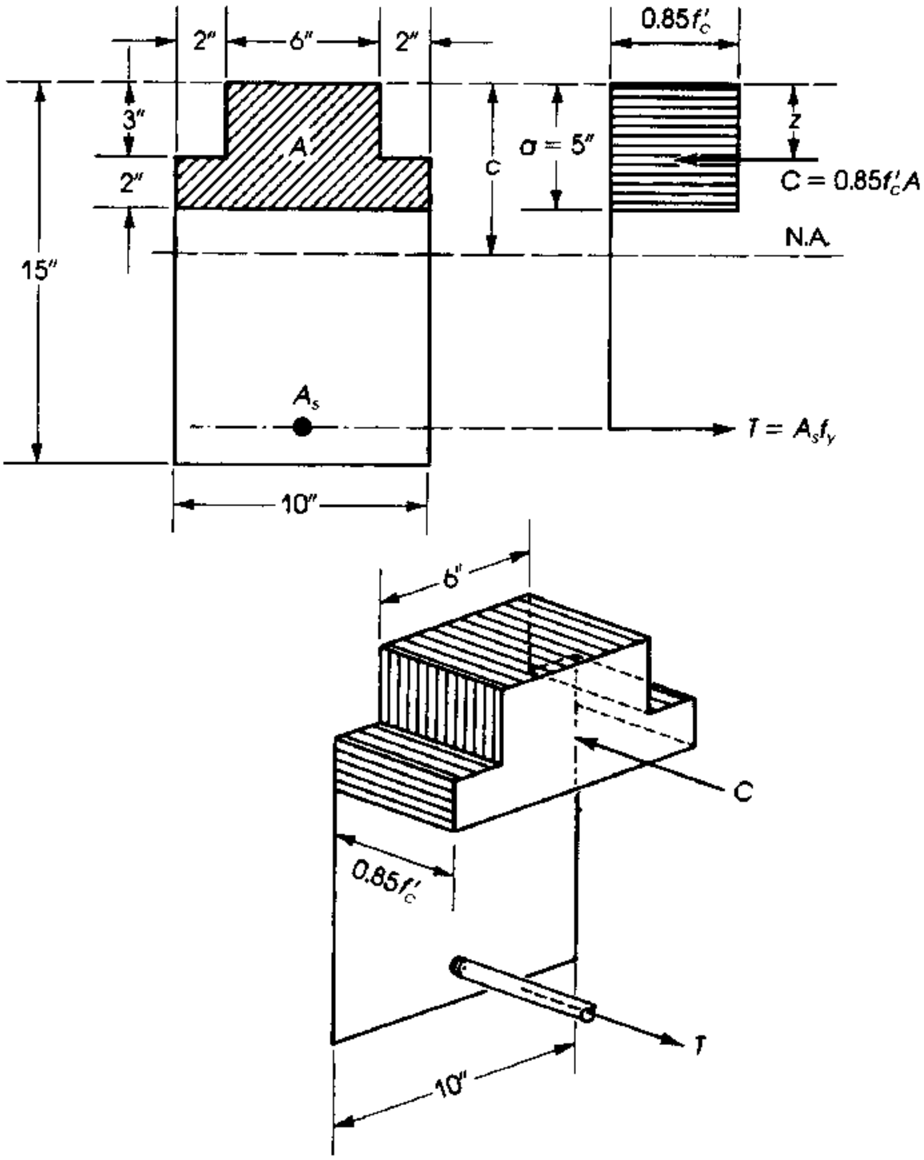


Figure 3.10 Ultimate forces in a nonrectangular section.

where

b = width of the compression face of the member
d = distance from the extreme compression fiber to the centroid of the longitudinal tension reinforcement

Two basic equations for the analysis and design of structural members are the two equations of equilibrium that are valid for any load and any section:

1. The compression force should be equal to the tension force; otherwise, a section will have linear displacement plus rotation:

$$C = T \tag{3.12}$$

2. The internal nominal bending moment, M_n, is equal to either the compressive force, C, multiplied by its arm or the tension force, T, multiplied by the same arm:

$$M_n = C(d - z) = T(d - z)$$

$$(M_u = \phi M_n \text{ after reduction by the factor } \phi) \tag{3.13}$$

The use of these equations can be explained by considering the case of a rectangular section with tension reinforcement (Fig. 3.8). The section may be balanced, under-reinforced, or over-reinforced, depending on the percentage of steel reinforcement used.

3.9.1 The Balanced Section

Let us consider the case of a balanced section, which implies that at ultimate load the strain in concrete equals 0.003 and that of steel equals the first yield stress at distance d_t divided by the modulus of elasticity of steel, f_y/E_s. This case is explained by the following steps.

Step 1. From the strain diagram of Fig. 3.11,

$$\frac{c_b}{d - c_b} = \frac{0.003}{f_y/E_s}$$

From triangular relationships (where is c_b is c for a balanced section) and by adding the numerator to the denominator,

$$\frac{c_b}{d} = \frac{0.003}{0.003 + f_y/E_s}$$

Substituting $E_s = 29 \times 10^3$ ksi,

$$c_b = \left(\frac{87}{87 + f_y}\right) d \tag{3.14}$$

where f_y is in ksi.

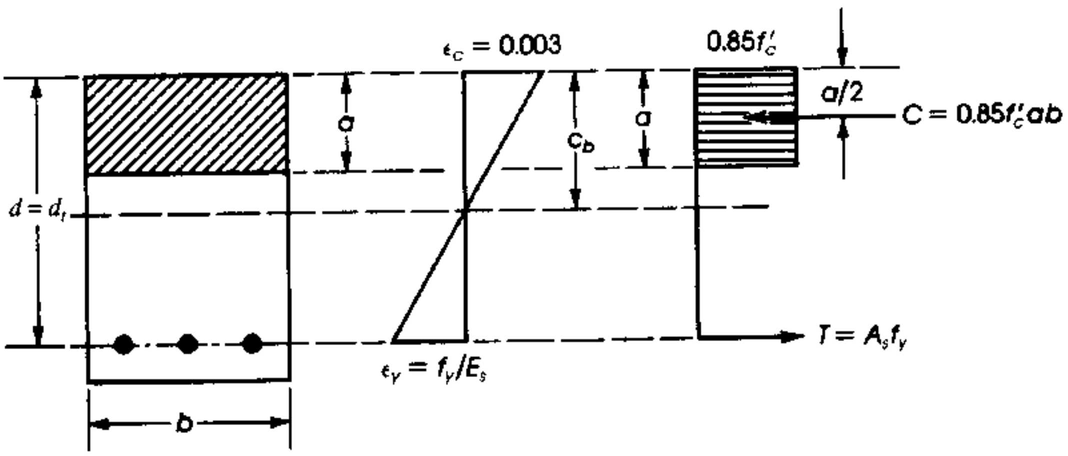


Figure 3.11 Rectangular balanced section.

Step 2. From the equilibrium equation,

$$C = T$$

$$0.85 f'_c ab = A_s f_y \tag{3.15}$$

$$a = \frac{A_s f_y}{0.85 f'_c b} \tag{3.16}$$

Here a is the depth of the compressive block, equal to $\beta_1 c$, where $\beta_1 = 0.85$ for $f'_c \leq 4000$ psi (27.6 MPa) and decreases linearly by 0.05 per 1000 psi (6.9 MPa) for higher concrete strengths (Fig. 3.9). Because the balanced steel reinforcement ratio is used,

$$\rho_b = \frac{A_s(\text{balanced})}{bd} = \frac{A_{sb}}{bd} \tag{3.17}$$

and substituting the value of A_{sb} in Eq. 3.15,

$$0.85 f'_c ab = f_y \rho_b bd$$

Therefore,

$$\rho_b = \frac{0.85 f'_c}{f_y d} a = \frac{0.85 f'_c}{f_y d}(\beta_1 c_b)$$

Substituting the value of c_b from Eq. 3.14, the general equation of the balanced steel ratio becomes

$$\rho_b = 0.85 \beta_1 \frac{f'_c}{f_y} \left(\frac{87}{87 + f_y} \right) \tag{3.18}$$

Step 3. The internal nominal moment, M_n, is calculated by multiplying either C or T by the distance between them:

$$M_n = C(d - z) = T(d - z) \tag{3.13}$$

For a rectangular section, the distance $z = a/2$ as the line of application of the force C lies at the center of gravity of the area ab, where

$$a = \frac{A_s f_y}{0.85 f'_c b}$$

$$M_n = C\left(d - \frac{a}{2}\right) = T\left(d - \frac{a}{2}\right)$$

For a balanced or an under-reinforced section, $T = A_s f_y$. Then

$$M_n = A_s f_y \left(d - \frac{a}{2}\right) \tag{3.19}$$

To get the usable design moment ϕM_n, the previously calculated M_n must be reduced by the capacity reduction factor, ϕ,

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b}\right) \tag{3.19a}$$

Equation 3.19a can be written in terms of the steel percentage ρ:

$$\rho = \frac{A_s}{bd} \quad A_s = \rho bd$$

$$\phi M_n = \phi f_y \rho bd \left(d - \frac{\rho bd f_y}{1.7 f_c' b}\right) = \phi \rho f_y bd^2 \left(1 - \frac{\rho f_y}{1.7 f_c'}\right) \tag{3.20}$$

Equation 3.20 can be written as

$$\phi M_n = R_u bd^2 \tag{3.21}$$

where

$$R_u = \phi \rho f_y \left(1 - \frac{\rho f_y}{1.7 f_c'}\right) \tag{3.22}$$

The ratio of the equivalent compressive stress block depth, a, to the effective depth of the section, d, can be found from Eq. 3.15:

$$0.85 f_c' ab = \rho bd f_y \tag{3.23}$$

$$\frac{a}{d} = \frac{\rho f_y}{0.85 f_c'}$$

3.9.2 Upper Limit of Steel Percentage

The upper limit or the maximum steel percentage, ρ_{max}, that can be used in a singly reinforced concrete section in bending is based on the net tensile strain in the tension steel, the balanced steel ratio, and the grade of steel used. The relationship between the steel percentage, ρ, in the section, and the net tensile strain, ε_t, is as follows:

$$\varepsilon_t = \left(\frac{0.003 + f_y/E_s}{\rho/\rho_b}\right) - 0.003 \tag{3.24}$$

For $f_y = 60$ ksi, and assuming $f_y/E_s = 0.002$,

$$\varepsilon_t = \left(\frac{0.005}{\rho/\rho_b}\right) - 0.003 \tag{3.25}$$

These expressions are obtained by referring to Fig. 3.12. For a balanced section,

$$c_b = \frac{a_b}{\beta_1} = \frac{A_{sb} f_y}{0.85 f_c' b \beta_1} = \frac{\rho_b f_y d}{0.85 f_c' \beta_1}$$

Similarly, for any steel ratio, ρ,

$$c = \frac{\rho f_y d}{0.85 f_c' \beta_1} \quad \text{and} \quad \frac{c}{c_b} = \frac{\rho}{\rho_b}$$

Divide both sides by d to get

$$\frac{c}{d} = \left(\frac{\rho}{\rho_b}\right)\left(\frac{c_b}{d}\right) \tag{3.26}$$

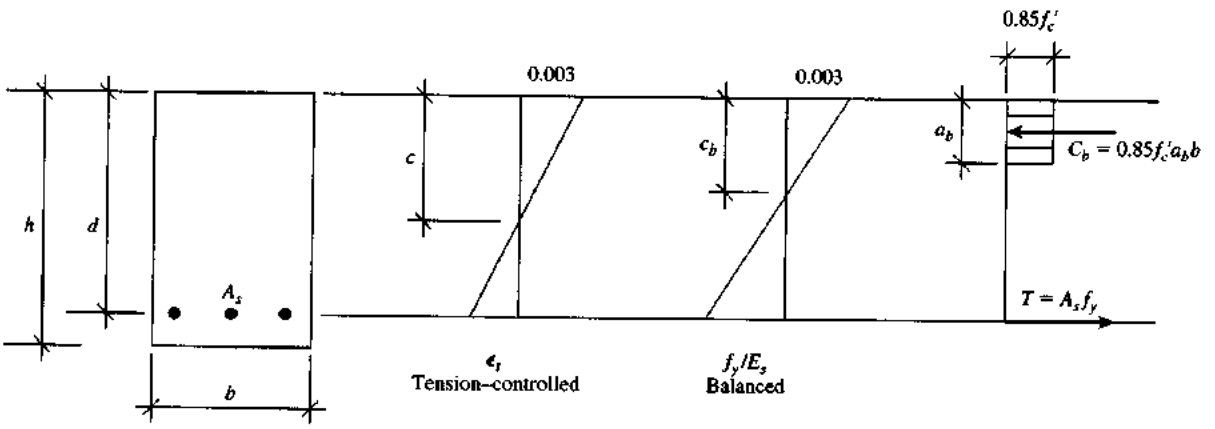


Figure 3.12 Strains in tension-controlled and balanced conditions.

From the triangles of the strain diagrams,

$$\frac{c}{d} = \frac{0.003}{0.003 + \varepsilon_t}$$
$$\varepsilon_t = \frac{0.003}{(c/d)} - 0.003 \tag{3.27}$$

Similarly,

$$\frac{c_b}{d} = \frac{0.003}{0.003 + f_y/E_s} \tag{3.28}$$

From Eqs. 3.26 and 3.28,

$$\frac{c}{d} = \left(\frac{\rho}{\rho_b}\right)\left(\frac{c_b}{d}\right) = \left(\frac{\rho}{\rho_b}\right)\left(\frac{0.003}{0.003 + f_y/E_s}\right)$$

Substitute this value in Eq. 3.27 to get

$$\varepsilon_t = \frac{0.003}{(c/d)} - 0.003 = \left[\frac{0.003 + f_y/E_s}{\rho/\rho_b}\right] - 0.003 \tag{3.24}$$

For grade 60 steel, $f_y = 60$ ksi, $E_s = 29{,}000$ ksi, and $f_y/E_s = 0.00207$, then

$$\varepsilon_t = \left(\frac{0.00507}{\rho/\rho_b}\right) - 0.003 \tag{3.25}$$

To determine the upper limit or the maximum steel percentage, ρ, in a singly reinforced concrete section, refer to Fig. 3.6. It can be seen that concrete sections subjected to flexure or axial load and bending moment may lie in compression-controlled, transition, or tension-controlled zones. When $\varepsilon_t \leq 0.002$ (or $c/d_t \geq 0.6$), compression controls, whereas when $\varepsilon_t \geq 0.005$ (or $c/d_t \leq 0.375$), tension controls. The transition zone occurs when $0.002 < \varepsilon_t < 0.005$ or $0.6 > c/d_t > 0.375$.

For members subjected to flexure, the relationship between the steel ratio, ρ, was given in Eq. 3.24:

$$\varepsilon_t + 0.003 = \frac{0.003 + f_y/E_s}{\rho/\rho_b} \tag{3.24}$$

or

$$\frac{\rho}{\rho_b} = \frac{0.003 + f_y/E_s}{0.003 + \varepsilon_t} \tag{3.29}$$

For $f_y = 60$ ksi and $E_s = 29{,}000$ ksi, f_y/E_s may be assumed to be 0.00207.

$$\frac{\rho}{\rho_b} = \frac{0.00507}{0.003 + \varepsilon_t} \tag{3.30}$$

The limit for tension to control is $\varepsilon_t \geq 0.005$. For $\varepsilon_t = 0.005$, Eq. 3.30 becomes

$$\frac{\rho}{\rho_b} = \frac{0.005}{0.008} = \frac{5}{8} = 0.625 \tag{3.30a}$$

or $\rho \leq 0.63375\rho_b$ for tension-controlled sections if $\varepsilon_t = 0.00507 = f_y/E_s$. Both values can be used for practical analysis and design. The small increase in ρ will slightly increase the moment capacity of the section. For example, if $f'_c = 4$ ksi and $f_y = 60$ ksi, $\rho_b = 0.0285$ and $\rho \leq 0.01806$ for tension to control (as in the case of flexural members). The ϕ factor in this case is 0.9. This value is less than $\rho_{\max} = 0.75\rho_b = 0.0214$ allowed by the ACI Code for flexural members when $\phi = 0.9$ can be used.

Design of beams and other flexural members can be simplified using the limit of $\varepsilon_t = 0.005$.

$$\frac{\rho}{\rho_b} = \frac{0.003 + f_y/E_s}{0.008} \tag{3.31}$$

In this case, $\rho = \rho_{\max}$ = upper limit for tension-controlled sections.

$$\rho_{\max} = \left(\frac{0.003 + f_y/E_s}{0.008}\right)\rho_b \tag{3.31a}$$

Note that when ρ used $\leq \rho_{\max}$, tension controls and $\phi = 0.9$. When $\rho > \rho_{\max}$, section will be in the transition region with $\phi < 0.9$.
And for $f_y = 60$ ksi and $f_y/E_s = 0.00207$,

$$\frac{\rho_{\max}}{\rho_b} = 0.63375 \tag{3.32}$$

This steel ratio will provide adequate ductility before beam failure.

Similarly,

$$\text{for } f_y = 40 \text{ ksi}, \quad \rho_{\max} = 0.5474\rho_b \tag{3.32a}$$

$$\text{for } f_y = 50 \text{ ksi}, \quad \rho_{\max} = 0.5905\rho_b \tag{3.32b}$$

$$\text{for } f_y = 75 \text{ ksi}, \quad \rho_{\max} = 0.6983\rho_b \tag{3.32c}$$

It was established that $\phi M_n = R_u bd^2$ (Eq. 3.21), where $R_u = \phi\rho f_y(1 - \rho f_y/1.7f'_c)$ (Eq. 3.22). Once f'_c and f_y are known, then ρ_b, ρ, R_u, and bd^2 can be calculated. For example, for $f'_c = 4$ ksi, $f_y = 60$ ksi, $\phi = 0.9$, $\varepsilon_t = 0.005$, and one row of bars in the section,

$$\rho_b = 0.0285 \quad \rho = 0.01806 \quad R_u = 820 \text{ psi}$$

Note that for one row of bars in the section, it can be assumed that $d = d_t = h - 2.5$ in., whereas for two rows of bars, $d = h - 3.5$ in., and $d_t = h - 2.5$ in. $= d + 1.0$ in. (Refer to Figs. 3.4 and 3.5 and Section 4.3.3.)

Table 3.2 Values of ρ_{max} and $R_u = M_u/bd^2$ for Flexural Tension-Controlled Sections with One Row of Bars, $\varepsilon_t = 0.005$

f'_c (ksi)	f_y (ksi)	ρ_b	$\rho_{max} = 0.63375\,\rho_b$	R_u (psi) (Eq. 3.22)
3	60	0.0214	0.01356	615
4	60	0.0285	0.01806	820
5	60	0.0335	0.02123	975
6	60	0.0377	0.02389	1109

Table 3.2 gives the values of ρ, ρ_b, and $R_u = M_u/bd^2$ for flexural tension-controlled sections with one row of bars.

For reinforced concrete flexural members with $\rho > \rho_{max}$, ε_t will be less than 0.005. Section 10.3 of the ACI Code specifies that ε_t should not be less than 0.004 in the transition region to maintain adequate ductility and warning before failure.

For this limitation of $\varepsilon_t = 0.004$, the general equation (3.29) becomes

$$\frac{\rho}{\rho_b} = \frac{0.003 + f_y/E_s}{0.007} \tag{3.33}$$

For $f_y = 60$ ksi,

$$\frac{\rho}{\rho_b} = \frac{0.003 + 0.00207}{0.007} = 0.724 \tag{3.34}$$

and the limit in the transition region is

$$\rho_{max\,t} = 0.724\,\rho_b \tag{3.34a}$$

Note that the t here refers to the transition region. In this case, limit of ϕ is

$$\phi_t = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.817 < 0.9 \tag{3.35}$$

For $f_y = 60$ ksi and $f'_c = 4$ ksi, $\rho_b = 0.0285$, $\rho_{max\,t} = 0.02063$, $R_n = 1012$ psi (from Eq.3.22, and $R_u = \phi R_n = 0.817(1012) = 826$ psi.

This steel ratio in Eq. 3.33 is the upper limit ($\rho_{max\,t}$) for a singly reinforced concrete section in the transition region with $\phi < 0.9$.

It can be noticed that the aforementioned $R_u = 826$ psi calculated for $\varepsilon_t = 0.004$, is very close to $R_u = 820$ psi for $\rho_{max} = 0.63375\rho_b$ and $\phi = 0.9$. Therefore, adding reinforcement beyond ρ_{max} (for $\varepsilon_t = 0.005$, Table 3.2) reduces ϕ because of the reduced ductility resulting in little or nonsubstantial gain in design strength. Adding compression reinforcement in the section is a better solution to increase the design moment, keeping the section in the tension-controlled region with $\phi = 0.9$. (Refer to Section 3.14.)

Table 3.3 gives the values of ρ_t(limit), ρ_b, and R_u for flexural members in the transition region for $f_y = 60$ ksi and $\varepsilon_t = 0.004$ and one row of bars. In this case $\phi = 0.817$ (Eq. 3.35) and $\rho/\rho_b = 0.724$. It is clear that for $f_y = 60$ ksi, the design R_u in both cases, when $\varepsilon_t = 0.005$ with $\phi = 0.9$ and when $\varepsilon_{max} = 0.004$ with $\phi = 0.816$, are quite close.

Table 3.3 Values of ρ_t and R_u for Sections in the Transition Region with $\varepsilon_t = 0.004$, $f_y = 60$ ksi, and One Row of Bars ($\phi = 0.817$)

f'_c (ksi)	ρ_b	ρ_t (limit)	R_u (psi)
3	0.0214	0.0155	617
4	0.0285	0.0206	822
5	0.0335	0.0243	980
6	0.0377	0.0273	1116

Example 3.1

For the section shown in Fig. 3.13, calculate

a. The balanced steel reinforcement
b. The maximum reinforcement area allowed by the ACI Code for tension-controlled section and in the transition region
c. The position of the neutral axis and the depth of the equivalent compressive stress block for the tension-controlled section in b.

Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

a. $\rho_b = 0.85\,\beta_1 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right)$

Because $f'_c = 4000$ psi, $\beta_1 = 0.85$:

$$\rho_b = (0.85)^2\left(\frac{4}{60}\right)\left(\frac{87}{87+60}\right) = 0.0285$$

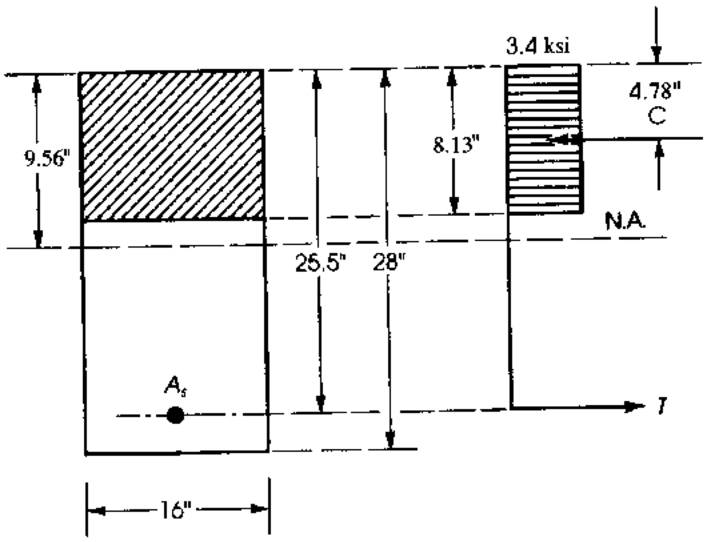


Figure 3.13 Example 3.1.

The area of steel reinforcement to provide a balanced condition is

$$A_{sb} = \rho_b bd = 0.0285 \times 16 \times 25.5 = 11.63 \text{ in.}^2$$

b. For a tension-controlled section, $\rho_{max} = 0.63375\ \rho_b = 0.63375 \times 0.0285 = 0.01806$ or, from Eq. 3.32,

$$A_{s\,max} = \rho_{max} bd = 0.01806 \times 16 \times 25.5 = 7.37 \text{ in.}^2 \text{ for } \phi = 0.9.$$

For the transition region, $\rho_{max\,t} = 0.724\ \rho_b = 0.0206$. For the case of $\varepsilon_t = 0.004$, $A_{s\,max\,t} = 0.0206(16 \times 25.5) = 8.41 \text{ in.}^2$ for $\phi = 0.817$

c. The depth of the equivalent compressive block using $A_{s\,max}$ is

$$a_{max} = \frac{A_{s\,max} f_y}{0.85 f'_c b} = \frac{7.37 \times 60}{0.85 \times 4 \times 16} = 8.13 \text{ in.}$$

The distance from the top fibers to the neutral axis is $c = \alpha/\beta_1$. Because $f'_c = 4000$ psi, $\beta_1 = 0.85$; thus,

$$c = \frac{8.13}{0.85} = 9.56 \text{ in.}$$

or $c/d = 0.375$ and $c = 0.375(25.5) = 9.56$ in.

Example 3.2

Determine the design moment strength and the position of the neutral axis of the rectangular section shown in Fig. 3.14 if the reinforcement used is three no. 9 bars. Given: $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. The area of three no. 9 bars is 3.0 in.^2

$$\rho = \frac{A_s}{bd} = \frac{3.0}{21 \times 12} = 0.0119$$

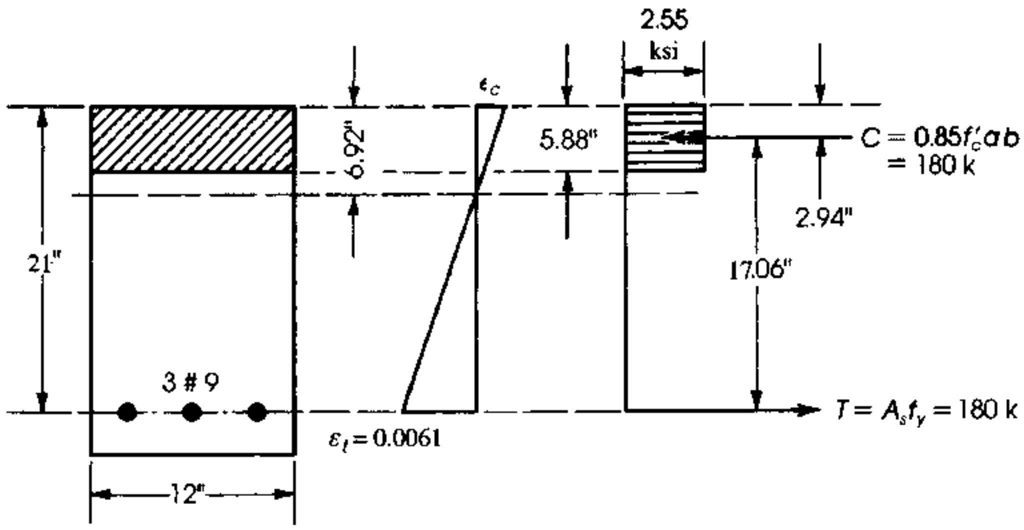


Figure 3.14 Example 3.2.

2. $\rho_{max} = 0.01356 > \rho$, tension-controlled section, $\phi = 0.9$ or check ε_t:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{3(60)}{0.85 \times 3 \times 12} = 5.88 \text{ in.}$$

$$c = \frac{a}{0.85} = 6.92 \text{ in.}$$

$$d_t = d = 21 \text{ in.}$$

$$\varepsilon_t = \left(\frac{21 - 6.92}{6.92}\right) 0.003$$

$$= 0.0061 > 0.005, \quad \phi = 0.9$$

$$\text{or } \frac{c}{d_t} = 0.33 < 0.375 \quad \text{(o.k.)}$$

3. $\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{3.0 \times 60}{0.85 \times 3 \times 12} = 5.88 \text{ in.}$$

$$\phi M_n = 0.9 \times 3.0 \times 60 \left(21 - \frac{5.88}{2}\right) = 2926 \text{ K·in.} = 243.8 \text{ K·ft}$$

Discussion

In this example, the section is tension-controlled, which implies that the steel will yield before the concrete reaches its ultimate strain. A simple check can be made from the strain diagram (Fig. 3.14). From similar triangles,

$$\frac{\varepsilon_c}{\varepsilon_y} = \frac{c}{(d - c)} \text{ and } \varepsilon_y = \frac{f_y}{E_s} = \frac{60}{29000} = 0.00207$$

$$\varepsilon_c = \frac{6.92}{(21 - 6.92)} \times 0.00207 = 0.00102$$

which is much less than 0.003. Therefore, steel yields before concrete reaches its limiting strain of 0.003.

Example 3.3

Repeat Example 3.2 using three no. 10 bars as the tension steel (Fig. 3.15).

Solution

1. Check ε_t:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{3.81(60)}{0.85 \times 3 \times 12} = 7.47 \text{ in.}$$

$$c = \frac{a}{0.85} = 8.79 \text{ in.} \quad d_t = d = 21 \text{ in.} \quad \frac{c}{d_t} = 0.419 > 0.375$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = \left(\frac{21 - 8.79}{8.79}\right) 0.003 = 0.004168$$

This value is less than 0.005 but greater than 0.004 (transition region), $\phi < 0.9$.

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.831$$

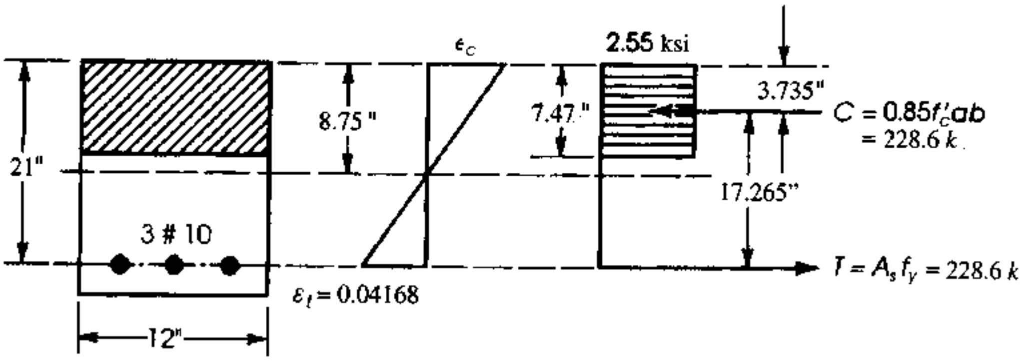


Figure 3.15 Example 3.3.

2. Calculate ϕM_n:

$$\phi M_n = 0.831(3.81)(60)\left[21 - \frac{7.47}{2}\right] = 3278\ \text{K·in.} = 273\ \text{K·ft}$$

Discussion

For a tension-controlled section, $\varepsilon_t = 0.005$ and $\rho = 0.63375\ \rho_b = 0.01356$ (Table 3.2), $\phi = 0.9$.

$$A_s \max = 0.01356(12 \times 21) = 3.417\ \text{in.}^2 < 3.81\ \text{in.}^2$$

$$a = \frac{3.417 \times 60}{0.85 \times 3 \times 12} = 6.7\ \text{in.}$$

$$\phi M_n = 0.9 \times 3.417 \times 60\left(21 - \frac{6.7}{2}\right) = 271.4\ \text{K·ft}$$

which is close to the above ϕM_n. This is a somewhat conservative approach.

3.10 LOWER LIMIT OR MINIMUM PERCENTAGE OF STEEL

If the factored moment applied on a beam is very small and the dimensions of the section are specified (as is sometimes required architecturally) and are larger than needed to resist the factored moment, the calculation may show that very small or no steel reinforcement is required. In this case, the maximum tensile stress due to bending moment may be equal to or less than the modulus of rupture of concrete: $f_r = \lambda 7.5\sqrt{f_c'}$. If no reinforcement is provided, sudden failure will be expected when the first crack occurs, thus giving no warning. The ACI Code, 10.5, specifies a minimum steel area, A_s,

$$A_{s\,\min} = \left(\frac{3\sqrt{f_c'}}{f_y}\right) b_w d \geq \left(\frac{200}{f_y}\right) b_w d$$

or the minimum steel ratio, $\rho_{\min} = (3\sqrt{f_c'}/f_y) \geq 200/f_y$, where the units of f_c' and f_y are in psi. This ρ ratio represents the lower limit. The first term of the preceding equation was specified to accommodate a concrete strength higher than 5 ksi. The two minimum ratios are equal when $f'_c = 4440$ psi. This indicates that

$$\rho_{min} = \frac{200}{f_y} \text{ when } f'_c < 4500 \text{ psi}$$

$$\rho_{min} = \frac{3\sqrt{f'_c}}{f_y} \text{ when } f'_c \geq 4500 \text{ psi}$$

For example, if $f_y = 60$ ksi, $\rho_{min} = 0.00333$ when $f'_c < 4500$ psi, whereas $\rho_{min} = 0.00353$ when $f'_c = 5000$ psi and 0.00387 when $f'_c = 6000$ psi.

In the case of a rectangular section, use $b = b_w$ in the preceding expressions. For statically determinate T-sections with the flange in tension, as in the case of cantilever beams, the value of $A_{s\,min}$ should be equal to or greater than the *smaller* of (a) and (b):

(a) $$A_{s\,min} = \left(\frac{6\sqrt{f'_c}}{f_y}\right) b_w d$$

(b) $$A_{s\,min} = \left(\frac{3\sqrt{f'_c}}{f_y}\right) b_w d \geq \left(\frac{200}{f_y}\right) b_w d$$

where b_w and b are the width of the beam web and flange, respectively, and f'_c and f_y are in psi. For example, if $b = 48$ in., $b_w = 16$ in., $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi, than $A_{s\,min} = 2.02$ in.^{2} in (a) controls, which is smaller than the value of $A_{s\,min}$ in (b) (3.2 in.^{2}).

3.11 ADEQUACY OF SECTIONS

A given section is said to be *adequate* if the internal moment strength of the section is equal to or greater than the externally applied factored moment, M_u, or $\phi M_n \geq M_u$. The procedure can be summarized as follows:

1. Calculate the external applied factored moment, M_u.

$$M_u = 1.2M_D + 1.6M_L$$

2. Calculate ϕM_n for the basic singly reinforced section:
   a. Check that $\rho_{min} \leq \rho \leq \rho_{max}$.
   b. Calculate $a = A_s f_y/(0.85 f'_c b)$ and check ε_t for ϕ.
   c. Calculate $\phi M_n = \phi\, A_s f_y(d - a/2)$.
3. If $\phi M_n \geq M_u$, then the section is adequate; Fig. 3.16 shows a typical tension-controlled section.

Example 3.4

An 8-ft-span cantilever beam has a rectangular section and reinforcement as shown in Fig. 3.17. The beam carries a dead load, including its own weight, of 1.5 K/ft and a live load of 0.9 K/ft. Using $f'_c = 4$ ksi and $f_y = 60$ ksi, check if the beam is safe to carry the above loads.

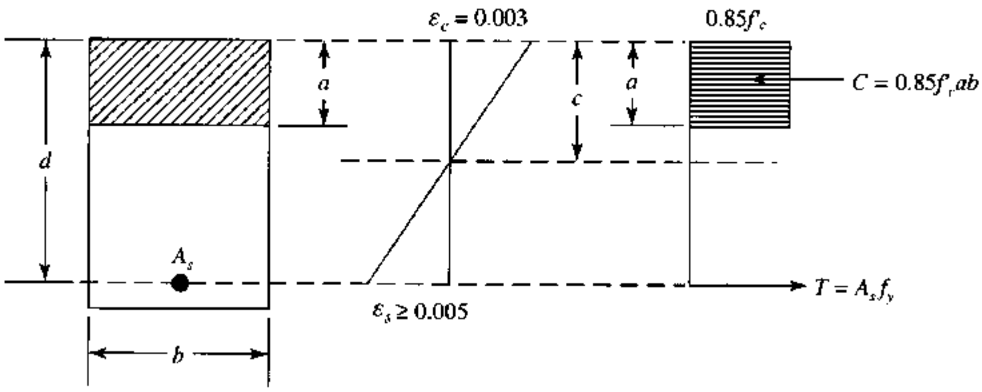


Figure 3.16 Tension-controlled rectangular section.

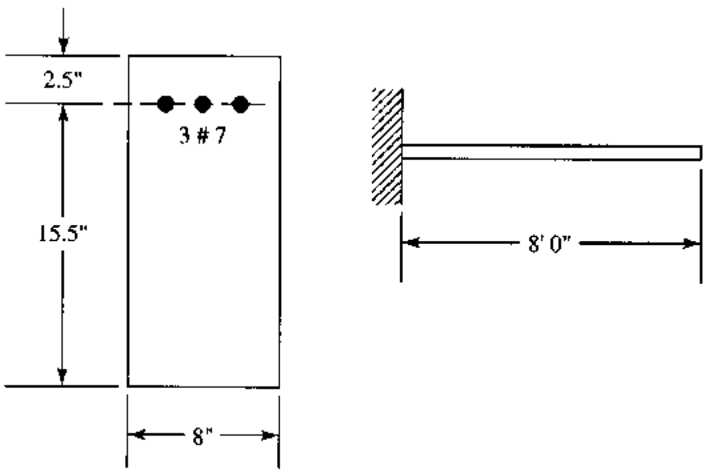


Figure 3.17 Example 3.4.

Solution

1. Calculate the external factored moment:

$$W_u = 1.2D + 1.6L = 1.2(1.5) + 1.6(0.9) = 3.24 \text{ K/ft}$$

$$M_u = W_u \frac{L^2}{2} = 3.24 \frac{8^2}{2} = 103.68 \text{ K·ft} = 1244 \text{ K·in.}$$

2. Check ε_t:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{1.8 \times 60}{0.85 \times 4 \times 8} = 3.97 \text{ in.}$$

$$c = \frac{a}{0.85} = 4.67 \text{ in.} \quad d_t = d = 15.5 \text{ in.} \quad \frac{c}{d_t} = 0.3 < 0.375$$

Also,

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = \left(\frac{15.5 - 4.67}{4.67}\right) 0.003 = 0.007 > 0.005, \quad \phi = 0.9$$

or check

$$\rho = \frac{A_s}{bd} = \frac{1.8}{8 \times 15.5} = 0.0145 < \rho_{max} = 0.01806$$

(from Table 3.2). Therefore, it is a tension-controlled section and $\phi = 0.9$.

3. Calculate ϕM_n:

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$= 0.9(1.8)(60)\left(15.5 - \frac{3.97}{2}\right) = 1312 \text{ K-in.} > M_u$$

Then section is adequate.

Example 3.5

A simply supported beam has a span of 20 ft. If the cross section of the beam is as shown in Fig. 3.18, $f'_c = 3$ ksi, and $f_y = 60$ ksi, determine the allowable uniformly distributed service live load on the beam assuming the dead load is that due to beam weight. Given: $b = 12$ in., $d = 17$ in., total depth $h = 20$ in., and reinforced with three no. 8 bars ($A_s = 2.37$ in.^{2}).

Solution

1. Determine the design moment strength:

$$\rho = \frac{A_s}{bd} = \frac{3 \times 0.79}{12 \times 17} = 0.0116$$

$$\rho_{max} = 0.01356 \text{ (Table 3.2)}$$

$$\rho < \rho_{max}$$

Therefore it is a tension-controlled section and $\phi = 0.9$

Also, $\rho > \rho_{min} = \dfrac{200}{f_y} = 0.00333$.

2. $$\phi M_n = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b}\right)$$

$$= 0.9 \times 2.37 \times 60\left(17 - \frac{2.37 \times 60}{1.7 \times 3 \times 12}\right) = 1878 \text{ K-in.} = 156.5 \text{ K-ft}$$

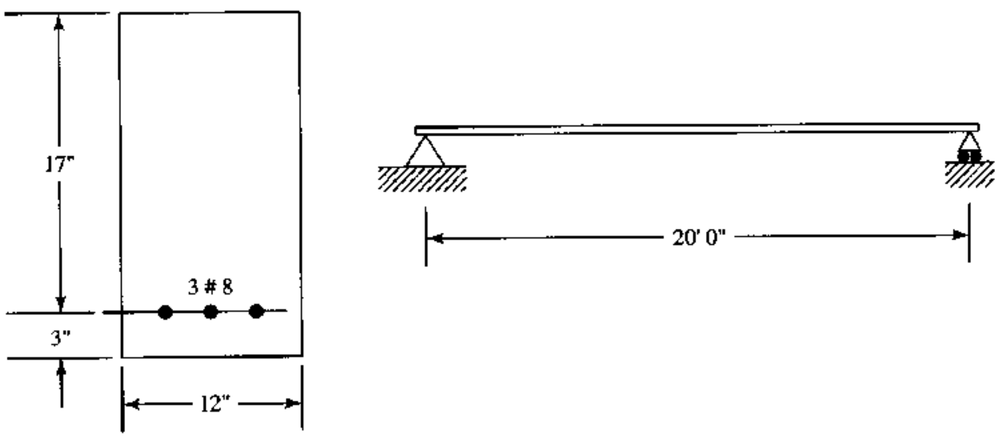


Figure 3.18 Example 3.5.

3. The dead load acting on the beam is self-weight (assumed):

$$w_D = \frac{12 \times 20}{144} \times 150 = 250 \text{ lb/ft} = 0.25 \text{ K/ft}$$

where 150 is the weight of reinforced concrete in pcf.

4. The external factored moment is

$$M_u = 1.2M_D + 1.6M_L$$

$$= 1.2\left(\frac{0.25}{8} \times 20^2\right) + 1.6\left(\frac{w_L}{8} \times 20^2\right) = 15.0 + 80w_L$$

where w_L = uniform service live load on the beam in K/ft.

5. Internal design moment equals the external factored moment:

$$156.5 = 15.0 + 80w_L \quad \text{and} \quad w_L = 1.77 \text{ K/ft}$$

The allowable uniform service live load on the beam is 1.77 K/ft.

Example 3.6: Minimum Steel Reinforcement

Check the design adequacy of the section shown in Fig. 3.19 to resist a factored moment $M_u = 30$ K-ft, using $f'_c = 3$ ksi and $f_y = 40$ ksi.

Solution

1. Check ρ provided in the section:

$$\rho = \frac{A_s}{bd} = \frac{3 \times 0.2}{10 \times 18} = 0.00333$$

2. Check $\rho_{\min}$ required according to the ACI Code:

$$\rho_{\min} = \frac{200}{f_y} = 0.005 > \rho = 0.00333$$

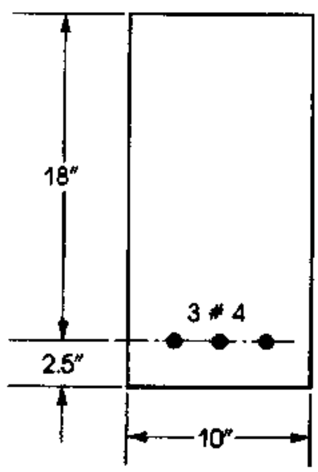


Figure 3.19 Example 3.6.

Therefore, use $\rho = \rho_{\min} = 0.005$.

$$A_{s\,\min} = \rho_{\min} bd = 0.005 \times 10 \times 18 = 0.90 \text{ in.}^2$$

Use three no. 5 bars ($A_s = 0.91$ in.^{2}), because three no. 4 bars are less than the minimum specified by the code.

3. Check moment strength: $\phi M_n = \phi A_s f_y (d - a/2)$.

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{0.91 \times 40}{0.85 \times 3 \times 10} = 1.43 \text{ in.}$$

$$\phi M_n = 0.9 \times 0.91 \times 40 \left(18 - \frac{1.43}{2}\right) = 566 \text{ K·in.} = 47.2 \text{ K·ft}$$

4. An alternative solution, according to the ACI Code, Section 10.5, for three no. 4 bars, $A_s = 0.6$ in.^{2} is

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{0.6 \times 40}{0.85 \times 3 \times 10} = 0.94 \text{ in.}$$

$$\phi M_n = \frac{0.9}{12} \times 0.6 \times 40 \left(18 - \frac{0.94}{2}\right) = 31.55 \text{ K·ft}$$

$$A_s \text{ required for 30 K·ft} = \frac{30}{31.55} \times 0.6 = 0.57 \text{ in.}^2$$

The minimum A_s required according to the ACI Code, Section 10.5, is at least one-third greater than 0.57 in.^{2}:

$$\text{Minimum } A_s \text{ required} = 1.33 \times 0.57 = 0.76 \text{ in.}^2$$

which exceeds the 0.6 in.^{2} provided by the no. 4 bars. Use three no. 5 bars, because $A_s = 0.91$ in.^{2} is greater than the 0.76 in.^{2} required.

3.12 BUNDLED BARS

When the design of a section requires the use of a large amount of steel—for example, when $\rho_{\max}$ is used—it may be difficult to fit all bars within the cross-section. The ACI Code, 7.6, allows the use of parallel bars placed in a bundled form of two, three, or four bars, as shown in Fig. 3.20. Up to four bars (no. 11 or smaller) can be bundled when they are enclosed by stirrups.

The same bundled bars can be used in columns, provided that they are enclosed by ties. All bundled bars may be treated as a single bar for checking the spacing and concrete cover requirements. The single bar diameter shall be derived from the equivalent total area of the bundled bars.

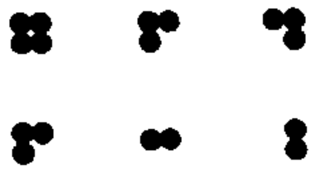


Figure 3.20 Bundled bar arrangement.

Summary: Singly Reinforced Rectangular Section

The procedure for determining the design moment of a singly reinforced rectangular section according to the ACI Code limitations can be summarized as follows:

1. Calculate the steel ratio in the section, $\rho = A_s/bd$.
2. Calculate the balanced and maximum steel ratios, Eqs. 3.18 and 3.31 or Table 3.2, for tension-controlled section. Also, calculate $\rho_{min} = 200/f_y$ when $f'_c < 4500$ psi (f'_c and f_y are in psi units) and $\rho_{min} = 3\sqrt{f'_c}/f_y$ when $f'_c \geq 4500$ psi.
3. If $\rho_{min} \leq \rho \leq \rho_{max}$, then the section meets the ACI Code limitations for tension-controlled section. If $\rho \leq \rho_{min}$, the section is not acceptable (unless a steel ratio $\rho \geq \rho_{min}$ is used). If $\rho \geq \rho_{max}$, $\phi = 0.9$.
4. Calculate $a = \dfrac{A_s f_y}{0.85 f'_c b}$, c, ε_t, and ϕ.
5. Calculate $\phi M_n = \phi A_s f_y \left(d - \dfrac{a}{2}\right)$.

Flow charts representing this section and other sections are given on www.wiley.com/college/hassoun.

3.13 SECTIONS IN THE TRANSITION REGION (ϕ < 0.9)

In the case when the NTS, ε_t in the extreme tension steel lies between the compression-controlled strain limit (0.002 for $f_y = 60$ ksi) and the tension-controlled strain limit of 0.005, the strength reduction factor, ϕ, will be less than 0.9. Consequently, the design moment strength of the section ϕM_n will be smaller than ϕM_n with $\phi = 0.9$ (refer to Fig. 3.6). In the transition region, ε_t should not be less than 0.004 for flexural members (ACI Code, Section 10.3). (See Example 3.8.)

Example 3.7

Determine the design moment strength of a rectangular concrete section reinforced with four no. 9 bars in one row (Fig. 3.21).

Given: $b = 12$ in., $d = 16.5$ in., $h = 19$ in., $f'_c = 4$ ksi, and $f_y = 60$ ksi.

Solution

1. By the ACI Code provisions, for $f'_c = 4$ ksi, $f_y = 60$ ksi, and tension-controlled conditions ($\rho_b = 0.0285$ and $\rho_{max} = 0.01806$), check $\rho = A_s/bd = 4/(12 \times 6) = 0.02083 > \rho_{max}$. This indicates that the section is in the transition region and $\phi < 0.9$.

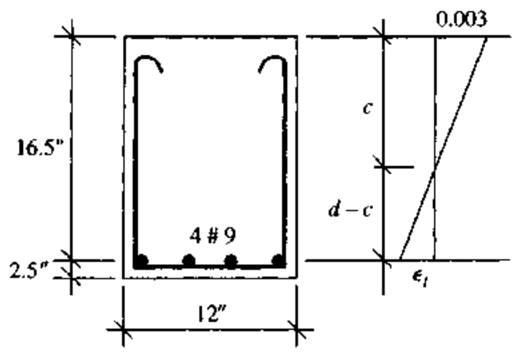


Figure 3.21 Example 3.7 ($d = d_t$).

2. Calculate a, c, and ε_t:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{4 \times 60}{0.85 \times 4 \times 12} = 5.882 \text{ in.}$$

$$c = \frac{a}{0.85} = 6.92 \text{ in.} \quad d_t = d = 16.5 \text{ in.} \quad \frac{c}{d_t} = 0.42 > 0.375$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = \left(\frac{16.5 - 6.92}{6.92}\right) 0.003 = 0.004153 > 0.004$$

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.829$$

3. Calculate:

$$\phi M_n = \phi A_s f_y (d - a/2)$$

$$= 0.829(4)(60)\frac{\left(16.5 - \frac{5.882}{2}\right)}{12} = 224.9 \text{ K·ft}$$

Discussion

A slightly conservative approach can be used assuming tension-controlled section, $\rho = \rho_{max} = 0.01806$ and $\phi = 0.9$. $A_{s\,max} = 0.01806(12 \times 16.5) = 3.576 \text{ in.}^2$, $a = 5.259$ in., and $\phi M_n = 223.2$ K·ft (almost equal to the above ϕM_n).

Example 3.8: Two Rows of Bars

Determine the design moment strength of a rectangular concrete section reinforced with six no. 9 bars in two rows (Fig. 3.22).

Given: $b = 12$ in., $d = 23.5$ in., $h = 27$ in., $d_t = 24.5$ in., $f'_c = 4$ ksi, and $f_y = 60$ ksi.

Solution

1. For tension-controlled condition, $\varepsilon_t = 0.005$, $\rho_{max} = 0.01806$ (Table 3.2) and $\rho_b = 0.0285$. Check

$$\rho = \frac{A_s}{bd} = \frac{6}{12 \times 23.5} = 0.02128 > \rho_{max}$$

Section is in the transition region.

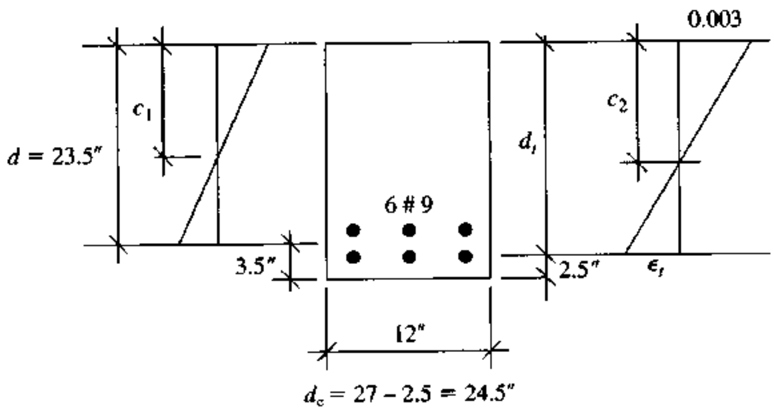


Figure 3.22 Example 3.8.

2. Calculate a, c, and ε_t:

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{6 \times 60}{0.85 \times 4 \times 12} = 8.824 \text{ in.}$$

$$c = \frac{a}{0.85} = 10.38 \text{ in.} \quad d_t = h - 2.5 = 27 - 2.5 = 24.5$$

$$\frac{c}{d_t} = 0.424 > 0.375$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right)0.003 = \left(\frac{24.5 - 10.38}{10.38}\right)0.003 = 0.00408 > 0.004$$

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.823$$

3. Calculate

$$\phi M_n = \phi A_s f_y (d - a/2)$$

$$= 0.823(6)(60)\frac{\left(23.5 - \frac{8.824}{2}\right)}{12} = 471 \text{ K·ft}$$

Discussion

For a tension-controlled section limitation, $\rho_{max} = 0.01806$ and $R_u = 820$ psi,

$$\phi M_n = R_u b d^2 = 0.82(12)\frac{(23.5)^2}{12} = 452.8 \text{ K·ft}$$

This is a conservative value: It is advisable to choose adequate reinforcement to produce tension-controlled condition with $\phi = 0.9$.

3.14 RECTANGULAR SECTIONS WITH COMPRESSION REINFORCEMENT

In concrete sections proportioned to resist the bending moments resulting from external loading on a structural member, the internal moment is equal to or greater than the external moment, but a concrete section of a given width and effective depth has a minimum capacity when ρ_{max} is used. If the external factored moment is greater than the design moment strength, more compressive and tensile reinforcement must be added.

Compression reinforcement is used when a section is limited to specific dimensions due to architectural reasons, such as a need for limited headroom in multistory buildings. Another advantage of compression reinforcement is that long-time deflection is reduced, as is explained in Chapter 6. A third use of bars in the compression zone is to hold stirrups, which are used to resist shear forces.

Two cases of doubly reinforced concrete sections will be considered, depending on whether compression steel yields or does not yield.

3.14.1 When Compression Steel Yields

Internal moment can be divided into two moments, as shown in Fig. 3.23. M_{u_1} is the moment produced by the concrete compressive force and an equivalent tension force in steel, A_{s_1}, acting as

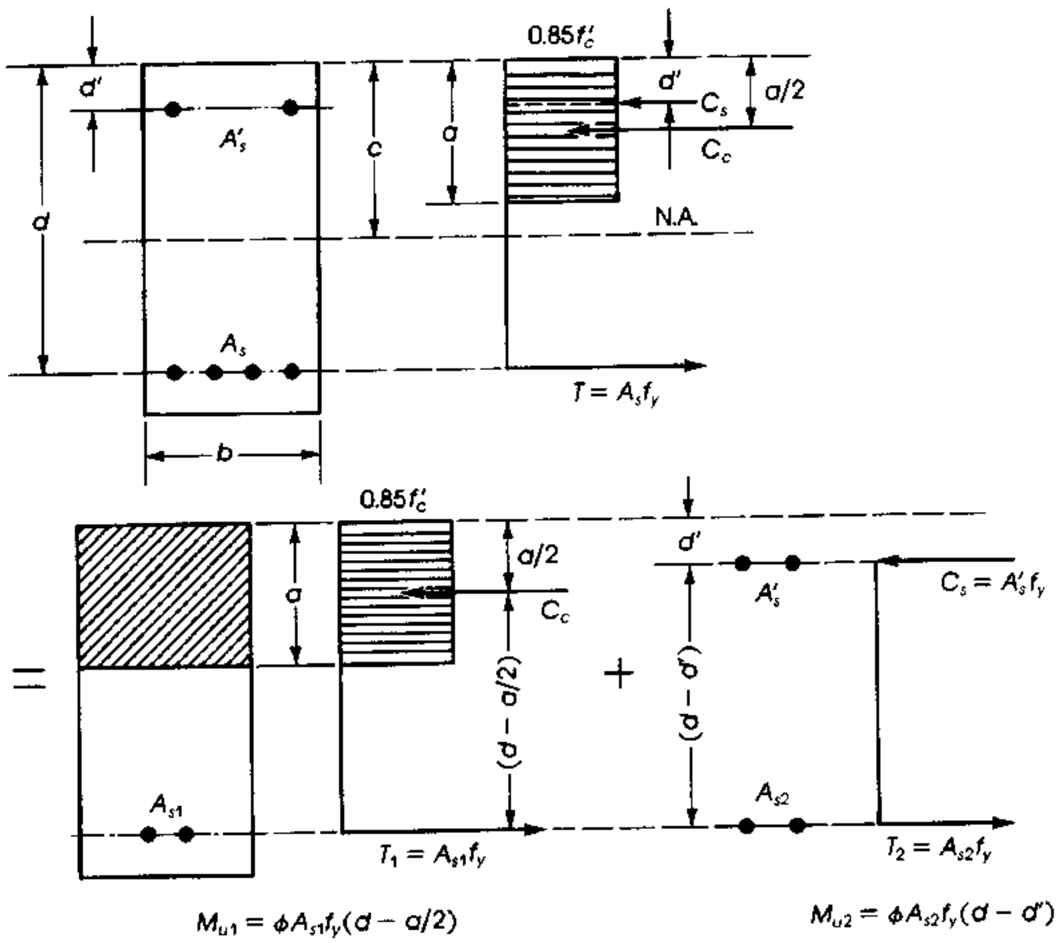


Figure 3.23 Rectangular section with compression reinforcement.

a basic section. M_{u_2} is the additional moment produced by the compressive force in compression steel A_s' and the tension force in the additional tensile steel, A_{s_2}, acting as a steel section.

The moment M_{u_1} is that of a singly reinforced concrete basic section,

$$T_1 = C_c \tag{3.36}$$

$$A_{s_1} f_y = C_c = 0.85 f_c' ab \tag{3.37}$$

$$a = \frac{A_{s_1} f_y}{0.85 f_c' b} \tag{3.38}$$

$$M_{u_1} = \phi A_{s_1} f_y \left(d - \frac{a}{2}\right) \tag{3.39}$$

The restriction for M_{u_1} is that $\rho_1 < A_{s_1}/bd$ shall be equal to or less than $\rho_{\max}$ for singly reinforced tension-controlled sections, as given in Eq. 3.31*a*.

Considering the moment M_{u_2} and assuming that the compression steel designated as A_s' yields,

$$M_{u_2} = \phi A_{s_2} f_y (d - d') \tag{3.40a}$$

$$M_{u_2} = \phi A_s' f_y (d - d') \tag{3.40b}$$

In this case $A_{s_2} = A'_s$, producing equal and opposite forces, as shown in Fig. 3.23. The total resisting moment, M_u, is then the sum of the two moments M_{u_1} and M_{u_2}:

$$\phi M_n = M_{u_1} + M_{u_2} = \phi\left[A_{s_1} f_y \left(d - \frac{a}{2}\right) + A'_s f_y (d - d')\right] \tag{3.41}$$

The total steel reinforcement used in tension is the sum of the two steel amounts A_{s_1} and A_{s_2}. Therefore,

$$A_s = A_{s_1} + A_{s_2} = A_{s_1} + A'_s \tag{3.42}$$

and

$$A_{s_1} = A_s - A'_s$$

Then, substituting $(A_s - A'_s)$ for A_{s_1} in Eqs. 3.38, 3.39, and 3.41,

$$a = \frac{(A_s - A'_s) f_y}{0.85 f'_c b} \tag{3.43}$$

$$\phi M_n = \phi\left[(A_s - A'_s) f_y \left(d - \frac{a}{2}\right) + A'_s f_y (d - d')\right] \tag{3.44}$$

and

$$(\rho - \rho') \le \rho_{\max} = \rho_b \left(\frac{0.003 + f_y/E_s}{0.008}\right) \tag{3.45}$$

For $f_y = 60$ ksi, $(\rho - \rho') \le 0.63375\ \rho_b$, $\phi = 0.9$, and $\varepsilon_t = 0.005$. Equation 3.45 must be fulfilled in doubly reinforced concrete sections, which indicates that the difference between total tension steel and the compression steel should not exceed the maximum steel for singly reinforced concrete tension-controlled sections. Failure due to yielding of the total tensile steel will then be expected, and sudden failure of concrete is avoided.

If $\rho_1 = (\rho - \rho') > \rho_{\max}$, the section will be in the transition region with a limit of $(\rho - \rho') \le \rho_{\max t}$ (Eq. 3.34*a*). In this case, $\phi < 0.9$ for M_{u_1} and $\phi = 0.9$ for M_{u_2}. Equation 3.44 becomes

$$\phi M_n = \phi\left[(A_s - A'_s) f_y \left(d - \frac{a}{2}\right)\right] + 0.9 A'_s f_y (d - d') \tag{3.44a}$$

Note that $(A_s - A'_s) \le \rho_{\max t}(bd)$.

In the compression zone, the force in the compression steel is $C_s = A'_s(f_y - 0.85 f'_c)$, taking into account the area of concrete displaced by A'_s. In this case,

$$T = A_s f_y = C_c + C_s = 0.85 f'_c ab + A'_s(f_y - 0.85 f'_c)$$

$$A_s f_y - A'_s f_y + 0.85 f'_c A'_s = 0.85 f'_c ab = C_c = A_{s_1} f_y \text{ (for the basic section)}$$

Dividing by bdf_y,

$$\rho - \rho'\left(1 - 0.85\frac{f'_c}{f_y}\right) = \rho_1, \qquad \text{where } \rho_1 = \frac{A_{s_1}}{bd} \le \rho_{\max}$$

Therefore,

$$\rho - \rho'\left(1 - 0.85\frac{f'_c}{f_y}\right) \le \rho_{\max} = \rho_b \left(\frac{0.003 + f_y/E_s}{0.008}\right) \tag{3.46}$$

Although Eq. 3.46 is more accurate than Eq. 3.45, it is quite practical to use both equations to check the condition for maximum steel ratio in rectangular sections when compression steel yields.

For example, if $f'_c = 3$ ksi and $f_y = 60$ ksi, Eq. 3.46 becomes $(\rho - 0.9575\rho') \leq 0.016$; if $f'_c = 4$ ksi and $f_y = 60$ ksi, then $(\rho - 0.9575\rho') \leq 0.02138$.

The maximum total tensile steel ratio, ρ, that can be used in a rectangular section when compression steel yields is as follows:

$$\text{Max } \rho = (\rho_{\max} + \rho') \tag{3.47}$$

where $\rho_{\max}$ is maximum tensile steel ratio for the basic singly reinforced tension-controlled concrete section. This means that maximum total tensile steel area that can be used in a rectangular section when compression steel yield is as follows:

$$\text{Max } A_s = bd(\rho_{\max} + \rho') \tag{3.47a}$$

In the preceding equations, it is assumed that compression steel yields. To investigate this condition, refer to the strain diagram in Fig. 3.24. If compression steel yields, then

$$\varepsilon'_s \geq \varepsilon_y = \frac{f_y}{E_s}$$

From the two triangles above the neutral axis, substitute $E_s = 29{,}000$ ksi and let f_y be in ksi. Then

$$\frac{c}{d'} = \frac{0.003}{0.003 + \frac{f_y}{E_s}} = \frac{87}{87 - f_y}$$

$$c = \left(\frac{87}{87 - f_y}\right)d' \tag{3.48}$$

From Eq. 3.37,

$$A_{s_1} f_y = 0.85 f'_c ab \tag{3.47}$$

but

$$A_{s_1} = A_s - A'_s \quad \text{and} \quad \rho_1 = (\rho - \rho')$$

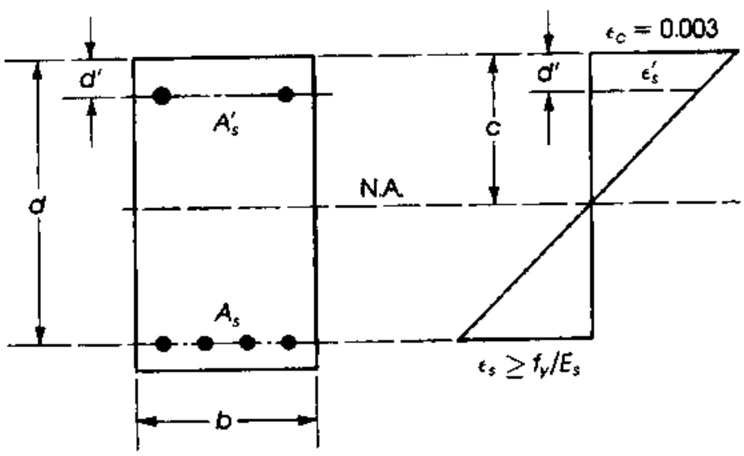


Figure 3.24 Strain diagram in doubly reinforced section.

Therefore, Eq. 3.37 becomes $(A_s - A'_s)f_y = 0.85f'_c ab$:

$$(\rho - \rho')bdf_y = 0.85f'_c ab$$

$$(\rho - \rho') = 0.85\left(\frac{f'_c}{f_y}\right)\left(\frac{a}{d}\right)$$

Also,

$$a = \beta_1 c = \beta_1\left(\frac{87}{87 - f_y}\right)d'$$

Therefore,

$$(\rho - \rho') = 0.85\beta_1\left(\frac{f'_c}{f_y}\right)\left(\frac{d'}{d}\right)\left(\frac{87}{87 - f_y}\right) = K \tag{3.49}$$

The quantity $(\rho - \rho')$ is the steel ratio, or $(A_s - A'_s)/bd = A_{s_1}/bd = \rho_1$ for the singly reinforced basic section.

If $(\rho - \rho')$ is greater than the value of the right-hand side in Eq. 3.49, then compression steel will also yield. In Fig. 3.25 we can see that if A_{s_1} is increased, T_1 and, consequently, C_1

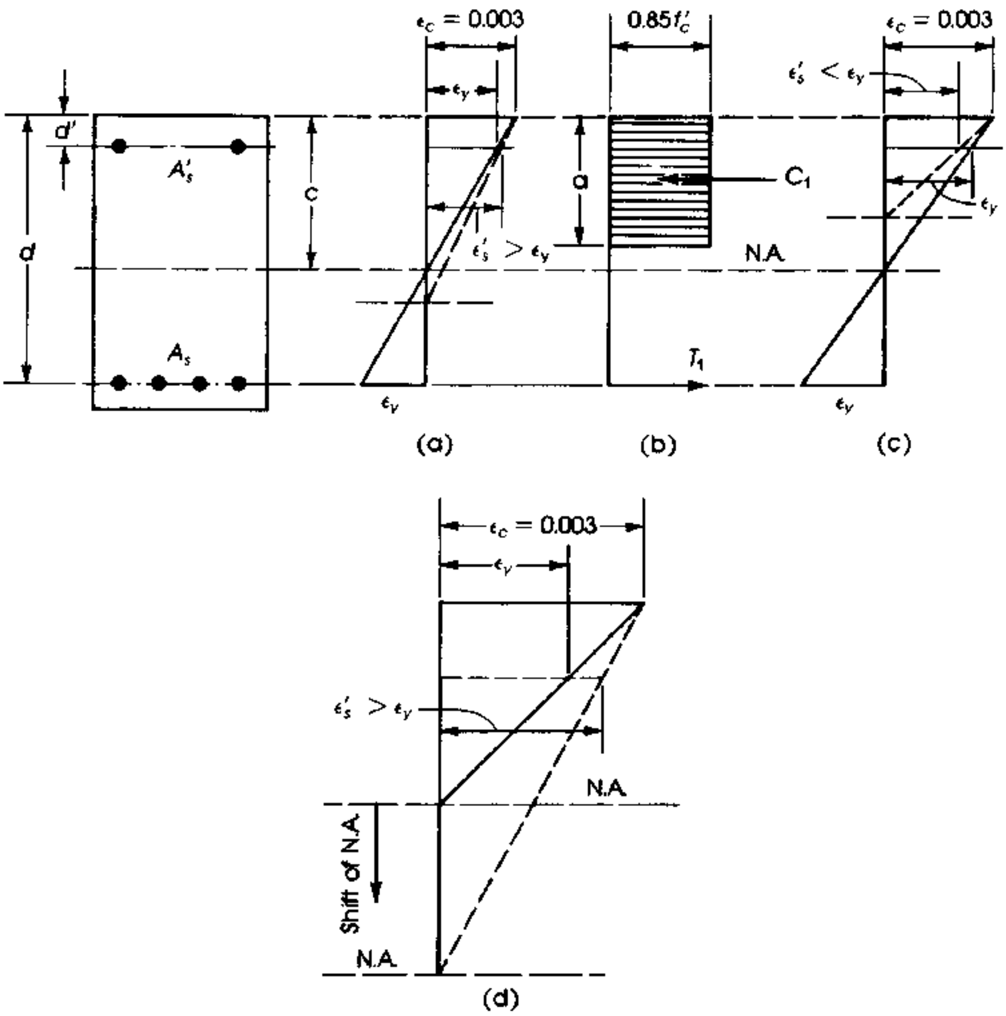


Figure 3.25 Yielding and nonyielding cases of compression reinforcement. Diagram (*d*), a close-up of (*a*), shows how the neutral axis responds to an increase in A_{s1}.

Table 3.4 Values of K for Different f'_c and f_y

f'_c (ksi)	f_y (ksi)	K	K (for $d' = 2.5$ in.)
3	40	$0.1003d'/d$	$0.251/d$
3	60	$0.1164d'/d$	$0.291/d$
4	60	$0.1552d'/d$	$0.388/d$
5	60	$0.1826d'/d$	$0.456/d$

will be greater and the neutral axis will shift downward, increasing the strain in the compression steel and ensuring its yield condition. If the tension steel used (A_{s_1}) is less than the right-hand side of Eq. 3.49, then T_1 and C_1 will consequently be smaller, and the strain in compression steel, ε'_s, will be less than or equal to ε_y, because the neutral axis will shift upward, as shown in Fig. 3.25*c*, and compression steel will not yield.

Therefore, Eq. 3.49 can be written

$$(\rho - \rho') \geq 0.85\,\beta_1 \frac{f'_c}{f_y} \times \frac{d'}{d} \times \frac{87}{87 - f_y} = K \qquad (3.49a)$$

where f_y is in ksi, and this is the condition for compression steel to yield.

For example, the values of K for different values of f'_c and f_y are as shown in Table 3.4.

Example 3.9

A rectangular beam has a width of 12 in. and an effective depth of $d = 22.5$ in. to the centroid of tension steel bars. Tension reinforcement consists of six no. 9 bars in two rows; compression reinforcement consists of two no. 7 bars placed as shown in Fig. 3.26. Calculate the design moment strength of the beam if $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Check if compression steel yields:

$$A_s = 6.0 \text{ in.}^2 \quad \rho = \frac{A_s}{bd} = \frac{6.0}{12 \times 22.5} = 0.02222$$

$$A'_s = 1.2 \text{ in.}^2 \quad \rho' = \frac{A'_s}{bd} = \frac{1.2}{12 \times 22.5} = 0.00444$$

$$A_s - A'_s = 4.8 \text{ in.}^2 \quad \rho - \rho' = 0.01778$$

For compression steel to yield,

$$(\rho - \rho') \geq 0.85\,\beta_1 \frac{f'_c}{f_y} \times \frac{d'}{d} \times \frac{87}{87 - f_y} = K$$

β_1 is 0.85 because $f'_c = 4000$ psi; therefore,

$$K = (0.85)^2 \left(\frac{4}{60}\right)\left(\frac{2.5}{22.5}\right)\left(\frac{87}{87-60}\right) = 0.0175$$

$$(\rho - \rho') = 0.01778 > 0.0175$$

Therefore, compression steel yields.

2. Check that $(\rho - \rho') \leq \rho_{max}$ (Eq. 3.45): For $f'_c = 4$ ksi and $f_y = 60$ ksi, $\rho_b = 0.0285$ and $\rho_{max} = 0.01806$ (Table 3.2). $(\rho - \rho') \leq 0.01778 < \rho_{max}$, and $\phi = 0.9$ (a tension-controlled condition).

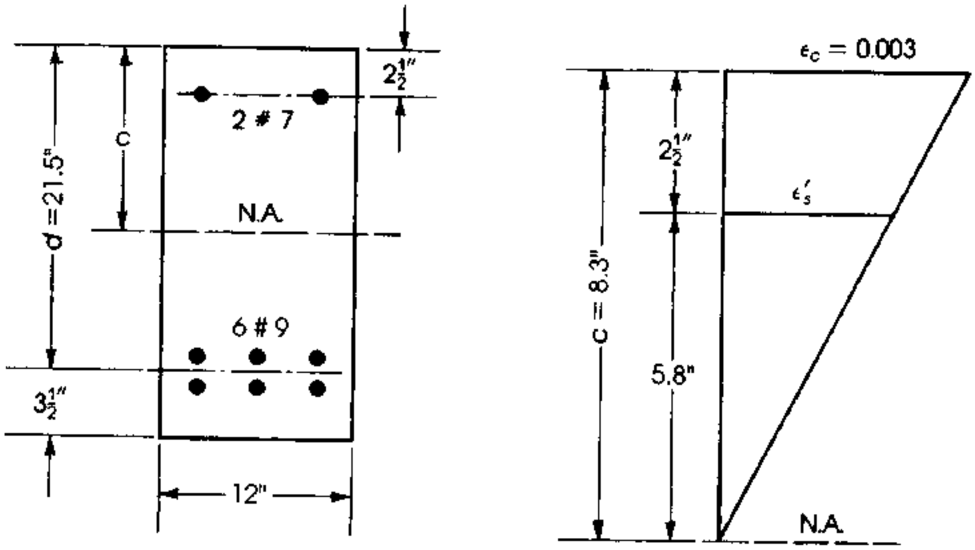


Figure 3.26 Example 3.9.

3. ϕM_n can be calculated by Eq. 3.44:

$$\phi M_n = \phi\left[(A_s - A'_s)f_y\left(d - \frac{a}{2}\right) + A'_s f_y(d - d')\right]$$

$$a = \frac{(A_s - A'_s)f_y}{0.85 f'_c b} = \frac{4.8 \times 60}{0.85 \times 4 \times 12} = 7.06 \text{ in.}$$

$$\phi M_n = (0.9)\left[4.8 \times 60\left(22.5 - \frac{7.06}{2}\right) + 1.2 \times 60(22.5 - 2.5)\right]$$

$$= 6213 \text{ K·in.} = 517.8 \text{ K·ft}$$

4. An alternative approach for checking if compression steel yields can be made as follows:

$$c = \frac{a}{0.85} = \frac{7.06}{0.85} = 8.3 \text{ in.}$$

$$\varepsilon'_s = \frac{5.8}{8.3} \times 0.003 = 0.0021 \qquad \varepsilon_y = \frac{f_y}{E_s} = \frac{60}{29{,}000} = 0.00207$$

Because ε'_s, exceeds ε_y, compression steel yields.

5. Check ε_t:$c = 8.3$ in., $d_t = 26 - 2.5 = 23.5$ in.

$$\varepsilon_t = \left(\frac{23.5 - 8.3}{8.3}\right)0.003 = 0.0055 > 0.005$$

$$\text{or } \frac{c}{d} = 0.353 < 0.375 \qquad \text{(o.k.)}$$

6. The maximum total tension steel for this section, max A_s, is equal to

$$\text{Max } A_s = bd(\rho_{\max} + \rho') = 12 \times 22.5(0.01806 + 0.00444)$$

$$= 6.08 \text{ in.}^2 > A_s = 6.0 \text{ in.}^2 \qquad \text{(used in the section)}$$

3.14.2 When Compression Steel Does Not Yield

As was explained earlier, if

$$(\rho - \rho') < \left(0.85\,\beta_1 \times \frac{f_c'}{f_y} \times \frac{d'}{d} \times \frac{87}{87 - f_y}\right) = K \tag{3.50}$$

then compression steel does not yield. This indicates that if $(\rho - \rho') < K$, the tension steel will yield before concrete can reach its maximum strain of 0.003, and the strain in compression steel, ε_s', will not reach ε_y at failure (Fig. 3.25). Yielding of compression steel will also depend on its position relative to the extreme compressive fibers d'. A higher ratio of d'/c will decrease the strain in the compressive steel, ε_s', as it places compression steel A_s' nearer to the neutral axis.

If compression steel does not yield, a general solution can be performed by analysis based on statics. Also, a solution can be made as follows: Referring to Figs. 3.23 and 3.24,

$$\varepsilon_s' = 0.003\left(\frac{c - d'}{c}\right) \qquad f_s' = E_s\varepsilon_s' = 29{,}000(0.003)\left(\frac{c - d'}{c}\right) = 87\left(\frac{c - d'}{c}\right)$$

Let $C_c = 0.85 f_c'\beta_1 cb$:

$$C_s = A_s'(f_s' - 0.85 f_c') = A_s'\left[87\frac{(c - d')}{c}0.85 f_c'\right]$$

Because $T = A_s f_y = C_c + C_s$, then

$$A_s f_y = (0.85 f_c'\beta_1 cb) + A_s'\left[87\left(\frac{c - d'}{c}\right) - 0.85 f_c'\right]$$

Rearranging terms yields

$$(0.85 f_c'\beta_1 b)c^2 + [(87A_s') - (0.85 f_c'A_s') - A_s f_y]c - 87A_s'd' = 0$$

This is similar to $A_1c^2 + A_2c + A_3 = 0$, where

$$A_1 = 0.85 f_c'\beta_1 b$$

$$A_2 = A_s'(87 - 0.85 f_c') - A_s f_y$$

$$A_3 = -87A_s'd'$$

Solve for c:

$$c = \frac{1}{2A_1}\left[-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}\right] \tag{3.51}$$

Once c is determined, then calculate f_s', a, C_c and C_s.
$f_s' = 87[(c - d')/c]$; $a = \beta_1 c$; $C_c = 0.85 f_c'ab$; and $C_s = A_s'(f_s' - 0.85 f_c')$.

$$\phi M_n = \phi\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] \tag{3.52}$$

When compression steel does not yield, $f_s' < f_y$, and the maximum total tensile steel reinforcement needed for a rectangular section is

$$\text{Max } A_s = \rho_{\max}bd + A_s'\frac{f_s'}{f_y} = bd\left(\rho_{\max} + \frac{\rho' f_s'}{f_y}\right) \tag{3.53}$$

Using steel ratios and dividing by bd:

$$\text{Max } \rho = \frac{\max A_s}{bd} \leq \rho_{\max} + \rho' \frac{f_s'}{f_y} \tag{3.54}$$

or

$$\left(\rho - \rho' \frac{f_s'}{f_y}\right) \leq \rho_{\max} \tag{3.55}$$

where $\rho_{\max}$ is the maximum steel ratio for the tension-controlled singly reinforced rectangular section (Eq. 3.31).

In this case,

$$a = \frac{A_s f_y - A_s' f_s'}{0.85 f_c' b} \tag{3.56}$$

$$\phi M_n = \phi \left[(A_s f_y - A_s' f_s') \left(d - \frac{a}{2}\right) + A_s' f_s' (d - d') \right] \tag{3.57}$$

In summary, the procedure for analyzing sections with compression steel is as follows:

1. Calculate ρ, ρ', and $(\rho - \rho')$. Also calculate $\rho_{\max}$ and $\rho_{\min}$.
2. Calculate

$$K = 0.85 \beta_1 \left(\frac{f_c'}{f_y}\right) \left(\frac{87}{87 - f_y}\right) \left(\frac{d'}{d}\right)$$

   Use ksi units.
3. If $(\rho - \rho') \geq K$, then compression steel yields, and $f_s' = f_y$; if $(\rho - \rho') < K$, then compression steel does not yield, and $f_s' < f_y$.
4. If compression steel yields, then
   a. Check that $\rho_{\max} \geq (\rho - \rho') \geq \rho_{\min}$ (to use $\phi = 0.9$) or check $\varepsilon_t \geq 0.005$.
   b. Calculate

$$a = \frac{(A_s - A_s') f_y}{0.85 f_c' b}.$$

   c. Calculate

$$\phi M_n = \phi \left[(A_s - A_s') f_y \left(d - \frac{a}{2}\right) + A_s' f_y (d - d') \right].$$

   d. The maximum A_s that can be used in the section is Max $A_s = bd(\rho_{\max} + \rho') \geq A_s$ (used).
5. If compression steel does *not* yield, then
   a. Calculate the distance to the neutral axis c by using analysis (Example 3.10) or by using the quadratic equation (3.51).
   b. Calculate

$$f_s' = 87 \left(\frac{c - d'}{c}\right) \text{(ksi)}.$$

c. Check that $(\rho - \rho' f'_s/f_y) \le \rho_{max}$ or max A_s that can be used in the section is greater than or equal to the A_s used.

$$\text{Max } A_s = bd\left(\rho_{max} + \frac{\rho' f'_s}{f_y}\right) \ge A_s \quad \text{(used)}$$

d. Calculate

$$a = \frac{(A_s f_y - A'_s f'_s)}{0.85 f'_c b} \quad \text{or } a = \beta_1 c.$$

e. Calculate

$$\phi M_n = \phi\left[(A_s f_y - A'_s f'_s)\left(d - \frac{a}{2}\right) + A'_s f'_s(d - d')\right].$$

For flow charts, refer to Chapter 21.

Example 3.10

Determine the design moment strength of the section shown in Fig. 3.27 using $f'_c = 5$ ksi, $f_y = 60$ ksi, $A'_s = 2.37$ in.^{2} (three no. 8 bars), and $A_s = 7.62$ in.^{2} (six no. 10 bars).

Solution

1. Calculate ρ and ρ':

$$\rho = \frac{A_s}{bd} = \frac{7.62}{14 \times 22.5} = 0.0242 \quad \rho' = \frac{A'_s}{bd} = \frac{2.37}{14 \times 22.5} = 0.00753$$

$$(\rho - \rho') = 0.01667$$

2. Apply Eq. 3.50, assuming $\beta_1 = 0.8$ for $f'_c = 5000$ psi.

$$K = 0.85\beta_1 \times \frac{f'_c}{f_y} \times \frac{d'}{d} \times \frac{87}{87 - f_y} = 0.85 \times 0.8\left(\frac{5}{60}\right)\left(\frac{2.5}{22.5}\right)\left(\frac{87}{87-60}\right) = 0.0203$$

(or from Table 3.3, $K = 0.456/d = 0.0203$).

$$(\rho - \rho') = 0.01667 < 0.0203$$

Therefore, compression steel does not yield, and $f'_s < 60$ ksi.

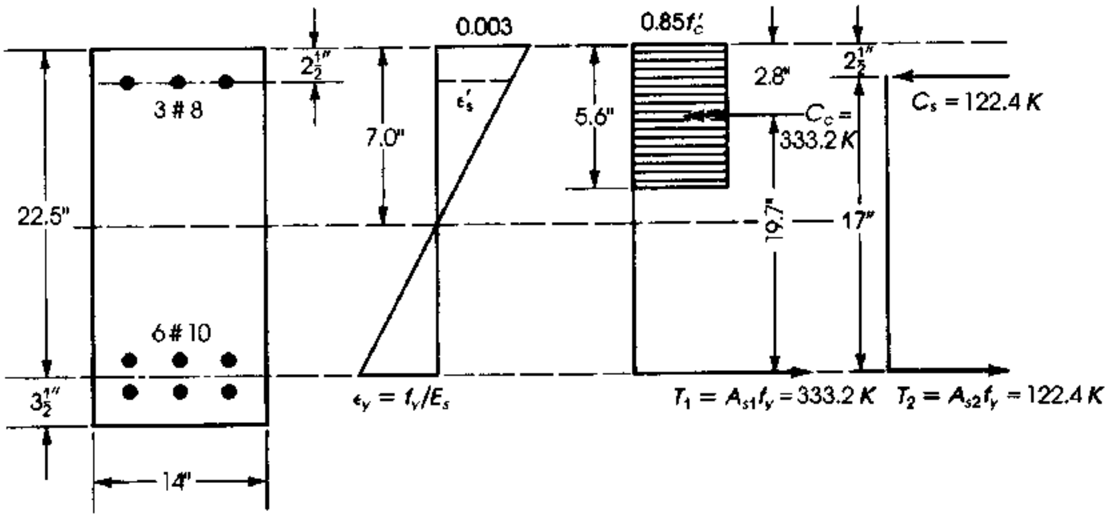


Figure 3.27 Example 3.10 analysis solution.

For $f'_c = 5$ ksi and $f_y = 60$ ksi, $\rho_b = 0.0335$ and $\rho_{max} = 0.02123$ [Table 3.2 $(\rho - \rho') < \rho_{max}$, for the basic section]. $\phi = 0.9$, so this is a tension-controlled condition.

3. Calculate ϕM_n by analysis. Internal forces:

$$C_c = 0.85 f'_c ab \qquad a = \beta_1 c = 0.8c$$

$$C_c = 0.85 \times 5(0.8c) \times 14 = 47.7c$$

$$\begin{aligned} C_s &= \text{the force in compression steel} \\ &= A'_s f'_s - \text{the force in displaced concrete} \\ &= A'_s(f'_s - 0.85 f'_c) \end{aligned}$$

From strain triangles,

$$\varepsilon'_s = 0.003 \frac{c - d'}{c}$$

$$\begin{aligned} f'_s &= E_s \varepsilon'_s \text{ (since steel is in the elastic range)} \\ &= 29{,}000 \left[\frac{0.003(c - d')}{c}\right] = \frac{87(c - d')}{c} \text{ (ksi)} \end{aligned}$$

Therefore,

$$C_s = 2.37[87 \frac{(c - d')}{c} - (0.85 \times 5)](\text{kips}) = \left[\frac{206.2(c - 2.5)}{c}\right] - 10.07$$

$$T = T_1 + T_2 = (A_{s1} + A_{s2}) f_y = S_s f_y = 7.62(60) = 457.2 \text{ kips}$$

4. Equate internal forces to determine the position of the neutral axis (the distance c):

$$T = C = C_c + C_s$$

$$457.2 = 47.6c + \frac{206.2(c - 2.5)}{c} - 10.07$$

$$c^2 - 5.48c - 10.83 = 0$$

$$c = 7.0 \text{ in.} \qquad a = 0.8c = 5.6 \text{ in.}$$

Equation 3.51 can also be used to calculate c and a.

5. Calculate f'_s, C_c, and C_s:

$$f'_s = \frac{87(c - 2.5)}{c} = \frac{87(7.0 - 2.5)}{7.0} = 55.9 \text{ ksi}$$

which confirms that compression steel does not yield.

$$C_c = 47.6c = 47.6(7.0) = 333.2 \text{ kips}$$

$$C_s = (A'_s f'_s - 10.07) = 2.37(55.90) - 10.07 = 122.40 \text{ kips}$$

6. To calculate ϕM_n, take moments about the tension steel A_s:

$$\begin{aligned} \phi M_n &= \phi \left[C_c \left(d - \frac{a}{2}\right) + C_s(d - d')\right] = 0.9[333.2(22.5 - 2.8) + 122.40(22.5 - 2.5)] \\ &= 8110.8 \text{ K·in.} = 675.9 \text{ K·ft} \end{aligned}$$

7. Check that $(\rho - \rho' f'_s/f_y) \leq \rho_{max}$ (Eq. 3.55):

$$0.0242 - 0.00754\left(\frac{55.9}{60}\right) = 0.0171 < \rho_{max} = 0.02123$$

The maximum total tension steel that can be used in this section is calculated by Eq. 3.50

$$\max A_s = bd\left(\rho_{max} + \frac{\rho' f'_s}{f_y}\right)$$

$$= 14(22.5)\left(0.02123 + \frac{0.00753 \times 55.9}{60}\right) = 8.9 \text{ in.}^2 > 7.62 \text{ in.}^2 \qquad \text{(o.k.)}$$

8. ε_t can be checked as follows: $c = 7.0$ in., $d_t = 23.5$ in.

$$\frac{c}{d_t} = 0.3 < 0.375 \quad \text{or}$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right)0.003 = \left(\frac{23.5 - 7}{7}\right)0.003 = 0.0071 > 0.005$$

Tension-controlled section.

3.15 ANALYSIS OF T- AND I-SECTIONS

3.15.1 Description

It is normal to cast concrete slabs and beams together, producing a monolithic structure. Slabs have smaller thicknesses than beams. Under bending stresses, those parts of the slab on either side of the beam will be subjected to compressive stresses, depending on the position of these parts relative to the top fibers and relative to their distances from the beam. The part of the slab acting with the beam is called the flange, and it is indicated in Fig. 3.28*a* by area bt. The rest of the section confining the area $(h - t)b_w$ is called the *stem*, or *web*.

In an I-section there are two flanges, a compression flange, which is actually effective, and a tension flange, which is ineffective, because it lies below the neutral axis and is thus neglected completely. Therefore, the analysis and design of an I-beam is similar to that of a T-beam.

3.15.2 Effective Width

In a T-section, if the flange is very wide, the compressive stresses are at a maximum value at points adjacent to the beam and decrease approximately in a parabolic form to almost 0 at a distance x from the face of the beam. Stresses also vary vertically from a maximum at the top fibers of the flange to a minimum at the lower fibers of the flange. This variation depends on the position of the neutral axis and the change from elastic to inelastic deformation of the flange along its vertical axis.

An equivalent stress area can be assumed to represent the stress distribution on the width b of the flange, producing an equivalent flange width, b_e, of uniform stress (Fig. 3.28*c*).

Analysis of equivalent flange widths for actual T-beams indicate that b_e is a function of span length of the beam [7]. Other variables that affect the effective width b_e are (Fig. 3.29).

- Spacing of beams
- Width of stem (web) of beam b_w
- Relative thickness of slab with respect to the total beam depth

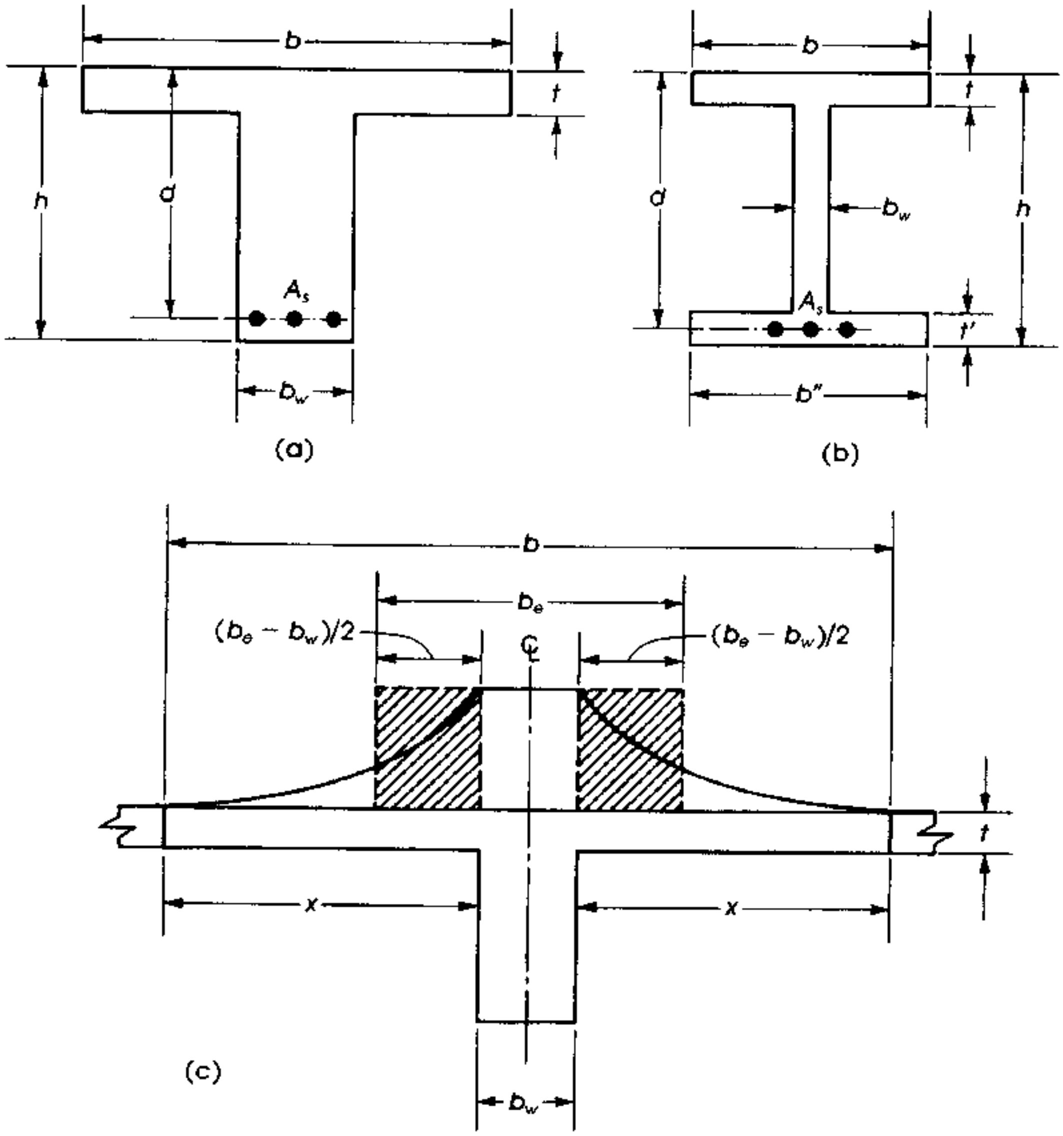


Figure 3.28 (*a*) T-section and (*b*) I-section, with (*c*) illustration of effective flange width b_e.

- End conditions of the beam (simply supported or continuous)
- The way in which the load is applied (distributed load or point load)
- The ratio of the length of the beam between points of zero moment to the width of the web and the distance between webs

The ACI Code, Section 8.10.2, prescribes the following limitations on the effective flange width b_e, considering that the span of the beam is equal to L:

1. $b_e = L/4$
2. $b_e = 16t + b_w$
3. $b_e = b$, where b is the distance between centerlines of adjacent slabs

The *smallest* of the aforementioned three values must be used.

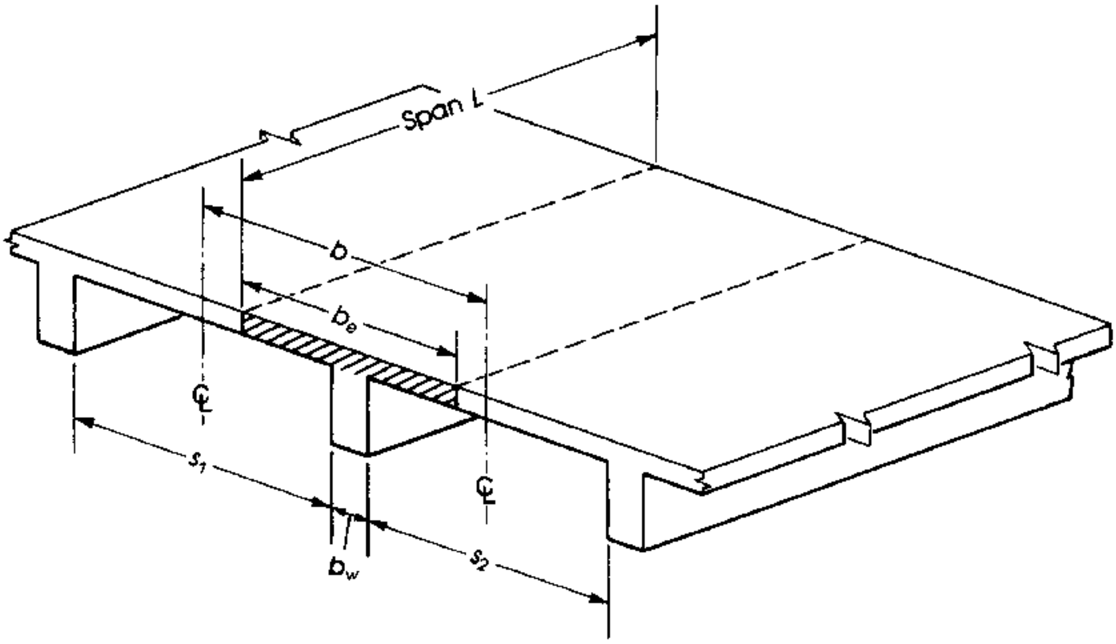


Figure 3.29 Effective flange width of T-beams.

These values are conservative for some cases of loading and are adequate for other cases. A similar effective width of flange can be adopted for I-beam sections. Investigations indicate that the effective compression flange increases as load is increased toward the ultimate value [6]. Under working loads, stress in the flange is within the elastic range.

A T-shaped or I-shaped section may behave as a rectangular section or a T-section. The two cases are investigated as follows.

3.15.3 T-Sections Behaving as Rectangular Sections

In this case, the depth of the equivalent stress block a lies within the flange, with extreme position at the level of the bottom fibers of the compression flange ($a \leq t$). When the neutral axis lies within the flange (Fig. 3.30a), the depth of the equivalent compressive distribution stress lies within the flange, producing a compressed area equal to $b_e a$. The concrete below the neutral axis is assumed ineffective, and the section is considered singly reinforced, as explained earlier, with b_e replaced by b. Therefore,

$$a = \frac{A_s f_y}{0.85 f_c' b_e} \tag{3.58}$$

and

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) \tag{3.59}$$

If the depth a is increased such that $a = t$, then the factored moment capacity is that of a singly reinforced concrete section:

$$\phi M_n = \phi A_s f_y \left(d - \frac{t}{2}\right) \tag{3.60}$$

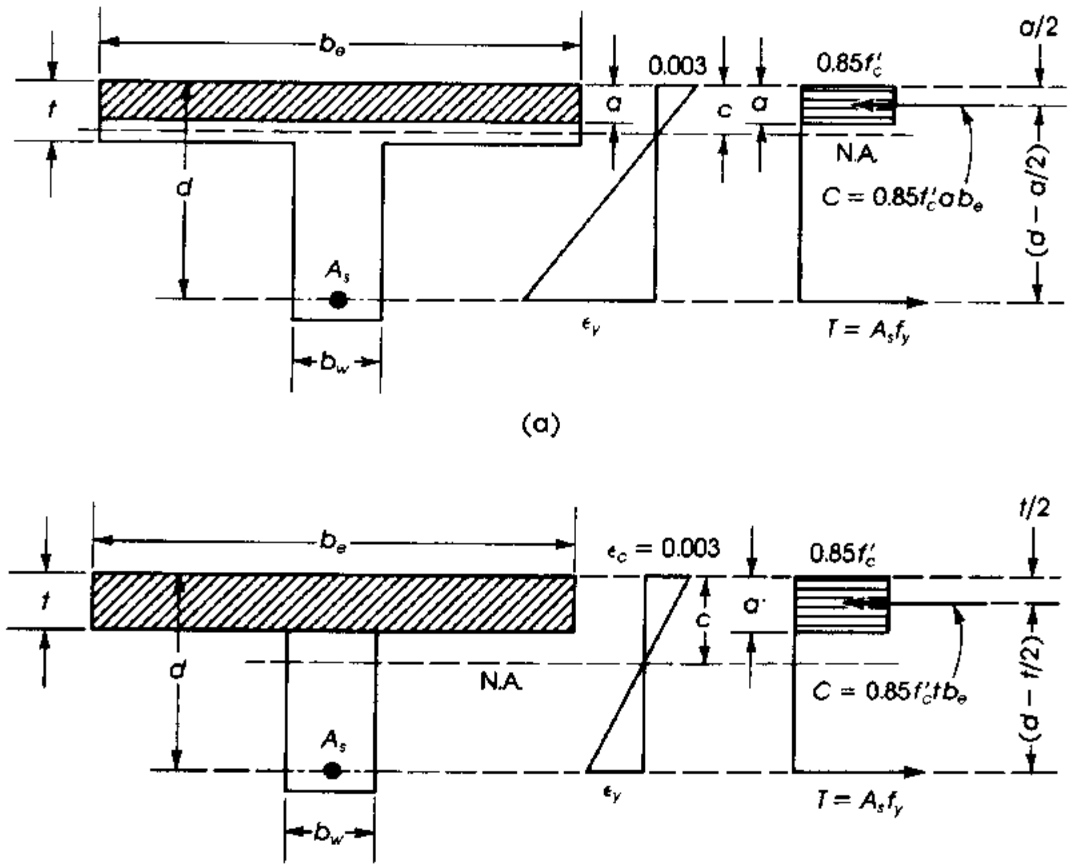


Figure 3.30 Rectangular section behavior (*a*) when the neutral axis lies within the flange and (*b*) when the stress distribution depth equals the slab thickness.

In this case

$$t = \frac{A_s f_y}{0.85 f'_c b_e} \text{ or } A_s = \frac{0.85 f'_c b_e t}{f_y} \tag{3.61}$$

In this analysis, the limit of the steel area in the section should apply: $A_s \leq A_{s\,\max}$, and $\varepsilon_t \geq 0.005$.

3.15.4 Analysis of a T-Section

In this case the depth of the equivalent compressive distribution stress lies below the flange; consequently, the neutral axis also lies in the web. This is due to an amount of tension steel A_s more than that calculated by Eq. 3.61. Part of the concrete in the web will now be effective in resisting the external moment. In Fig. 3.31, the compressive force C is equal to the compression area of the flange and web multiplied by the uniform stress of 0.85 f'_c:

$$C = 0.85 f'_c[b_e t + b_w(a - t)]$$

The position of C is at the centroid of the T-shaped compressive area at a distance z from top fibers.

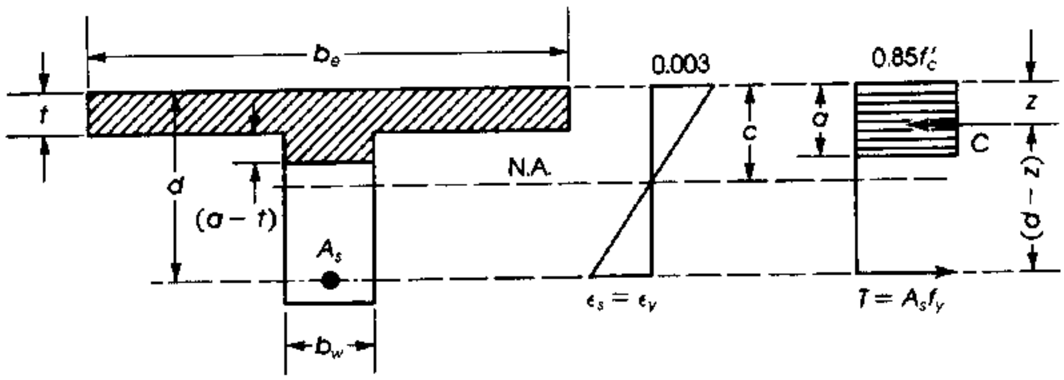


Figure 3.31 T-section behavior.

The analysis of a T-section is similar to that of a doubly reinforced concrete section, considering an area of concrete $(b_e - b_w)t$ as equivalent to the compression steel area A'_s. The analysis is divided into two parts, as shown in Fig. 3.32:

1. A singly reinforced rectangular basic section, $b_w d$, and steel reinforcement A_{s1}. The compressive force, C_1, is equal to $(0.85\ f'_c a b_w)$, the tensile force, T_1, is equal to $A_{s1} f_y$, and the moment arm is equal to $(d - a/2)$.
2. A section that consists of the concrete overhanging flange sides $2 \times [(b_e - b_w)t]/2$ developing the additional compressive force (when multiplied by $0.85\ f'_c$) and a moment arm equal to $(d - t/2)$. If A_{sf} is the area of tension steel that will develop a force equal to the

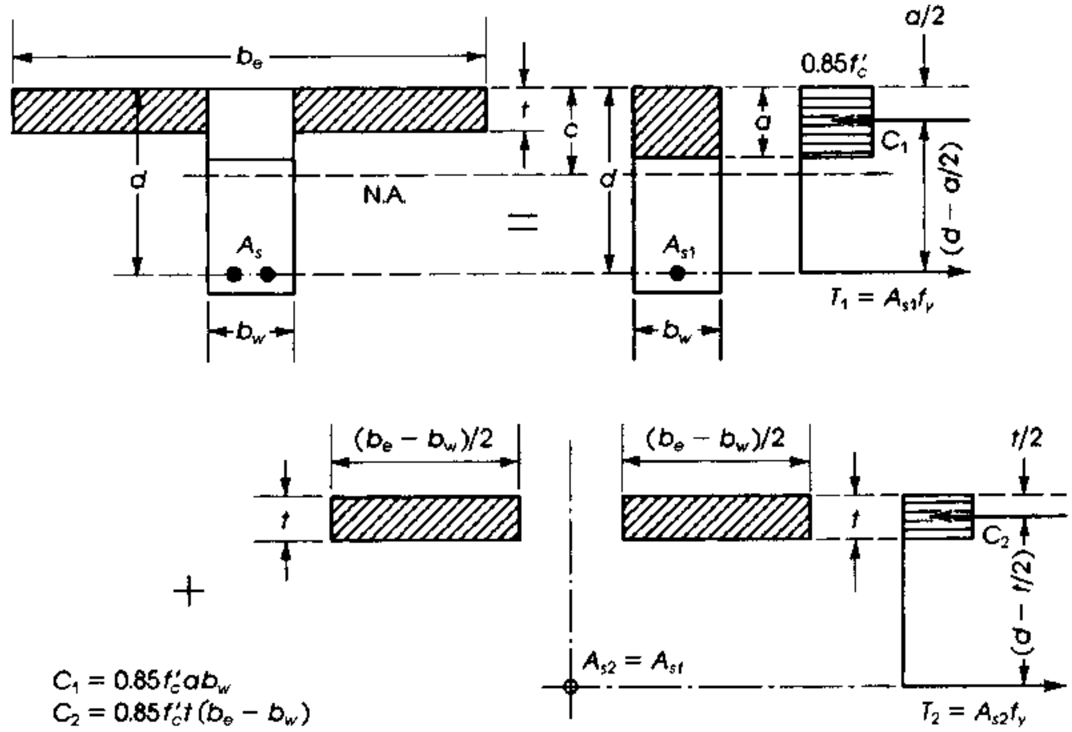


Figure 3.32 T-section analysis.

compressive strength of the overhanging flanges, then

$$A_{sf} f_y = 0.85 f_c'(b_e - b_w)t$$

$$A_{sf} = \frac{0.85 f_c' t(b_e - b_w)}{f_y} \tag{3.62}$$

The total steel used in the T-section A_s is equal to $A_{s1} + A_{sf}$, or

$$A_{s1} = A_s - A_{sf} \tag{3.63}$$

The T-section is in equilibrium, so $C_1 = T_1$, $C_2 = T_2$, and $C_1 = C_1 + C_2 = T_1 + T_2 = T$. Considering equation $C_1 = T_1$ for the basic section, then $A_{s1} f_y = 0.85\ f_c' a b_w$ or $(A_s - A_{sf}) f_y = 0.85\ f_c' a b_w$; therefore,

$$a = \frac{(A_s - A_{sf}) f_y}{0.85 f_c' b_w} \tag{3.64}$$

Note that b_w is used to calculate a. The factored moment capacity of the section is the sum of the two moments M_{u1} and M_{u2}.

$$\phi M_n = M_{u1} + M_{u2}$$

$$M_{u1} = \phi A_{s1} f_y \left(d - \frac{a}{2}\right) = \phi (A_s - A_{sf}) f_y \left(d - \frac{a}{2}\right)$$

where

$$A_{s1} = (A_s - A_{sf}) \quad \text{and} \quad a = \frac{(A_s - A_{sf}) f_y}{0.85\ f_c' b_w}$$

$$M_{u2} = \phi A_{sf} f_y \left(d - \frac{t}{2}\right) \tag{3.65}$$

$$\phi M_n = \phi \left[(A_s - A_{sf}) f_y \left(d - \frac{a}{2}\right) + A_{sf} f_y \left(d - \frac{t}{2}\right) \right]$$

Considering the web section $b_w d$, the net tensile strain (NTS), ε_t, can be calculated from a, c, and d_t as follows:
If $c = a/\beta_1$ (from Eq. 3.64) and $d_t = h - 2.5\,\text{in.}$, then $\varepsilon_t = 0.003\ (c - d_t)/c$. For tension-controlled section in the web, $\varepsilon_t \geq 0.005$.

The design moment strength of a T-section or I-section can be calculated from the preceding equation(3.65). It is necessary to check the following:

1. The total tension steel ratio relative to the web effective area is equal to or greater than $\rho_{\min}$.

$$\rho_w = A_s / b_w d \geq \rho_{\min}$$

$$\rho_{\min} = (3\sqrt{f_c'})/f_y \geq 200/f_y \tag{3.66}$$

2. Also, check that the NTS is equal to or greater than 0.005 for tension-controlled sections.
3. The maximum tension steel (Max A_s), in a T-section must be equal to or greater than the steel ratio used, A_s, for tension-controlled sections, with $\phi = 0.9$.

$$\text{Max } A_s = A_{sf}\ \text{(flange)}\ + \rho_{\max}(b_w d)\ \text{(web)} \tag{3.67}$$

$$\text{Max } A_s = (1/f_y)[0.85 f_c' t(b - b_w)] + \rho_{\max}(b_w d) \tag{3.68}$$

In steel ratios, relative to the web only, divide Eq. 3.67 by $b_w d$:

$$\rho_w = A_s/b_w d \leq (\rho_{\max} + A_{sf}/b_w d) \quad (3.69)$$

Or

$$(\rho_w - \rho_f) \leq \rho_{\max} \text{ (web)} \quad (3.70)$$

where $\rho_{\max}$ is the maximum steel ratio for the basic singly reinforced web section (Table 3.2), and $\rho_f = A_{sf}/b_w d$.

A general equation for calculating (Max A_s) in a T-section when $a > t$ can be developed as follows:

$$C = 0.85 f_c'[(b_e - b_w)t + ab_w]$$

For $\varepsilon_c = 0.003$ and $\varepsilon_t = 0.005$, then $c/d = 0.003/(0.003 + 0.005) = 0.375$ (for the web). Hence, $a = \beta_1 c = 0.375\beta_1 d$.

The maximum steel area is equal to C/f_y and therefore

$$\text{Max } A_s = (0.85 f_c'/f_y)[(b_e - b_w)t + 0.375\beta_1 b_w d] \quad (3.71)$$

where Max A_s is the maximum tension steel area that can be used in a T-section when $a > t$. For example for $f_c' = 3$ ksi and $f_y = 60$ ksi, the preceding equation is reduced to:

$$\text{Max } A_s = 0.0425[(b_e - b_w)t + 0.319 b_w d] \quad (3.72)$$

For $f_c' = 4$ ksi and $f_y = 60$ ksi,

$$\text{Max } A_s = 0.0567[(b_e - b_w)t + 0.319 b_w d] \quad (3.73)$$

In summary, the procedure to analyze a T-section or inverted L-section is as follows:

1. Determine the effective width of the flange b_e (refer to Section 3.15.3). Calculate $\rho_{\max}$ and $\rho_{\min}$ (or take from tables).
2. Check if $a \leq t$ as follows: $a' = A_s f_y/(0.85 f_c' b_e)$
3. If $a' \leq t$, it is a rectangular section analysis.
   a. Calculate $\phi M_n = \phi A_s f_y(d - a/2)$, $a = a'$
      Note that $c = a/\beta_1$ and $\varepsilon_t = 0.003\,(d_t - c)/c \geq 0.005$ for tension-controlled section and $\phi = 0.9$.
   b. Check that $\rho_w = A_s/b_w d \geq \rho_{\min}$.
   c. Max A_s can be calculated from Eq. 3.68 and should be $\geq A_s$ used. When $a < t$, normally this condition is met.
4. If $a' > t$, it is a T-section analysis:
   a. Calculate $A_{sf} = 0.85 f_c' t (b_e - b_w)/f_y$
   b. Check that $(\rho_w - \rho_f) \leq \rho_{\max}$ (relative to the web area), where

   $$\rho_w = A_s/b_w d \quad \text{and} \quad \rho_f = A_{sf}/b_w d$$

   Or check that Max $A_s \geq A_s$ used in the section, for $\phi = 0.9$, (Eq. 3.71).
   c. Check that $\rho_w = A_s/b_w d \geq \rho_{\min}$. This condition is normally met when $a > t$.
   d. Calculate $a = (A_s - A_{sf})/0.85 f_c' b_w$ (for the web section).
   e. Calculate ϕM_n from Eq. 3.65.

Example 3.11

A series of reinforced concrete beams spaced at 7 ft, 10 in. on centers have a simply supported span of 15 ft. The beams support a reinforced concrete floor slab 4 in. thick. The dimensions and reinforcement of the beams are shown in Fig. 3.33. Using $f'_c = 3$ ksi and $f_y = 60$ ksi, determine the design moment strength of a typical interior beam.

Solution

1. Determine the effective flange width b_e. The effective flange width is the smallest of

$$16t + b_w = (16 \times 4) + 10 = 74 \text{ in.}$$

$$\text{Span}/4 = 15 \times 12/4 = 45 \text{ in.}$$

$$\text{Center to center of beams} = (7 \times 12) + 10 = 94 \text{ in.}$$

Therefore, $b_e = 45$ in. controls.

2. Check the depth of the stress block. If the section behaves as a rectangular one, then the stress block lies within the flange (Fig. 3.30). In this case, the width of beam used is equal to 45 in.

$$a' = A_s f_y/(0.85 f'_c b) = 2.37 \times 60/(0.85 \times 3 \times 45) = 1.24 \text{ in.} < t$$

Therefore, it is a rectangular section with $a = a' = 1.24$ in.

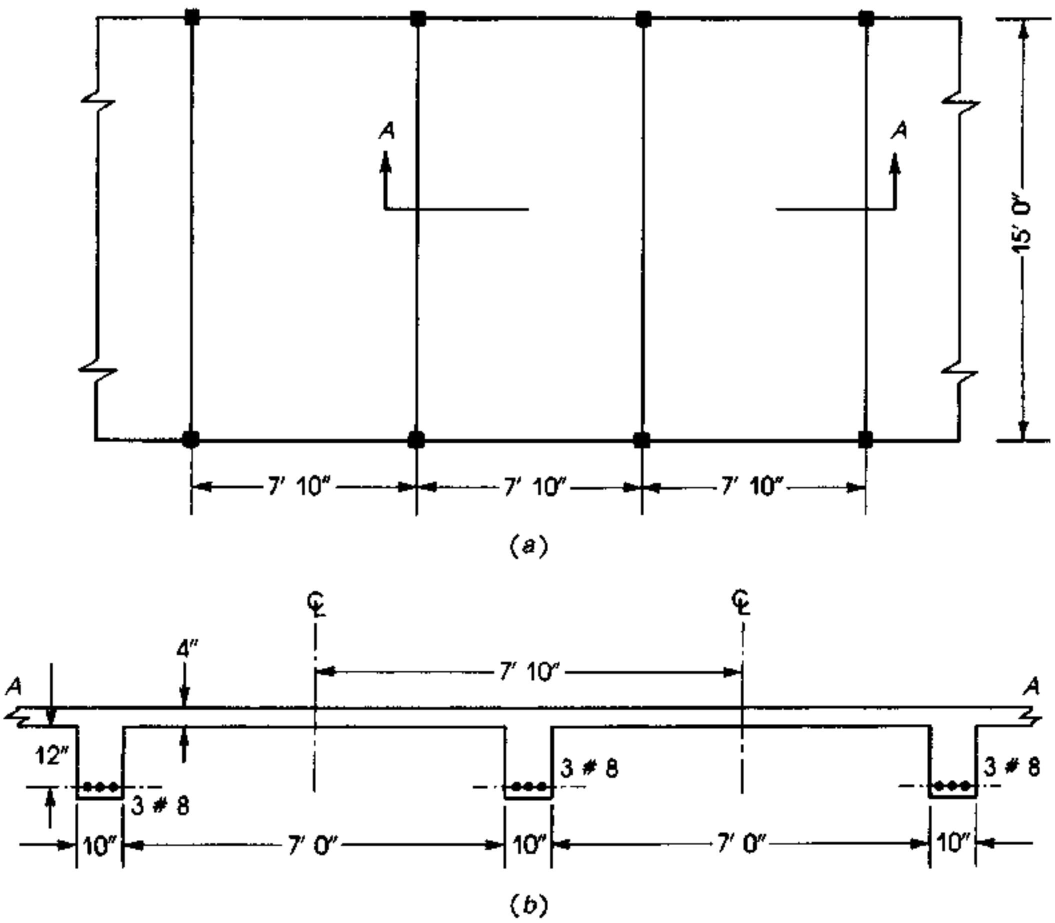


Figure 3.33 Example 3.11: (*a*) plan of slab-beam roof and (*b*) section *A*-*A*.

3. Check that

$$\rho_w = \frac{A_s}{b_w d} \geq \rho_{\min} = 0.00333$$

$$\rho_w = \frac{2.37}{(10 \times 16)} = 0.0148 > 0.00333$$

4. Check ε_t : $a = 1.24$ in., $c = 1.24/0.85 = 1.46$ in., $d_t = d = 16$ in.

$$\varepsilon_t = 0.003(d_t - c)/c = 0.003(16 - 1.46)/1.46 = 0.0299 > 0.005, \phi = 0.9$$

5. Calculate $\phi M_n = \phi A_s f_y(d - a/2) = 0.9(2.37)(60)(16 - 1.24/2)$

$$= 1968 \text{ K·in.} = 164 \text{ K·ft.}$$

6. You may check that A_s used is less than or equal to A_s (Eq. 3.72), which is not needed when $a < t$:

$$\text{Max } A_s = 0.0425[(45 - 10) + 0.31 \times 10 \times 16] = 8.11 \text{ in.}^2 > 2.37 \text{ in.}^2$$

Example 3.12

Calculate the design moment strength of the T-section shown in Fig. 3.34 using $f'_c = 3.5$ ksi and $f_y = 60$ ksi.

Solution

1. Given $b = b_e = 36$ in., $b_w = 10$ in., $d = 17$ in., and $A_s = 6.0$ in.^{2}, check if $a \leq t$:

$$a' = A_s f_y/(0.85 f'_c b) = 6 \times 60/(0.85 \times 3.5 \times 36) = 3.36 \text{ in.}$$

Since $a' > t$, it is a T-section analysis.

2. $A_{sf} = 0.85 f'_c t(b - b_w)/f_y = 0.85 \times 3.5 \times 3(36 - 10)/60 = 3.87$ in.^{2}.$(A_s - A_{sf}) = A_{s1}$ (web) $= 6 - 3.87 = 2.13$ in.^{2}
3. Check ε_t:a (web) $= A_{s1} f_y/(0.85 f'_c b_w) = 2.13 \times 60/(0.85 \times 3.5 \times 10) = 4.3$ in. c $= 4.3/0.85 = 5.06$ in., $d_t = 20.5 - 2.5 = 18$ in., and $c/d_t = 0.281 < 0.375$ Or, $\varepsilon_t = 0.003(d_t - c)/c = 0.0077 > 0.005$, then $\phi = 0.9$
4. Check that $A_s > A_{s\min}$, $\rho_{\min} = 0.00333$

$$A_{s\min} = 0.00333 \times 10 \times 17 = 0.57 \text{ in.}^2$$

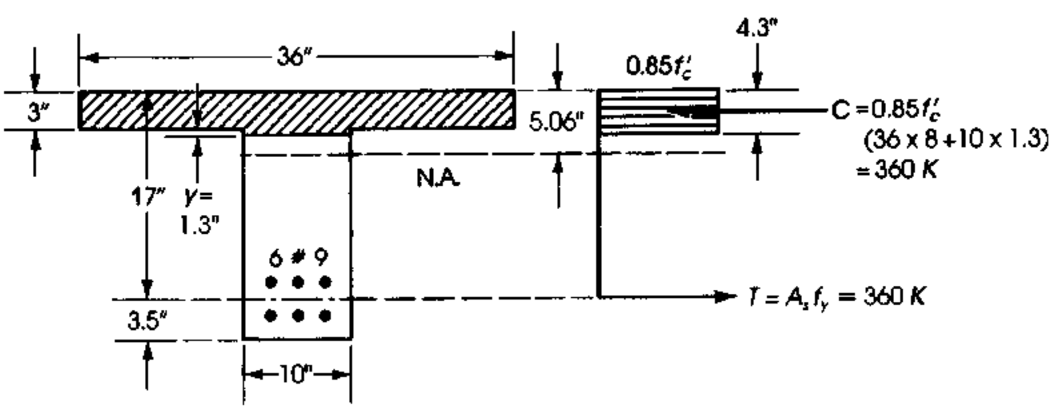


Figure 3.34 Example 3.12.

5. Calculate ϕM_n using Eq. 3.65:

$$\phi M_n = \phi\left[(A_s - A_{sf})f_y\left(d - \frac{a}{2}\right) + A_{sf}f_y\left(d - \frac{t}{2}\right)\right]$$

$$= 0.9\left[2.13 \times 60\left(17 - \frac{4.3}{2}\right) + 3.87 \times 60\left(17 - \frac{3}{2}\right)\right]$$

$$= 4947 \text{ K·in.} = 412.3 \text{ K·ft}$$

Another approach to check whether $a \leq t$ is to calculate the tension force, $T = A_s f_y$, and compare it to the compressive force in the total flange (Fig. 3.34).

$$T = A_s f_y = 60 \times 60 = 360 \text{ K}$$

$$C = 0.85 f_c' t b_e = 0.85 \times 3.5 \times 3 \times 36 = 321.3 \text{ K}$$

Since T exceeds C, then $a \leq t$, and the section acts as a T-section.

An additional area of concrete should be used to provide the difference of (360 − 321.3) = 38.7 K. This area has a width of 10 in. and a depth of y. Therefore,

$$b_w y(0.85 f_c') = 38.7 \text{ K or } 10(y)(0.85 \times 3.5) = 38.7 \text{ K}$$

$y = 1.3$ in., and $a = y + t = 1.3 + 3 = 4.3$ in., as calculated earlier.

3.16 DIMENSIONS OF ISOLATED T-SHAPED SECTIONS

In some cases, isolated beams with the shape of a T-section are used in which additional compression area is provided to increase the compression force capacity of sections. These sections are commonly used as prefabricated units.

The ACI Code, Section 8.10.4, specifies the size of isolated T-shaped sections as follows:

1. Flange thickness, t, shall be equal to or greater than one-half of the width of the web, b_w.
2. Total flange width b shall be equal to or less than four times the width of the web, b_w (Fig. 3.35).

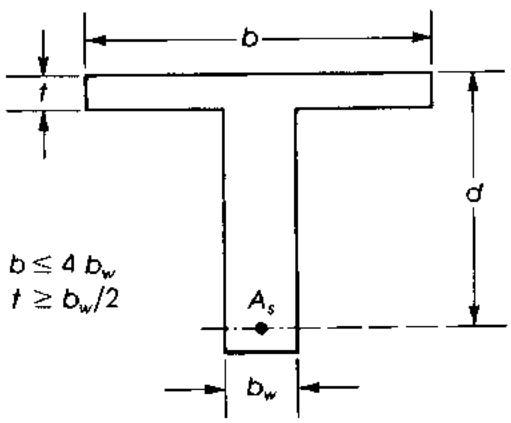


Figure 3.35 Isolated T-shaped sections.

3.17 INVERTED L-SHAPED SECTIONS

In slab-beam girder floors, the end beam is called a *spandrel beam*. This type of floor has part of the slab on one side of the beam and is cast monolithically with the beam. The section is unsymmetrical under vertical loading (Fig. 3.36*a*). The loads on slab S_1 cause torsional moment uniformly distributed on the spandrel beam B_1. Design for torsion is explained later. The over-hanging flange width $(b - b_w)$ of a beam with the flange on one side only is limited by the ACI Code, Section 8.10.2, to the smallest of the following:

1. One-twelfth of the span of the beam
2. Less than or equal to six times the thickness of the slab
3. Less than or equal to one-half the clear distance to the next beam.

If this is applied to the spandrel beam in Fig. 3.36*b*, then

1. $(b - 12) \leq (20 \times 12)/12 = 20$ in. (controls)
2. $(b - 12) \leq 6 \times 6 = 36$ in.
3. $(b - 12) \leq 3.5 \times 12 = 42$ in.

Therefore, the effective flange width is $b = 20 + 12 = 32$ in., and the effective dimensions of the spandrel beam are as shown in Fig. 3.36*d*.

3.18 SECTIONS OF OTHER SHAPES

Sometimes a section different from the previously defined sections is needed for special requirements of structural members. For instance, sections such as those shown in Fig. 3.37 may be used in the precast concrete industry. The analysis of such sections is similar to that of a rectangular section, taking into consideration the area of the removed or added concrete. The next example explains the analysis of such sections.

Example 3.13

The section shown in Fig. 3.38 represents a beam in a structure containing prefabricated elements. The total width and total depth are limited to 14 and 21 in., respectively. Tension reinforcement used is four no. 9 bars. Using $f'_c = 4$ ksi and $f_y = 60$ ksi., determine the design moment strength of the section.

Solution

1. Determine the position of the neutral axis based on $T = 4 \times 60 = 240$ K.

$$240 = 0.85 f'_c[2(4 \times 5) + 14(a - 4)]$$

where a = depth of the equivalent compressive block needed to produce a total compressive force of 240 K.

If $240 = (0.85 \times 4)(40 + 14a - 56)$, then $a = 6.18$ in. and $c = a/0.85 = 7.28$ in.

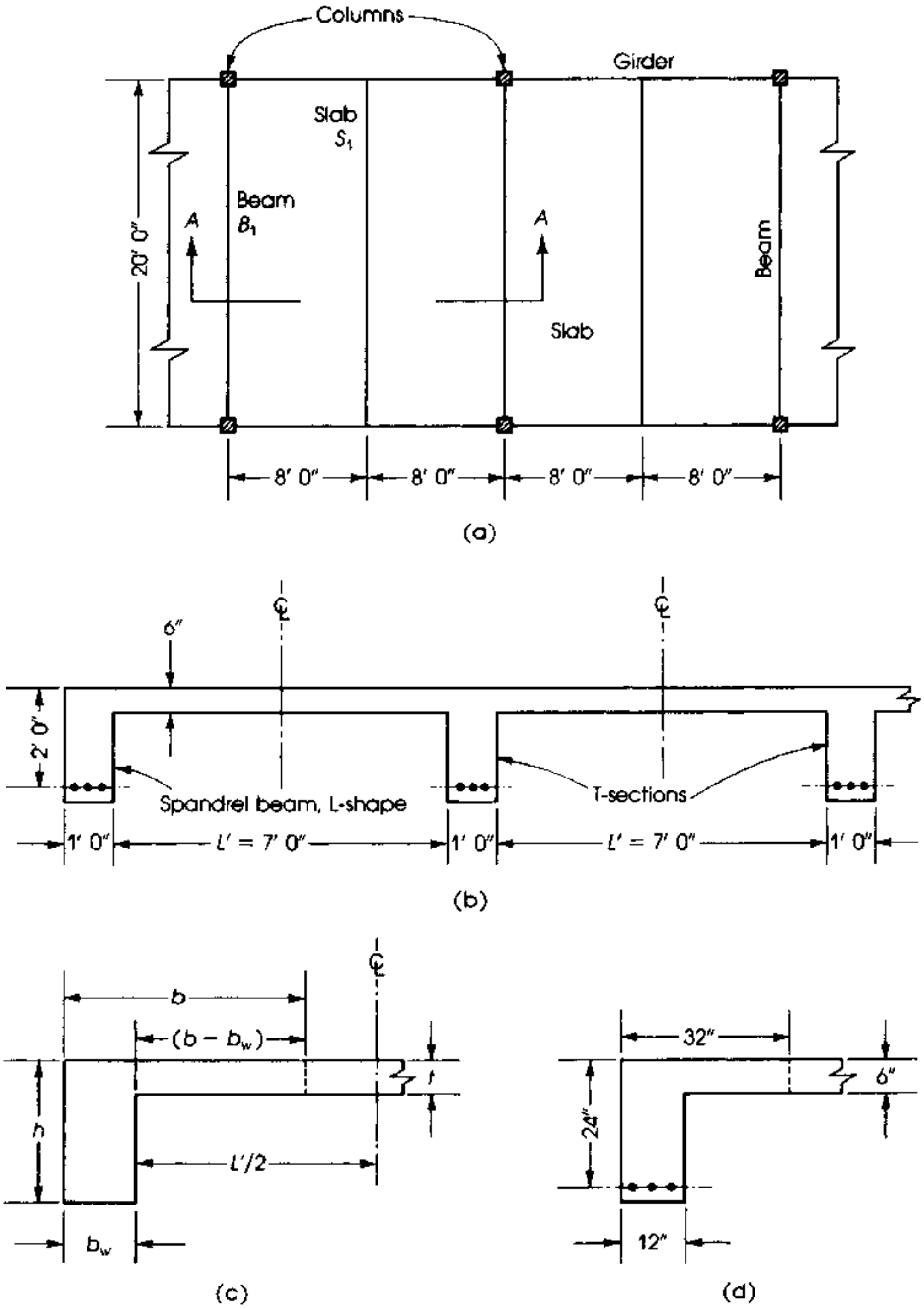


Figure 3.36 Slab-beam-girder floor, showing (*a*) plan, (*b*) section including spandrel beam, (*c*) dimensions of the spandrel beam, and (*d*) its effective flange width.

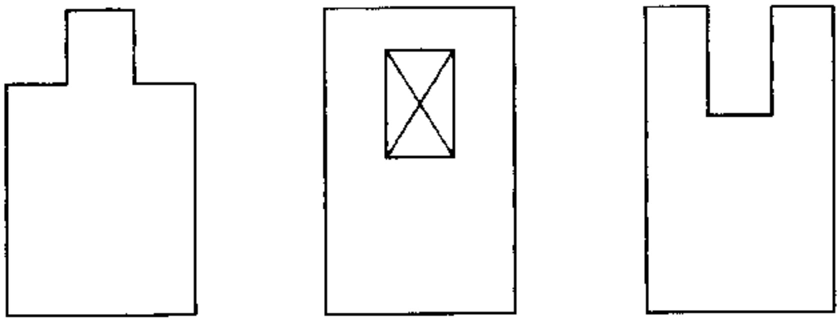


Figure 3.37 Sections of other shapes.

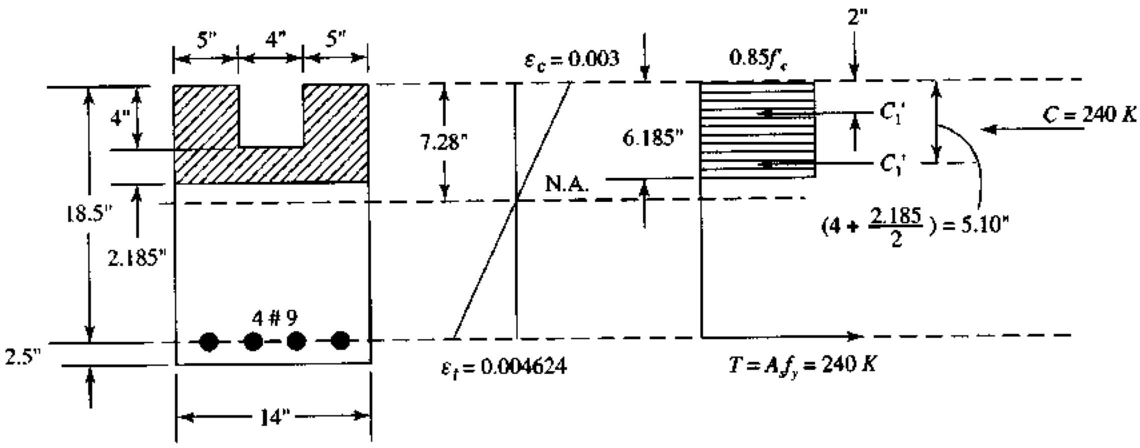


Figure 3.38 Example 3.13: (*a*) balanced and (*b*) under-reinforced sections.

2. Calculate M_n by taking moments of the two parts of the compressive forces (each by its arm), about the tension steel.

$$C_1' = \text{compressive force on the two small areas, } 4 \times 5 \text{ in.}$$
$$= 0.85 \times 4\,(2 \times 4 \times 5) = 136 \text{ K.}$$
$$C_1'' = \text{compressive force on area,} 14 \times 2.185$$
$$= 0.85 \times 4 \times 14 \times 2.185 = 104 \text{ K.}$$
$$M_n = C_1'(d - 2) + C_1''(d - 5.10)$$
$$= 136 \times 16.5 + 104 \times 13.4 = 3637.6 \text{ K·in.} = 303.1 \text{ K·ft}$$

3. Calculate $\varepsilon_t = 0.003(d_t - c)/c$, where $d_t = 18.5$ in.

$$\varepsilon_t = 0.003(18.5 - 7.28)/7.28 = 0.004624 < 0.005 \text{ but} > 0.004$$

Section is in the transition region and $\phi < 0.9$.

$$\phi = 0.48 + 83\varepsilon_t = 0.864$$
$$\phi M_n = 0.864(303.1) = 261.9 \text{ K·ft}$$

3.19 ANALYSIS OF SECTIONS USING TABLES

Reinforced concrete sections can be analyzed and designed using tables shown in Appendix A (for U.S. customary units) and Appendix B (for SI units). The tables give the value of R_u as related to the steel ratio, ρ, in addition to the maximum and minimum values for ρ and R_u. When the section dimensions are known, R_u is calculated; then ρ and A_s are determined from tables. The values in the tables are calculated based on tension-controlled sections with $\phi = 0.9$. If ϕ is less than 0.9 (transition region), the values of R_u should be multiplied by the ratio $\phi/0.9$.

$$\phi M_n = R_u bd^2 \qquad R_u = M_u/bd^2 = \phi\rho f_y[1 - \rho f_y/1.7f'_c]$$

$$A_s = \rho bd \quad \text{and} \quad \rho = A_s/bd$$

For any given value of ρ, R_u can be determined from tables. Then ϕM_n can be calculated. The values of ρ and R_u range between a minimum value of R_u (min) when ρ minimum is used, to a maximum value as limited by the ACI Code, when ρ is equal to ρ (max), for tension controlled sections with $\phi = 0.9$.

The use of tables will reduce the manual calculation time. The next example explains the use of tables.

Example 3.14

Calculate the design moment strength of the section shown in Example 3.2, Fig. 3.14 using tables. Use $b = 12$ in., $d = 21$ in., $f'_c = 3$ ksi, $f_y = 60$ ksi and three no. 9 bars.

Solution

1. Using three no. 9 bars, $A_s = 3.0\text{ in.}^2$, $\rho = A_s/bd = 3.0/(12 \times 21) = 0.0119$. From Table 3.2, $\rho_{max} = 0.01356 > \rho$ used. Therefore, $\phi = 0.9$, and it is a tension-controlled section.
   From Table A1, for $\rho = 0.0119$, $f'_c = 3$ ksi and $f_y = 60$ ksi, get $R_u = 553$ psi (by interpolation).
2. Calculate $\phi M_n = R_u bd^2 = 0.553\ (12)(21)^2 = 2926$ K·in. = 243.8 K·ft

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3.20 ADDITIONAL EXAMPLES

The following examples are introduced to enhance the understanding of the analysis and design applications.

Example 3.15

Calculate the design moment strength of the precast concrete section shown in Fig. 3.39 using $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. The section behaves as a rectangular section with $b = 14$ in., and $d = 21.5$ in. Note that the width b is that of the section on the compression side.
2. Check that $\rho = A_s/bd = 5/(14 \times 21.5) = 0.01661$, which is less than the maximum steel ratio of 0.018 for tension-controlled sections. Therefore, $\phi = 0.9$. Also $\rho > \rho_{min} = 0.00333$. Therefore, ρ is within the limits of a tension-controlled section.
3. Calculate a : $a = A_s f_y/(0.85 f'_c b) = 5 \times 60/(0.85 \times 4 \times 14) = 6.3$ in.

$$\phi M_n = \phi A_s f_y(d - a/2) = 0.9 \times 5 \times 60(21.5 - 6.3/2) = 4954.5 \text{ K·in} = 412.9 \text{ K·ft.}$$

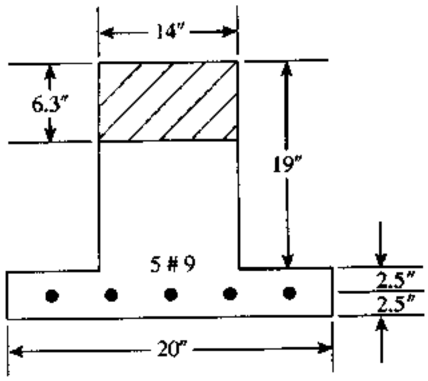

Figure 3.39 Example 3.15.

Example 3.16

A reinforced concrete beam was tested to failure and had a rectangular section, $b = 14$ in., and $d = 18.5$ in. At ultimate moment (failure), the strain in the tension steel was recorded and was equal to 0.004106. The strain in the concrete at failure may be assumed to be 0.003. If $f'_c = 3$ ksi and $f_y = 60$ ksi, it is required to:

1. Check if the tension steel has yielded.
2. Calculate the steel area provided in the section to develop the above strains. Then calculate the applied moment.
3. Calculate the design moment strength based on the ACI Code provisions. (Refer to Fig. 3.40.)

Solution

1. Check the strain in the tension steel relative to the yield strain. The yield strain $\varepsilon_y = f_y/E_s = 60/29{,}000 = 0.00207$. The measured strain in the tension steel is equal to 0.004106, which is much greater than 0.00207, indicating that the steel bars have yielded and in the elastoplastic range. The concrete strain was 0.003 indicating that the concrete has failed and started to crush. Therefore, the tension steel has yielded.
2. Calculate the depth of the neutral axis c from the strain diagram. (Fig. 3.40). From the triangles of the strain diagram,

$$c/d = 0.003/(0.003 + 0.004106) \quad \text{and} \quad c = 18.5\left(\frac{3}{7.106}\right) = 7.81 \text{ in.}$$

$$a = \beta_1 c = 0.85 \times 7.81 = 6.64 \text{ in.}$$

The compression force in the concrete, $C_c = 0.85\ f'_c ab = 0.85 \times 3 \times 6.64 \times 14 = 237$ K. The tension steel $A_s = C_c/f_y = 237/60 = 3.95\text{ in.}^2$ (section has five no. 8 bars).

$$M_n = A_s f_y\left(d - \frac{a}{2}\right) = 3.95 \times 60(18.5 - 6.64/2) = 3597.6 \text{ K·in} = 299.8 \text{ K·ft}$$

3. Check $\varepsilon_t = 0.003(d_t - c)/c$.

$$c = 7.81 \text{ in.}, d_t = h - 2.5 \text{ in.} = 22 - 2.5 = 19.5 \text{ in.}$$

$\varepsilon_t = 0.003(19.5 - 7.81)/7.81 = 0.0049$, which is less than 0.005 for tension-controlled sections, but greater than 0.004. Section is in the transition region, and $\phi < 0.9$.

$$\phi = 0.48 + 83\varepsilon_t = 0.853$$

The allowable design moment $= \phi M_n = 0.863 \times 299.8 = 255.6$ K·ft.

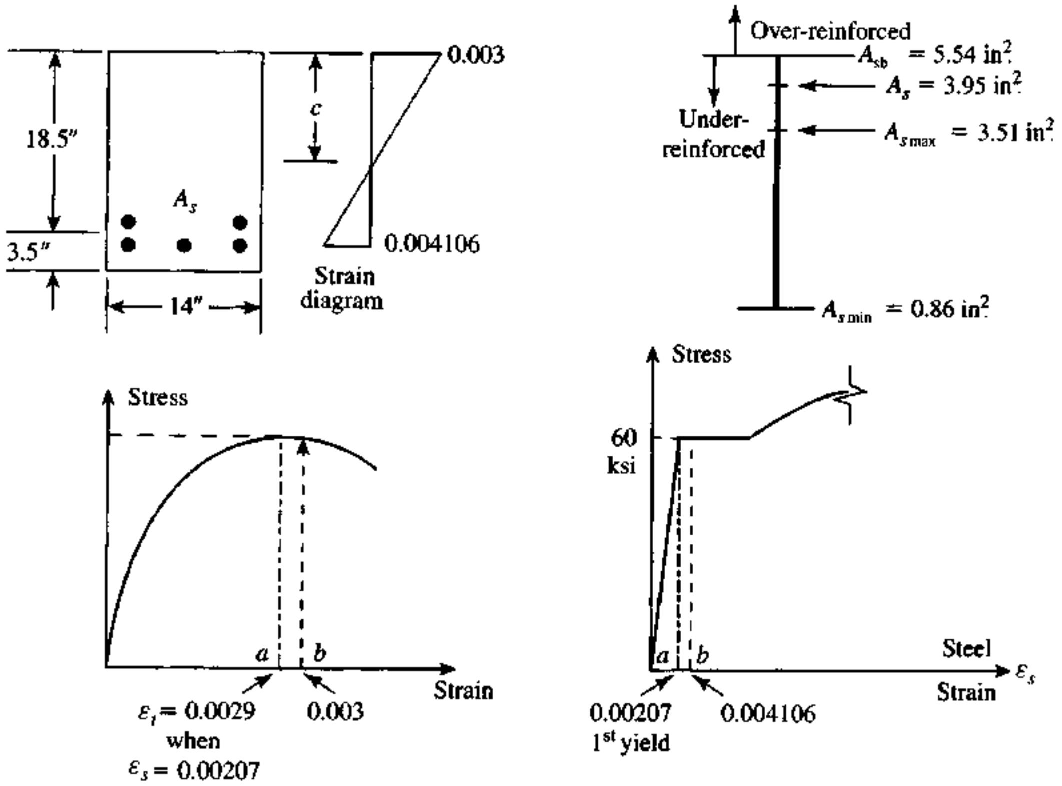

Figure 3.40 Example 3.16.

Discussion

From Table 3.2, $\rho_b = 0.0214$ and $\rho_{\max} = 0.01356$. For comparison, A_s (max) $= 0.01356(14 \times 18.5) = 3.51\text{ in.}^2$ for $\phi = 0.9$, and A_s (balanced) $= 5.54\text{ in.}^2$. The ratio of $A_s/A_{s\max} = 3.95/3.51 = 1.125$ and $A_s/A_{sb} = 0.713$. If $A_s = A_{\max} = 3.51\text{ in.}^2$ is used with $\phi = 0.9$, then

$$a = 3.51 \times 60/(0.85 \times 3 \times 14) = 5.9\text{ in.}$$

and

$$\phi M_n = 0.9 \times 3.51 \times 60(18.5 - 5.9/2) = 2947.2\text{ K·in.} = 245.6\text{ K·ft.}$$

which is equal to 96% of the moment calculated above. Figure 3.40 shows the behavior of the tested beam.

3.21 EXAMPLES USING SI UNITS

The following equations are some of those mentioned in this chapter but converted to SI units. The other equations, which are not listed here, can be used for both U.S. Customary and SI units. Note that f_c' and f_y are in MPa (N/mm^2).

$$\rho_b = 0.85\beta_1(f_c'/f_y)[600/(600 + f_y)] \tag{3.18}$$

For tension-controlled condition,

$$\rho_{max} = (0.003 + f_y/E_s)\rho_b/0.008 \tag{3.31}$$

$$(\rho - \rho') \geq 0.85\beta_1(f'_c/f_y)(d'/d)[600/(600 - f_y)] = K \tag{3.49}$$

Example 3.17

Determine the design moment strength and the position of the neutral axis of a rectangular section that has $b = 300$ mm, $d = 500$ mm, and is reinforced with five 20-mm-diameter bars. Given $f'_c = 20$ MPa and $f_y = 400$ MPa.

Solution

1. Area of five 20-mm bars is 1570 mm^2.

$$\rho = A_s/bd = 1570/(300 \times 500) = 0.01047 \quad \rho_{min} = 1.4/f_y = 0.0035$$

For $f'_c = 20$ MPa and $f_y = 400$ MPa, $\rho_b = 0.0217$ and $\rho_{max} = 0.01356$. Note that $E_s = 200{,}000$ MPa and $f_y/E_s = 0.002$. Because $\rho < \rho_{max}$, it is a tension-controlled section with $\phi = 0.9$. Also $\rho > \rho_{min}$.

2. Calculate the design moment strength:

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$a = A_s f_y/(0.85 f'_c b) = 1570 \times 400/(0.85 \times 20 \times 300) = 123 \text{ mm}$$

$$\phi M_n = 0.9 \times 1570 \times 400 \left(500 - \frac{123}{2}\right) \times 10^{-6} = 247.8 \text{ KN·m}$$

Note that the moment was multiplied by 10^{-6} to get the answer in KN·m. The distance to the neutral axis from the compression fibers $(c) = a/\beta_1$, where $\beta_1 = 0.85$ for $f'_c < 30$ MPa. Therefore, $c = 123/0.85 = 145$ mm.

Example 3.18

A 2.4-m-span cantilever beam has a rectangular section with $b = 300$ mm, $d = 490$ mm, and is reinforced with three bars, 25 mm in diameter. The beam carries a uniform dead load (including its own weight) of 25.5 KN/m and a uniform live load of 32 KN/m. Check the adequacy of the section if $f'_c = 30$ MPa and $f_y = 400$ MPa.

Solution

1. $U = 1.2D + 1.6L = 1.2 \times 25.5 + 1.6 \times 32 = 81.8$ KN/m. External factored moment $= M_u = UL^2/2 = 81.8(2.4^2)/2 = 235.6$ KN·m.
2. Calculate the design moment strength:

$$A_s = 1470 \text{ mm}^2 \quad \rho = A_s/bd = 1470/(300 \times 490) = 0.01$$

$$\rho_b = 0.85\beta_1(f'_c/f_y)[600/(600 + f_y)] = 0.0325$$

$$\rho_{max} = (0.005/0.008)\rho_b = \left(\frac{5}{8}\right)(0.0325) = 0.0203, \; \rho_{min} = \frac{1.4}{400} = 0.0035$$

Since $\rho < \rho_{max}$ but $> \rho_{min}$, it is a tension-controlled section and $\phi = 0.9$. $a = A_s f_y/(0.85 f'_c b) = 1470 \times 400/(0.85 \times 30 \times 300) = 77$ mm, $c = 90$ mm. $\phi M_n = \phi A_s f_y(d - a/2) = 0.9 \times 1470 \times 400(490 - 77/2) \times 10^{-6} = 238.9$ KN·m. $\varepsilon_t = 0.003(d_t - c)/c = 0.003\,(490 - 90)/90 = 0.01333 > 0.005$, $\phi = 0.9$ as assumed.

3. The internal design moment strength is greater than the external factored moment. Therefore, the section is adequate.

Example 3.19

Calculate the design moment strength of a rectangular section with the following details: $b = 250$ mm, $d = 440$ mm, $d' = 60$ mm, tension steel is six bars 25 mm in diameter (in two rows), compression steel is three bars 20 mm in diameter, $f'_c = 20$ MPa and $f_y = 350$ MPa.

Solution

1. Check if compression steel yields:

$$A_s = 490 \times 6 = 2940 \text{ mm}^2, \quad A'_s = 314 \times 3 = 942 \text{ mm}^2 \quad A_s - A'_s = 1998 \text{ mm}^2$$

$$\rho = 2940/(250 \times 440) = 0.0267 \quad \rho' = 942/(250 \times 440) = 0.00856$$

$$\rho - \rho' = 0.01814.$$

For compression steel to yield:

$$(\rho - \rho') \geq 0.85 \times 0.85 \times (20/350)(60/440)(600/600 - 350) = 0.01351$$

$$(\rho - \rho') = 0.01814 > 0.01351. \text{ Therefore, compression steel yields.}$$

2. Calculate M_n:

$$a = (A_s - As')/0.85 f'_c b = 1998/(0.85 \times 20 \times 250) = 164 \text{ mm}$$

$$M_n = \left[1998 \times 350\left(440 - \frac{164}{2}\right) + 942 \times 350(440 - 60)\right] \times 10^{-6} = 417.3 \text{ KN·m}$$

3. Check ϕ based on $\varepsilon_t \geq 0.005$.

$$\varepsilon_t = 0.003(d_t - c)/c \quad a = 164 \text{ mm} \quad c = 164/0.85 = 193 \text{ mm}$$

$$d_t = h - 65 \text{ mm} = d + 25 \text{ mm for two rows of tension bars.}$$

$$d_t = 440 + 25 = 465 \text{ mm}$$

$\varepsilon_t = 0.003(465 - 193)/193 = 0.04228$, which is less than 0.005, but greater than the 0.004 limit. $\phi = 0.48 + 83 \times \varepsilon_t, = 0.831$, and $\phi M_n = 0.831\ (417.3) = 346.8$ KN·m.

SUMMARY

Flow charts for the analysis of sections are given on www.wiley.com/college/hassoun.

Section 3.1–3.8

1. The type of failure in a reinforced concrete flexural member is based on the amount of tension steel used, A_s.
2. Load factors for dead and live loads are $U = 1.2D + 1.6L$. Other values are given in the text.
3. The reduction strength factor for beams $(\phi) = 0.9$ for tension controlled sections with $\varepsilon_t \geq 0.005$.
4. An equivalent rectangular stress block can be assumed to calculate the design moment strength of the beam section, ϕM_n.
5. Design provisions are based on four conditions, Section 3.5

Section 3.9–3.13: Analysis of a Singly Reinforced Rectangular Section

Given: f'_c, f_y, b, d, and A_s. Required: the design moment strength, ϕM_n.

To determine the design moment strength of a singly reinforced concrete rectangular section,

1. Calculate the compressive force, $C = 0.85 f'_c ab$ and the tensile force, $T = A_s f_y$. Calculate $a = A_s f_y/(0.85 f'_c b)$. Calculate $\phi M_n = \phi C(d - a/2) = \phi T(d - a/2) = \phi A_s f_y(d - a/2)$. Check $\varepsilon_t = 0.003(d_t - c)/c \geq 0.005$ for $\phi = 0.9$ (tension-controlled section). (See Section 3.6.)
2. Calculate the balanced, maximum, and minimum steel ratios:

$$\rho_b = 0.85\beta_1(f'_c/f_y)[87/(87+f_y) \qquad \rho_{\max} = (0.003 + f_y/E_s)\rho_b/0.008$$

$$\rho_{\min} = 0.2/f_y \text{ for } f'_c \leq 4.5 \text{ ksi}$$

   (where f'_c and f_y are in ksi. (See Section 3.9.2.) The steel ratio in the section is $\rho = A_s/bd$. Check that $\rho_{\min} \leq \rho \leq \rho_{\max}$.
3. Another form of the design moment strength is

$$M_n = \rho f_y(bd^2)(1 - \rho f_y/1.7 f'_c) = R_n bd^2$$

$$R_n = \rho f_y[1 - (\rho f_y/1.7 f'_c)] \quad \text{and} \quad R_u = \phi R_n$$

4. For $f_y = 60$ ksi and $f'_c = 3$ ksi (Table 3.2), $\rho_{\max} = 0.01356$, $\rho_{\min} = 0.00333$, $R_n = 686$ psi, and $R_u = 615$ psi.

   For $f_y = 60$ ksi and $f_c' = 4$ ksi, $\rho_{\max} = 0.01806$, $\rho_{\min} = 0.00333$, $R_n = 911$ psi, and $R_u = 820$ psi.

Section 3.14: Analysis of Rectangular Section with Compression Steel

Given: b, d, d', A_s, A'_s, f'_c, and f_y. Required: the deign moment strength, ϕM_n.

1. Calculate $\rho = A_s/bd$, $\rho' = A_s/bd$, and $(\rho - \rho')$.
2. Calculate ρ_b, $\rho_{\max}$, and $\rho_{\min}$ as given above (or see Section 3.10)
3. Calculate $K = 0.85\beta_1(f'_c/f_y)(d'/d)[87/(87 - f_y)]$. ($f'_c$ and f_y are in ksi.)
4. When compression steel yields,
   a. Check that $\rho \geq \rho_{\min}$.
   b. Check that $(\rho - \rho') \geq K$ for compression steel to yield. If not, then compression steel does not yield.
   c. If compression steel yields, then $f'_s = f_y$.
   d. Check that $\rho \leq (\rho_{\max} + \rho')$ or $(\rho - \rho') \leq \rho_{\max}$.
   e. Calculate $a = (A_s - A'_s)f_y/(0.85 f'_c b)$.
   f. Calculate $\phi M_n = \phi(A_s - A'_s)f_y(d - a/2) + \phi A'_s f_y(d - d')$.
   g. If $(\rho - \rho') > \rho_{\max}$ but $< \rho_{\max t}$ (for the transition region), then $\phi < 0.9$ for M_{u1} and $\phi = 0.9$ for M_{u2} (Eq. 3.44 a).
5. When compression steel does not yield,
   a. Compression steel does not yield when $(\rho - \rho') < K$. The value of f'_s is not known.
   b. Calculate c = the distance to the neutral axis from the compression fibers as follows:

$$A_1 c^2 + A_2 c + A_3 = 0,$$

where

$$A_1 = 0.85f'_c\beta_1 b$$
$$A_2 = A'_s(87 - 0.85f_c) - A_s f_y$$
$$A_3 = -87A'_s d'.$$

Solve for c.

An alternative solution to calculate c is as follows:

$$C + C' = T$$

$$C = 0.85f'_c(\beta_1 cb - A'_s) \quad C' = A'_s[87(c - d')/c] - 0.85f'_c A'_s$$

and

$$T = A_s f_y$$

Solve for c.

c. Calculate $f'_s = 87(c - d')/c \le f_y$(in ksi).

d. Check that $\rho \le [\rho_{\max} + \rho'(f'_s/f_y)]$ or $A_s \le [\rho_{\max}(bd) + A'_s(f'_s/f_y)]$.

e. Calculate a:

$$a = (A_s f_y - A'_s f'_s)/(0.85f'_c b) \quad \text{or} \quad a = \beta_1 c$$

f. Calculate ϕM_n:

$$\phi M_n = \phi[(A_s f_y - A'_s f'_s)(d - a/2) + A'_s f'_s(d - d')]$$

Note that $(A_s f_y - A'_s f_y) = A_{s1} = A_s - A_{s2} = A_s - (A'_s f'_s/f_y)$ and $A_{s2}f_y = A'_s f'_s$. Also, $a = A_{s1}f_y/(0.85f'_c b)$

Sections 3.15–3.17: Analysis of T-Sections

Given: f'_c, f_y, A_s, and section dimensions. Required: design moment strength, ϕM_n. Two possible cases may develop. (Determine the effective flange width, b_e, first.)

Case 1

1. If $a \le t$ (the slab thickness), then it is a T-section shape but acts as a singly reinforced rectangular section using $b = b_e$ (the flange effective width) to calculate ϕM_n.

$$a' = A_s f_y/(0.85f'_c b_e) \le t$$

   Or, check that A_c (the area of concrete in compression) $= A_s f_y/(0.85f'_c) \le bt$. If $A_c \ge bt$, then it is a T-section analysis.
2. If $a' \le t$ or $A_c \le bt$, then $a' = a$ and $\phi M_n = \phi A_s f_y(d - a/2)$.
3. Check that ρ_w (steel ratio in web) $= A_s/b_w d \ge \rho_{\min}$.
4. Check that $A_s \le A_{s\max}$ from Eq. 3.71. (Normally, this is o.k. for this case.)

$$A_{s\max} = 0.6375(f'_c/f_y)[t(b - b_w) + (0.375)b_w\beta_1 d]$$

5. Check that $\varepsilon_t \ge 0.005$ for $\phi = 0.9$. (Normally this is o.k. for this case.)
6. The effective flange width $b = b_e$ is the smallest of
   a. Span/4

b. Center to center of adjacent slabs
c. $(b_w + 16t)$, where t = slab thickness

Case 2

1. When $a > t$ or $A_c > bt$, it is a T-section analysis.
2. For the flange, $C_f = 0.85 f'_c t(b - b_w) = A_{sf} f_y$, calculate $A_{sf} = C_f / f_y$.
3. For the web,

$$A_{sw} = \text{tension steel in the web} = A_s - A_{sf}$$

$$a = (A_s - A_{sf}) f_y / (0.85 f'_c b_w)$$

$$C_w(\text{web}) = 0.85 f'_c a b_w = A_{sw} f_y$$

4. $$\begin{aligned}\phi M_n &= \phi[M_w(\text{web}) + M_f(\text{flange})] = \phi[C_w(d - a/2) + C_f(d - t/2)] \\ &= \phi[0.85 f'_c a b_w (d - a/2) + 0.85 f'_c t (b - b_w)(d - t/2)] \\ &= \phi[(A_s - A_{sf}) f_y (d - a/2) + A_{sf} f_y (d - t/2)]\end{aligned}$$

5. Check that $\varepsilon_t \geq 0.005$ for tension-controlled section and $\phi = 0.9$. (See Example 3.12).
6. Check that $A_{s\,\min} \leq A_s \leq A_{s\,\max}$. (See case 1.)

Sections 3.18–3.21

1. Analysis of nonuniform sections is explained in Example 3.13.
2. Tables in Appendix A may be used for the analysis of rectangular sections.
3. Examples in SI units are introduced.

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PROBLEMS

3.1 Singly reinforced rectangular sections. Determine the design moment strength of the sections given in the following table, knowing that $f'_c = 4$ ksi and $f_y = 60$ ksi. (Answers are given in the right column.)

No.	b (in.)	d (in.)	A_s (in.^{2})	ϕM_n (K·ft)
a	14	22.5	5.08 (4 no. 10)	441.2
b	18	28.5	7.62 (6 no. 10)	849.1
c	12	23.5	4.00 (4 no. 9)	370.1
d	12	18.5	3.16 (4 no. 8)	230.0
e	16	24.5	6.35 (5 no. 10)	600
f	14	26.5	5.00 (5 no. 9)	525.3
g	10	17.5	3.00 (3 no. 9)	200.5
h	20	31.5	4.00 (4 no. 9)	535.2

For problems in SI units, 1 in. = 25.4 mm, 1 in.^{2} = 645 mm^2, 1 ksi = 6.9 MPa (N/mm^2), and 1 M_u (K·ft) = 1.356 kN·m.

3.2 Rectangular section with compression steel. Determine the design moment strength of the sections given in the following table, knowing that $f'_c 4$ ksi, $f_y = 60$ ksi, and $d' = 2.5$ in. (Answers are given in the right column. In the first four problems, $f'_s = f_y$)

No.	b (in.)	d (in.)	A_s (in.^{2})	A'_s(in.^{2})	ϕM_n (K·ft)
a	15	22.5	8.0 (8 no. 9)	2.0 (2 no. 9)	692.2
b	17	24.5	10.08 (8 no. 10)	2.54 (2 no. 10)	950
c	13	22	7.00 (7 no. 9)	1.8 (3 no. 7)	590.2
d	10	21.5	5.08 (4 no. 10)	1.2 (2 no. 7)	464.7
e	14	20.5	7.62 (6 no. 10)	2.54 (2 no. 10)	597.9
f	16	20.5	9.0 (9 no. 9)	4.0 (4 no. 9)	716.3
g	20	18.0	12.0 (12 no. 9)	6.0 (6 no. 9)	820.3
h	18	20.5	10.16 (8 no. 10)	5.08 (4 no. 10)	813.7

For problems in SI units: 1 in. = 25.4 mm, 1 in.^{2} = 645 mm^2, 1 ksi = 6.9 MPa (N/mm^2), and 1 M_u(K·ft) = 1.356 kN·m.

3.3 T-sections. Determine the design moment strength of the T-sections given in the following table, knowing that $f'_c = 3$ ksi and $f_y = 60$ ksi. (Answers are given in the right column. In the first three problems, $a < t$.)

No.	b (in.)	b_w (in.)	t (in.)	d (in.)	A_s (in.^{2})	ϕM_n (K·ft)
a	54	14	3	17.5	5.08 (4 no. 10)	374.8
b	48	14	4	16.5	4.0 (4 no. 9)	279.4
c	72	16	4	18.5	10.16 (8 no. 10)	769.9
*d	32	16	3	15.5	6.0 (6 no. 9)	N.G.
e	44	12	4	20.5	8.0 (8 no. 9)	660.1
f	50	14	3	16.5	7.0 (7 no. 9)	466.8
g	40	16	3	16.5	6.35 (5 no. 10)	415.0
h	42	12	3	17.5	6.0 (6 no. 9)	425.8

For problems in SI units: 1 in. = 25.4 mm, 1 in.^{2} = 645 mm^2, 1 ksi = 6.9 MPa (N/mm^2), and 1 M_u (K·ft) = 1.356 kN·m.
*Answer = 325.5 K·ft if ρ_{max} is used.

3.4 Calculate ρ_b, ρ_{max}, R_u(max), R_u, a/d, and max (a/d) for a rectangular section that has a width of b = 12 in. (300 mm) and an effective depth of d = 20 in. (500 mm) for the following cases:

a. $f'_c = 3$ ksi, $f_y = 40$ ksi, A_s = four no. 8 bars
b. $f'_c = 4$ ksi, $f_y = 60$ ksi, A_s = four no. 7 bars
c. $f'_c = 4$ ksi, $f_y = 75$ ksi, A_s = four no. 9 bars
d. $f'_c = 5$ ksi, $f_y = 60$ ksi, A_s = four no. 9 bars
e. $f'_c = 30$ MPa, $f_y = 400$ MPa, $A_s = 3 \times 30$ mm
f. $f'_c = 20$ MPa, $f_y = 300$ MPa, $A_s = 3 \times 25$ mm
g. $f'_c = 30$ MPa, $f_y = 500$ MPa, $A_s = 4 \times 25$ mm
h. $f'_c = 25$ MPa, $f_y = 300$ MPa, $A_s = 4 \times 20$ mm

3.5 Using the ACI Code requirements, calculate the design moment strength of a rectangular section that has a width of b = 250 mm (10 in.) and an effective depth of d = 550 mm (22 in.) when $f'_c = 20$ MPa (3 ksi), $f_y = 420$ MPa (60 ksi), and the steel used is as follows:

a. 4 × 20 mm   b. 3 × 25 mm   c. 4 × 30 mm
d. 2 no. 9 bars   e. 6 no. 9 bars

3.6 A reinforced concrete simple beam has a rectangular section with a width of b = 8 in. (200 mm) and effective depth of d = 18 in. (450 mm). At design moment (failure), the strain in the steel was recorded and was equal to 0.0015. (The strain in concrete at failure may be assumed to be 0.003.) Use $f'_c = 3$ ksi (20 MPa) and $f_y = 50$ ksi (350 MPa) for all parts.

a. Check if the section is balanced, under-reinforced, or over-reinforced.
b. Determine the steel area that will make the section balanced.
c. Calculate the steel area provided in the section to produce the aforementioned strains, and then calculate its moment. Compare this value with the design moment strength allowed by the ACI Code using ρ_{max}.
d. Calculate the design moment strength of the section if the steel percentage used is $\rho = 1.4\%$.

3.7 A 10-ft.- (3-m-)span cantilever beam has an effective cross-section (bd) of 12 in. by 24 in. (300 by 600 mm) and is reinforced with five no. 8 (5 × 25 mm) bars. If the uniform load due to its own weight and the dead load are equal to 685 lb/ft (10 kN/m), determine the allowable uniform live load on the beam using the ACI load factors. Given: $f'_c = 3$ ksi (20 MPa) and $f_y = 60$ ksi (400 MPa).

3.8 The cross-section of a 17-ft- (5-m-) span simply supported beam is 10 by 28 in. (250 by 700 mm), and it is reinforced symmetrically with eight no. 6 bars (8 × 20 mm) in two rows. Determine the allowable concentrated live load at midspan considering the total acting dead load (including self-weight) is equal to 2.55 K/ft (37 kN/m). Given: $f'_c = 3$ ksi (20 MPa) and $f_y = 40$ ksi (300 MPa).

3.9 Determine the design moment strength of the sections shown in Fig. 3.41. Neglect the lack of symmetry in (b). Given: $f'_c = 4$ ksi (30 MPa) and $f_y = 60$ ksi (400 MPa).

3.10 A rectangular concrete section has a width of $b = 12$ in. (300 mm), an effective depth of $d = 18$ in. (450 mm), and $d' = 2.5$ in. (60 mm). If compression steel consisting of two no. 7 bars (2 × 20 mm) is used, calculate the allowable moment strength that can be applied on the section if the tensile steel, A_s, is as follows:

a. Four no. 7 (4 × 20 mm) bars **b.** Eight no. 7 (8 × 20 mm) bars
Given: $f'_c = 3$ ksi (20 MPa) and $f_y = 40$ ksi (300 MPa).

3.11 A 16-ft- (4.8-m-) span simply supported beam has a width of $b = 12$ in. (300 mm), $d = 22$ in. (500 mm), $d' = 2.5$ in. (60 mm), and A'_s = three no. 6 bars (3 × 20 mm). The beam carries a uniform dead load of 2 K/ft (30 kN/m), including its own weight. Calculate the allowable uniform live load that can be safely applied on the beam. Given: $f'_c = 4$ ksi (20 MPa) and $f_y = 60$ ksi (400 MPa). (Hint: Use ρ_{max} for the basic section to calculate M_u.)

3.12 Check the adequacy of a 10-ft- (3-m-)span cantilever beam, assuming a concrete strength of $f'_c = 4$ ksi (30 MPa) and a steel yield strength of $f_y = 60$ ksi (400 MPa) are used. The dimensions of the beam section are $b = 10$ in. (250 mm), $d = 20$ in. (500 mm), $d' = 2.5$ in. (60 mm), A_s = six no. 7 bars (6 × 20 mm), A'_s = two no. 5 bars (2 × 15 mm). The dead load on the beam, excluding its own weight, is equal to 2 K/ft (30 kN/m), and the live load equals 1.25 K/ft (20 kN/m). (Compare the internal M_u with the external factored moment.)

3.13 A series of reinforced concrete beams spaced at 9 ft (2.7 m) on centers are acting on a simply supported span of 18 ft (5.4 m). The beam supports a reinforced concrete floor slab 4 in. (100 mm) thick. If the width of the web is $b_w = 10$ in. (250 mm), $d = 18$ in. (450 mm), and the beam is reinforced with three

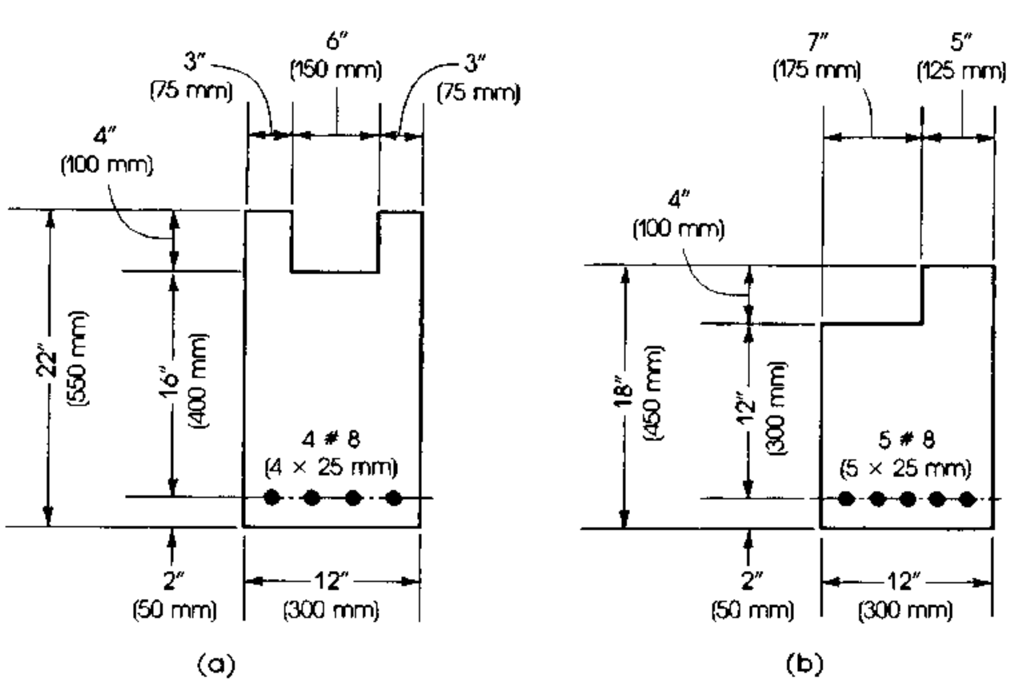


Figure 3.41 Problem 3.9.

no. 9 bars (3 × 30 mm), determine the moment strength of a typical interior beam. Given: $f_c' = 4$ ksi (30 MPa) and $f_y = 60$ ksi (400 MPa).

3.14 Calculate the design moment strength of a T-section that has the following dimensions:

- Flange width = 30 in. (750 mm)
- Flange thickness = 3 in. (75 mm)
- Web width = 10 in. (250 mm)
- Effective depth (d) = 18 in. (450 mm)
- Tension reinforcement: six no. 8 bars (6 × 25 mm)
- $f_c' = 3$ ksi (20 MPa)
- $f_y = 60$ ksi (400 MPa)

3.15 Repeat Problem 3.14 if $d = 24$ in. (600 mm).

3.16 Repeat Problem 3.14 if the flange is an inverted L shape with the same flange width projecting from one side only. (Neglect lack of symmetry.)

CHAPTER 4

FLEXURAL DESIGN OF REINFORCED CONCRETE BEAMS

Reinforced conrete office building, Amman, Jordan.

4.1 INTRODUCTION

In the previous chapter, the analysis of different reinforced concrete sections was explained: Details of the section were given, and we had to determine the design moment of the section. In this chapter, the process is reversed: The external moment is given, and we must find safe, economic, and practical dimensions of the concrete section and the area of reinforcing steel that provide adequate internal moment strength.

4.2 RECTANGULAR SECTIONS WITH REINFORCEMENT ONLY

From the analysis of rectangular singly reinforced sections (Section 3.9), the following equations were derived for tension-controlled sections, where f_c' and f_y are in ksi:

$$\rho_b = 0.85\ \beta_1\ \frac{f_c'}{f_y}\left(\frac{87}{87+f_y}\right) \tag{3.18}$$

$$\rho_{\max} = \rho_b\left(\frac{0.003+\dfrac{f_y}{E_s}}{0.008}\right) \tag{3.31}$$

For $f_y = 60$ ksi,

$$\rho_{\max} = 0.63375\rho_b\ (\text{or } 0.634\rho_b) \tag{3.32}$$

Also,

$$\text{for} f_y = 40 \text{ ksi}, \rho_{\max} = 0.5474\ \rho_b$$

$$\text{for} f_y = 50 \text{ ksi}, \rho_{\max} = 0.5905\ \rho_b$$

$$\text{for} f_y = 75 \text{ ksi}, \rho_{\max} = 0.6983\ \rho_b$$

The value of β_1 in ρ_b is 0.85 when $f_c' \leq 4000$ psi (30 N/mm^2) and decreases by 0.05 for every increase of 1000 psi (7 N/mm^2) in concrete strength. The steel percentage of balanced section, ρ_b, and the maximum allowable steel percentage, $\rho_{\max}$, can be calculated for different values of f_c' and f_y, as shown in Table 4.1 or Table A.4 in Appendix A.

It should be clarified that the designer has a wide range of choice between a large concrete section and relatively small percentage of steel, ρ, producing high ductility and a small section with a high percentage of steel with low ductility. A high value of the net tensile strain, ε_t, indicates a high ductility and a relatively low percentage of steel. The limit of the net tensile strain for tension-controlled sections is 0.005, with $\phi = 0.9$. The strain limit of 0.004 can be used with a reduction in ϕ. If the ductility index is represented by the ratio of the net tensile strain, ε_t, to the yield strain, $\varepsilon_y = f_y/E_s$, the relationship between ε_t, ρ/ρ_b, ϕ, and $\varepsilon_t/\varepsilon_y$ is shown in Table 4.2 for $f_y = 60$ ksi. Also, the ACI Code, Section 8.4, indicates that ε_t should be ≥ 0.0075 for the redistribution of moments in continuous flexural members producing a ductility index of 3.75. It can be seen that adopting $\varepsilon_t \geq 0.005$ is preferable to the use of a higher steel ratio, ρ/ρ_b, with $\varepsilon_t = 0.004$, because the increase in M_n is offset by a lower ϕ. The value of $\varepsilon_t = 0.004$ represents the use of minimum steel percentage of 0.00333 for $f_c' = 4$ ksi and $f_y = 60$ ksi. This case should be avoided. The value of ϕ between $\varepsilon_t = 0.005$ and 0.004 can be calculated from Eq. 3.8: $\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right)$.

Table 4.1 Suggested Design Steel Ratios, ρ_s

f_c' (ksi)	f_y (ksi)	% ρ_b	% $\rho_{\max}$	% ρ_s	Ratio ρ_s/ρ_b	Ratio $\rho_s/\rho_{\max}$	R_{us} (psi)	$R_{u\max}$ (psi)
3	40	3.71	2.031	1.4	0.38	0.69	450	614
	60	2.14	1.356	1.2	0.56	0.89	556	615
4	60	2.85	1.806	1.4	0.49	0.78	662	820
	75	2.07	1.445	1.2	0.58	0.83	702	820
5	60	3.35	2.123	1.6	0.48	0.75	766	975
	75	2.43	1.700	1.4	0.58	0.82	830	975

Table 4.2 Relation Between ε_t, ρ/ρ_b, ϕ, and $\varepsilon_t/\varepsilon_y$ ($f_y = 60$ ksi)

ε_t	0.004	0.005	0.006	0.007	0.0075	0.008	0.009	0.010	0.040
$\frac{\rho}{\rho_b}$	0.714	0.625	0.555	0.500	0.476	0.454	0.417	0.385	0.117
$\frac{\varepsilon_t}{\varepsilon_y}$	2.0	2.5	3.0	3.5	3.75	4.0	4.5	5.0	20
ϕ	0.82	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9

The design moment equations were derived in the previous chapter in the following forms:

$$\phi M_n = M_u = R_u bd^2 \tag{3.21}$$

where

$$R_u = \phi\rho f_y \left(1 - \frac{\rho f_y}{1.7 f'_c}\right) = \phi R_n \tag{3.22}$$

where $\phi = 0.9$, for tension-controlled sections and less than 0.9 for sections in the transition region.

$$\phi M_n = M_u = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b}\right) \tag{3.19a}$$

Also,

$$\phi M_n = M_u = \phi\rho f_y bd^2 \left(1 - \frac{\rho f_y}{1.7 f'_c}\right) \tag{3.20}$$

We can see that for a given factored moment and known f'_c and f_y, there are three unknowns in these equations: the width, b, the effective depth of the section, d, and the steel ratio, ρ. A unique solution is not possible unless values of two of these three unknowns are assumed. Usually ρ is assumed (using $\rho_{\max}$, for instance), and b can also be assumed.

Based on the preceding discussion, the following cases may develop for a given M_u, f'_c, and f_y:

1. If ρ is assumed, then R_u can be calculated from Eq. 3.22, giving $bd^2 = M_u/R_u$. The ratio of d/b usually varies between 1 and 3, with a practical ratio of 2. Consequently, b and d can be determined, and $A_s = \rho bd$. The ratio ρ for a singly reinforced rectangular section must be equal to or less than $\rho_{\max}$, as given in Eq. 3.31. It is a common practice to assume a value of ρ that ranges between $\frac{1}{2}\rho_{\max}$ and $\frac{1}{2}\rho_b$. Table 4.1 gives suggested values of the steel ratio ρ to be used in singly reinforced sections when ρ is not assigned. For example, if $f_y = 60$ ksi, the value $\rho_s = 1.4\%$ is suggested for $f'_c = 4$ ksi 1.6% for $f'_c = 5$ ksi and 1.2% for $f'_c = 3$ ksi. The designer may use ρ up to $\rho_{\max}$, which produces the minimum size of the singly reinforced concrete section. Using ρ_{min} will produce the maximum concrete section. If b is assumed in addition to ρ, then d can be determined as follows:

$$d = \sqrt{\frac{M_u}{R_u b}} \tag{4.1}$$

   If $d/b = 2$, then $d = \sqrt[3]{2M_u/R_u}$ and $b = d/2$, rounded to the nearest higher inch.

2. If b and d are given, then the required reinforcement ratio ρ can be determined by rearranging Eq. 3.20 to obtain

$$\rho = \frac{0.85 f'_c}{f_y}\left[1 - \sqrt{1 - \frac{4M_u}{1.7\phi f'_c bd^2}}\right] \tag{4.2}$$

$$= \frac{0.85f'_c}{f_y}\left[1 - \sqrt{1 - \frac{2R_n}{0.85f'_c}}\right] \tag{4.2a}$$

or

$$\rho = \frac{f'_c}{f_y}[0.85 - \sqrt{(0.85)^2 - Q}]$$

where

$$Q = \left(\frac{1.7}{\phi f'_c}\right)\frac{M_u}{bd^2} = \left(\frac{1.7}{\phi f'_c}\right)R_u \tag{4.3}$$

$$A_s = \rho bd = \left(\frac{f'_c}{f_y}\right)bd[0.85 - \sqrt{(0.85)^2 - Q}] \tag{4.4}$$

where all units are in kips (or pounds) and inches, and Q is dimensionless. For example, if $M_u = 2440$ K·in., $b = 12$ in., $d = 18$ in., $f'_c = 3$ ksi, and $f_y = 60$ ksi, then $\rho = 0.01389$ (from Eq. 4.2) and $A_s = \rho bd = 0.01389(12)(18) = 3.0\,\text{in.}^2$, or directly from Eq. 4.4, $Q = 0.395$ and $A_s = 3.0\,\text{in}^2$. When b and d are given, it is better to check if compression steel is or is not required because of a small d. This can be achieved as follows:

a. Calculate ρ_{max} and $R_u(\text{max}) = \phi\rho_{max}f_y[1 - (\rho_{max}f_y/1.7f'_c)]$.

b. Calculate $\phi M_n(max) = R_u bd^2 =$ the design moment strength of a singly reinforced concrete section.

c. If $M_u < \phi M_{n\,max}$, then no compression reinforcement is needed. Calculate ρ and A_s from the preceding equations.

d. $M_u > \phi M_{n\,max}$, then compression steel is needed. In this case, the design procedure is explained in Section 4.4.

3. If ρ and b are given, calculate R_u:

$$R_u = \phi\rho f_y\left(1 - \frac{\rho f_y}{1.7f'_c}\right)$$

Then calculate d from Eq. 4.1:

$$d = \sqrt{\frac{M_u}{R_u b}} \quad \text{and} \quad A_s = \rho bd$$

4.3 SPACING OF REINFORCEMENT AND CONCRETE COVER

4.3.1 Specifications

Figure 4.1 shows two reinforced concrete sections. The bars are placed such that the clear spacings shall be at least equal to nominal bar diameter D but not less than 1 in. (25 mm). Vertical clear spacings between bars in more than one layer shall not be less than 1 in. (25 mm), according to the ACI Code, Section 7.6.

The width of the section depends on the number, n, and diameter of bars used. Stirrups are placed at intervals; their diameters and spacings depend on shear requirements, to be explained

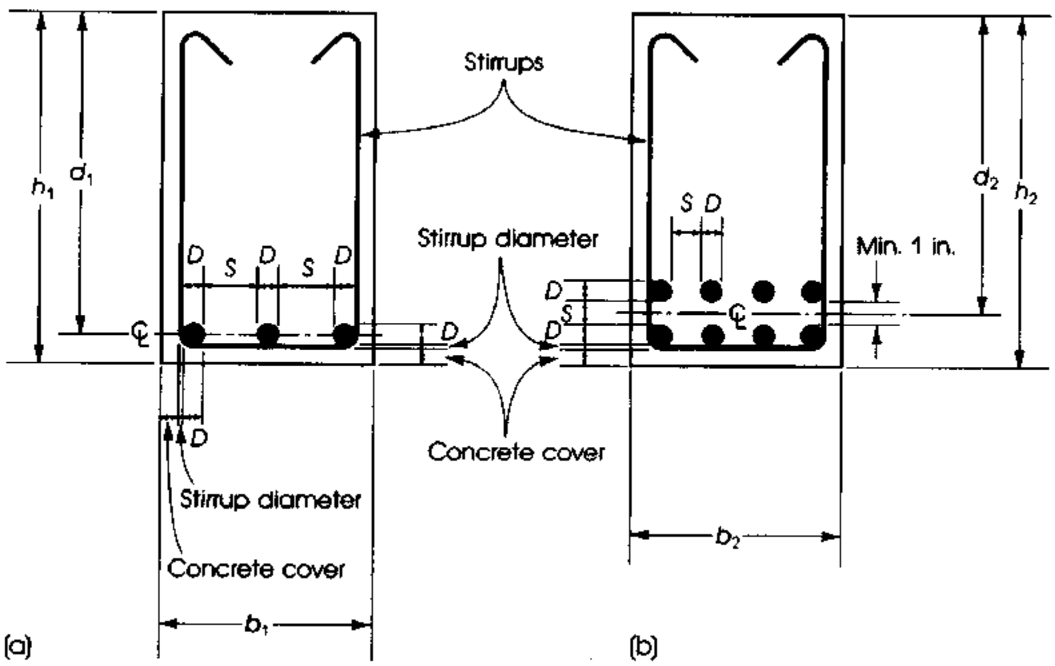


Figure 4.1 Spacing of steel bars (a) in one row or (b) two rows.

later. At this stage, stirrups of $\frac{3}{8}$ in. (10 mm) diameter can be assumed to calculate the width of the section. There is no need to adjust the width, b, if different diameters of stirrups are used. The specified concrete cover for cast-in-place and precast concrete is given in the ACI Code, Section 7.7. Concrete cover for beams and girders is equal to $\frac{3}{2}$ in. (38 mm), and that for slabs is equal to $\frac{3}{4}$ in. (20 mm), when concrete is not exposed to weather or in contact with ground.

4.3.2 Minimum Width of Concrete Sections

The general equation for the minimum width of a concrete section can be written in the following form:

$$b_{\min} = nD + (n-1)s + 2(\text{stirrup's diameter}) + 2(\text{concrete cover}) \qquad (4.5a)$$

where

n = number of bars
D = diameter of the largest bar used
s = spacing between bars (equal to D or 1 in., whichever is larger)

If the stirrup's diameter is taken equal to $\frac{3}{8}$ in. (10 mm) and concrete cover equals $\frac{3}{2}$ in. (38 mm), then

$$b_{\min} = nD + (n-1)s + 3.75 \text{ in. (95 mm)} \qquad (4.5b)$$

This equation, if applied to the concrete sections in Fig. 4.1, becomes

$$b_1 = 3D + 2S + 3.75 \text{ in. (95 mm)}$$

$$b_2 = 4D + 3S + 3.75 \text{ in. (95 mm)}$$

To clarify the use of Eq. 4.5, let the bars used in sections of Fig. 4.1 be no. 10 (32-mm) bars. Then

$b_1 = 5 \times 1.27 + 3.75 = 10.10$ in. $(s = D)$ say, 11 in.

$b_1 = 5 \times 32 + 95 = 225$ mm say, 250 mm

$b_2 = 7 \times 1.27 + 3.75 = 12.64$ in. say, 13 in.

$b_1 = 7 \times 232 + 95 = 319$ mm say, 320 mm

If the bars used are no. 6 (20 mm), the minimum widths become

$b_1 = 3 \times 0.75 + 2 \times 1 + 3.75 = 8.0$ in. $s = 1$ in.

$b_1 = 3 \times 20 + 2 \times 25 + 95 = 205$ mm say, 210 mm

$b_2 = 4 \times 0.75 + 3 \times 1 + 3.75 = 9.75$ in. say, 10 in.

$b_2 = 4 \times 20 + 3 \times 25 + 95 = 250$ mm

The width of the concrete section shall be increased to the nearest inch. Table A.7 in Appendix A gives the minimum beam width for different numbers of bars in the section.

4.3.3 Minimum Overall Depth of Concrete Sections

The effective depth, d, is the distance between the extreme compressive fibers of the concrete section and the centroid of the tension reinforcement. The minimum total depth is equal to d plus the distance from the centroid of the tension reinforcement to the extreme tension concrete fibers, which depends on the number of layers of the steel bars. In application to the sections shown in Fig. 4.1,

$$h_1 = d_1 + \frac{D}{2} + \frac{3}{8} \text{ in.} + \text{concrete cover} \tag{4.6a}$$

$$= d_1 + \frac{D}{2} + 1.857 \text{ in. (50 mm)}$$

for one row of steel bars and

$$h_2 = d_2 = 0.5 + D + \frac{3}{8} \text{ in.} + \text{concrete cover} \tag{4.6b}$$

$$= d_2 + D + 2.375 \text{ in. (60 mm)}$$

for two layers of steel bars. The overall depth, h, shall be increased to the nearest half inch (10 mm) or, better, to the nearest inch (20 mm in SI). For example, if $D = 1$ in. (25 mm), $d_1 =$ 18.9 in. (475 mm), and $d_2 = 20.1$ in. (502 mm),

$$\text{Minimum } h_1 = 18.9 + 0.5 + 1.875 = 21.275 \text{ in.}$$

say, 21.5 in. or 22 in.,

$$h_1 = 475 + 13 + 50 = 538 \text{ mm}$$

say, 540 mm or 550 mm, and

$$\text{Minimum } h_2 = 20.1 + 1.0 + 2.375 = 23.475 \text{ in.}$$

say, 23.5 in. or 24 in.,

$$h_2 = 502 + 25 + 60 = 587 \text{ mm}$$

say, 590 mm or 600 mm.

If no. 9 or smaller bars are used, a practical estimate of the total depth, h, can be made as follows:

$$h = d + 2.5 \text{ in. (65 mm)}, \quad \text{for one layer of steel bars}$$

$$h = d + 3.5 \text{ in. (90 mm)}, \quad \text{for two layers of steel bars}$$

For more than two layers of steel bars, a similar approach may be used.

It should be mentioned that the minimum spacing between bars depends on the maximum size of the coarse aggregate used in concrete. The nominal maximum size of the coarse aggregate shall not be larger than one-fifth of the narrowest dimension between sides of forms, nor one-third of the depth of slabs, nor three-fourths of the minimum clear spacing between individual reinforcing bars or bundles of bars (ACI Code, Section 3.3).

Example 4.1

Design a simply reinforced rectangular section to resist a factored moment of 361 K·ft using the maximum steel percentage ρ_{max} for tension-controlled sections. Given: $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

For $f'_c = 3$ ksi, $f_y = 60$ ksi, and $\beta_1 = 0.85$, ρ_{max} for a tension-controlled section is calculated as follows ($\phi = 0.9$):

$$\rho_b = (0.85)\beta_1 \left(\frac{f'_c}{f_y}\right)\left[\frac{87}{(87 + f_y)}\right],$$

$$\rho_b = (0.85)^2 \left(\frac{3}{60}\right)\left(\frac{87}{147}\right) = 0.0214$$

$$\rho_{max} = \rho_b \left(\frac{0.003 + \frac{f_y}{E_s}}{0.008}\right) = 0.63375\rho_b = 0.01356 \qquad \text{(Table 4.1)}$$

$$R_{u\,max} = \phi\rho_{max} f_y \left(1 - \frac{\rho_{max} f_y}{1.7 f'_c}\right)$$

$$= 0.9 \times 0.01356 \times 60 \times \left(1 - \frac{0.01356 \times 60}{1.7 \times 3}\right) = 0.615 \text{ ksi}$$

(Or, use the tables in Appendix A or Table 4.1.)
Since $M_u = R_u bd^2$,

$$bd^2 = \frac{M_u}{R_u} = \left(\frac{361 \times 12}{0.615}\right) = \frac{4332}{0.615} = 7043 \text{ in.}^3$$

Thus, for the following assumed b, calculate d and $A_s = \rho bd$:

$b = 10$ in.	$d = 26.5$ in.	$A_s = 4.24$ in.^{2}	
$b = 12$ in.	$d = 24.2$ in.	$A_s = 4.65$ in.^{2}	6 no. 8 bars ($A_s = 4.71$ in.^{2})
$b = 14$ in.	$d = 22.4$ in.	$A_s = 5.01$ in.^{2}	5 no. 9 bars ($A_s = 5.0$ in.^{2})
$b = 16$ in.	$d = 21.0$ in.	$A_s = 5.37$ in.^{2}	

The choice of the effective depth d depends on three factors:

1. The width b required. A small width will result in a deep beam that decreases the headroom available. Furthermore, a deep narrow beam may lower the design moment strength of the structural member due to possible lateral deformation.
2. The amount and distribution of reinforcing steel. A narrow beam may need more than one row of steel bars, thus increasing the total depth of the section.
3. The wall thickness. If cement block walls are used, the width b is chosen to be equal to the wall thickness. Exterior walls in buildings in most cases are thicker than interior walls. The architectural plan of the structure will show the different thicknesses.

A reasonable choice of d/b varies between 1 and 3, with practical value about 2. It can be seen from the previous calculations that the deeper the section, the more economical it is, as far as the quantity of concrete used, expressed by the area bd of a 1-ft length of the beam. Alternatively, calculate $bd^2 = M_u/R_u$ and then choose adequate b and d.

The area of the steel reinforcement, A_s, is equal to ρbd. The area of steel needed for the different choices of b and d for this example was shown earlier. Because the steel percentage required is constant ($\rho_{max} = 0.01356$), A_s is proportional to bd. For a choice of a 12 × 24.2-in. section, the required A_s is 4.65 in.^{2} Choose six no. 8 bars in two rows (actual $A_s = 4.71$ in.^{2}). The minimum b required for three no. 8 bars in one row is 8.9 in. < 12 in., and total $h = 24.2 + 3.5 = 27.7$ in., say, 28 in. (actual $d = 24.6$ in.). Another choice is a section with a 14 × 22.4-in. section with a total depth (h) of 25 in. and five no. 9 bars in one row. The choice of bars depends on

1. Adequate placement of bars in the section, normally in one or two rows, fulfilling the restrictions of the ACI Code for minimum spacing between bars
2. The area of steel bars chosen closest to the required calculated steel area

The final section is shown in Fig. 4.2.

Example 4.2

Solve Example 4.1 using a steel percentage ρ of about 1% and $b = 14$ in.

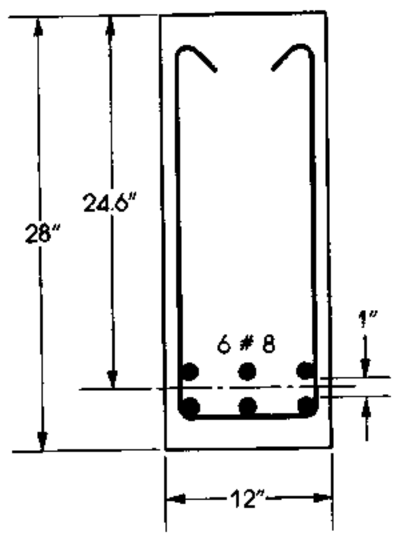


Figure 4.2 Example 4.1.

Solution

1. For $f'_c = 3$ ksi and $f_y = 60$ ksi, $\rho_{max} = 0.01356$ for a tension-controlled section:

$$R_u = \phi\rho f_y\left(1 - \frac{\rho f_y}{1.7f'_c}\right)$$

$$= 0.9 \times 0.01 \times 60\left(1 - \frac{0.01 \times 60}{1.7 \times 3}\right) = 0.476 \text{ ksi}$$

(From the tables in Appendix A, for $\rho = 0.01$, $R_u = 476$ psi.)

2. $bd^2 = M_u/R_u = 4332/0.476 = 9100\,\text{in.}^3$. Choosing $b = 14$ in. and $d = 25.5$ in.,

$$A_s = \rho bd = 0.01 \times 14 \times 25.5 = 3.57 \text{ in.}^2$$

Choose four no. 9 bars in one layer; $A_s = 4.00\,\text{in.}^2$

$$b_{min} = nD + (n-1)s + 3.75$$
$$= 7 \times 1.128 + 3.75 = 11.7 \text{ in.} < 14 \text{ in.}$$
$$h_{min} = d + \frac{D}{2} + 1.875$$
$$= 25.5 + \frac{1.138}{2} + 1.875 = 27.94 \text{ in.} \qquad \text{say, 28 in.} \qquad (d = 25.5 \text{ in.})$$

3. Because the actual A_s used is greater than the calculated A_s, a smaller depth can be adopted. Therefore, take $h = 26$ in. Then $d = 26 - 1.138/2 - 1.875 = 23.5$ in.

For small variation in depth, $A_s = 3.57(25.5/23.5) = 3.87\,\text{in.}^2$, which is less than the $4.00\,\text{in.}^2$ used (Fig. 4.3). A check of the design moment strength of the section can be made:

$$\text{actual } \rho = \frac{4}{14 \times 23.5} = 0.0121$$

Since $\rho < \rho_{max} = 0.01356$ for a tension-controlled section ($\phi = 0.9$),

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{4.0 \times 60}{0.85 \times 3 \times 14} = 6.72 \text{ in.}$$

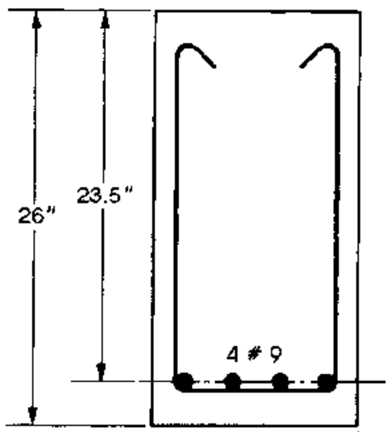


Figure 4.3 Example 4.2.

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$= 0.9 \times 4 \times 60 \left(23.5 - \frac{6.72}{2}\right) = 4350 \text{ K·in.} > 4332 \text{ K·in.}$$

which is acceptable.

4. Check the net tensile strain, ε_t. For $f_y = 60$ ksi,

$$\varepsilon_t = \left(\frac{0.005}{\frac{\rho}{\rho_b}}\right) - 0.003 \qquad (3.25)$$

$$\rho_b = 0.0214 \qquad \text{(Table 4.1)}$$

$$\frac{\rho}{\rho_b} = \frac{0.0121}{0.0214} = 0.5654$$

$$\varepsilon_t = \frac{0.005}{0.5654} - 0.003 = 0.00584 > 0.005 \qquad \text{(tension-controlled section)}$$

Or, alternatively, $c = a/0.85 = 7.9$ in., $d_t = 26 - 2.5 = 23.5$ in., $c/d_t = 0.336 < 0.375$, which is o.k.

Example 4.3

Find the necessary reinforcement for a given section that has a width of 10 in. and a total depth of 20 in. (Fig. 4.4) if it is subjected to an external factored moment of 163 K·ft. Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

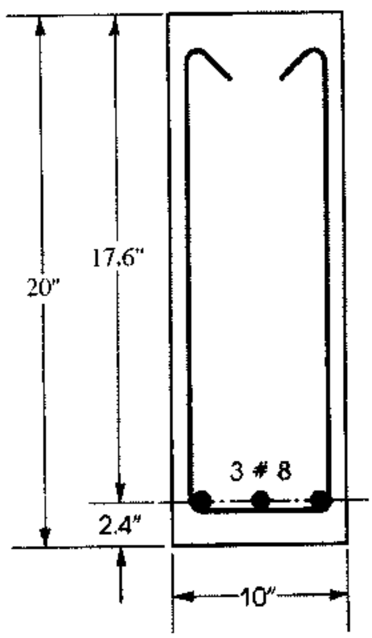


Figure 4.4 Example 4.3.

Solution

1. Assuming one layer of no. 8 steel bars (to be checked later), $d = 20 - 0.5 - 1.875 = 17.625$ in. (or $d = 20 - 2.5$ in. $= 17.5$ in.).
2. Check if the section is adequate without compression reinforcement. Compare the moment strength of the section (using $\rho_{\max}$ for tension-controlled condition). For $f'_c = 4$ ksi and $f_y = 60$ ksi, $\rho_{\max} = 0.01806$.

$$R_{u\,\max} = \phi\rho_{\max} f_y \left(1 - \frac{\rho_{\max} f_y}{1.7 f'_c}\right) = 820 \text{ psi} \qquad \text{(from Table 4.1)}$$

The moment strength of a singly reinforced basic section is

$$\phi M_{n\,\max} = R_{u\,\max} bd^2 = 0.82(10)(17.5)^2$$
$$= 2511 \text{ K·in.} > 163 \times 12 = 1956 \text{ K·in.}$$

Therefore, $\rho < \rho_{\max}$ and the section is singly reinforced, and tension controls ($\phi = 0.9$).

3. Calculate ρ from Eq 4.2 or 4.3:

$$Q = \left(\frac{1.7}{\phi f'_c}\right) \times \frac{M_u}{bd^2} = \left(\frac{1.7}{0.9 \times 4}\right) \times \left(\frac{1956}{10 \times 17.5^2}\right) = 0.302$$

$$\rho = \frac{f'_c}{f_y}(0.85 - \sqrt{(0.85)^2 - Q}) = 0.0134 < \rho_{\max} \qquad \text{(tension-controlled condition)}$$

$A_s = \rho bd = 0.0134(10)(17.5) = 2.345\text{ in.}^2$ Use three no. 8 bars ($A_s = 2.35\text{ in.}^2$) in one row, $b_{\min} < 10$ in. The final section is shown in Fig. 4.4.

Example 4.4

Find the necessary reinforcement for a given section, $b = 15$ in., if it is subjected to a factored moment of 313 K·ft. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. For $f'_c = 4$ ksi and $f_y = 60$ ksi, and from Table 4.1: $\rho_b = 0.0285$, $\rho_{\max} = 0.01806$ (tension-controlled section), $R_{u\,\max} = 820$ psi.
2. Using $\rho_{\max} = 0.01806$ and $R_u = 820$ psi,

$$bd^2 = \frac{M_u}{R_u} = \frac{313(12)}{0.820} = 4581 \text{ in.}^3$$

For $b = 15$ in. and $d = 17.50$,

$$A_s = \rho bd = 0.01806(15)(17.5) = 4.74 \text{ in.}^2$$

Choose four no. 10 bars, $A_s = 5.08\text{ in.}^2 > 4.74\text{ in.}^2$. Bars can be placed in one row, $b_{\min} = 12.7$ in. in Table A.7. Total depth (h) $= 17.5 + 2.5 = 20$ in.

Discussion

1. Since a steel area of 5.08 in.^2 used is greater than 4.74 in.^2 required (the limit for a tension-controlled section with $\phi = 0.9$), the section is in the transition zone. Actually, the section is under-reinforced and the nominal moment $= M_n = A_s f_y(d - a/2) = 368.6$ K·ft. ($A_s = 5.08\text{ in.}^2$ and $a = 5.976$ in.). If $\phi = 0.9$ is used then $\phi M_n = 331.7$ K·ft.

2. The ACI Code indicates that for sections in the transition zone, $\phi < 0.9$, and $\varepsilon_t \geq 0.004$.

Checking $\varepsilon_t = \left(\dfrac{0.005}{\dfrac{\rho}{\rho_b}}\right) - 0.003$,

$$\rho = \frac{5.08}{15 \times 17.5} = 0.01935 \qquad \frac{\rho}{\rho_b} = 0.679$$

$$\varepsilon_t = \left(\frac{0.00507}{0.679}\right) - 0.003 = 0.004467 > 0.004$$

Or, alternatively, calculate $a = 5.08 \times 60/(0.85 \times 4 \times 15) = 5.976$, $c = a/.85 = 7.03$, $d_t = d = 17.5$ in. Then $\varepsilon_t = 0.003(d_t - c)/c = 0.004467$. Calculate

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.856$$

$$\phi M_n = 0.856(368.6) = 315.4 \text{ K·ft}$$

3. It can be noticed that despite an additional amount of steel, $5.08 - 4.67 = 0.41 \text{ in.}^2$ (or about 9%), the design moment strength remained the same. This is because the strength reduction factor, ϕ, was decreased. Therefore, the design of sections within the tension-controlled zone with $\phi = 0.9$ gives a more economical design based on the ACI Code limitations.

4.4 RECTANGULAR SECTIONS WITH COMPRESSION REINFORCEMENT

A singly reinforced section has its moment strength when ρ_{max} of steel is used. If the applied factored moment is greater than the internal moment strength, as in the case of a limited cross-section, a doubly reinforced section may be used, adding steel bars in both the compression and the tension zones. Compression steel will provide compressive force in addition to the compressive force in the concrete area.

4.4.1 Assuming One Row of Tension Bars

The procedure for designing a rectangular section with compression steel when M_u, f'_c, b, d, and d' are given can be summarized as follows:

1. Calculate the balanced and the maximum steel ratio, ρ_{max}, using Eqs. 3.18 and 3.31.

$$\rho_b = 0.85\beta_1 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right)$$

Calculate $A_{s\,max} = A_{s1} = \rho_{max}bd$ (maximum steel area as singly reinforced).

2. Calculate $R_{u\,max}$ using ρ_{max} ($\phi = 0.9$):

$$R_{u\,max} = \phi\rho_{max} f_y \left(1 - \frac{\rho_{max} f_y}{1.7 f'_c}\right)$$

($R_{u\,max}$ can be obtained from the tables in Appendix A or Table 4.1.)

3. Calculate the moment strength of the section, M_{u1}, as singly reinforced, using ρ_{max} and $R_{u\,max}$.

$$M_{u1} = R_{u\,max} bd^2$$

If $M_{u1} < M_u$ (the applied moment), then compression steel is needed. Go to the next step. If $M_{u1} > M_u$, then compression steel is not needed. Use Eq. 4.2 to calculate ρ and $A_s = \rho bd$, as explained earlier.

4. Calculate $M_{u2} = M_u - M_{u1}$ = the moment to be resisted by compression steel.
5. Calculate A_{s2} from $M_{u2} = \phi A_{s2} f_y (d - d')$.
   Then calculate the total tension reinforcement, A_s:
$$A_s = A_{s1} + A_{s2}$$
6. Calculate the stress in the compression steel as follows:
   a. Calculate $f'_s = 87[(c - d')/c]$ ksi $\leq f_y$. (f'_s cannot exceed f_y.)
   b. Or, ε'_s can be calculated from the strain diagram, and $f'_s = (\varepsilon'_s \cdot E_s)$. If $\varepsilon'_s \geq \varepsilon_y$, then compression steel yields and $f'_s = f_y$.
   c. Calculate A'_s from $M_{u2} = \phi A'_s f'_s (d - d')$. If $f'_s = f_y$, then $A'_s = A_{s2}$. If $f'_s < f_y$, then $A'_s > A_{s2}$, and $A'_s = A_{s2}(f_y/f'_s)$.
7. Choose bars for A_s and A'_s to fit within the section width, b. In most cases, A_s bars will be placed in two rows, whereas A'_s bars are placed in one row.
8. Calculate $h = d + 2.5$ in. for one row of tension bars and $h = d + 3.5$ in. for two rows of tension steel. Round h to the next higher inch. Now check that $\lfloor \rho - \rho\prime(f'_s/f_y) \rfloor < \rho_{max}$ using the new d, or check that $A_{s\,max} = bd\lfloor \rho_{max} + \rho'(f'_s/f_y) \rfloor \geq A_s$ (used).
$$\rho = \frac{A_s}{(bd)} \quad \text{and} \quad \rho\prime = \frac{A'_s}{(bd)}$$
   This check may not be needed if ρ_{max} is used in the basic section.
9. If desired, the design moment strength of the final section, ϕM_n, can be calculated and compared with the applied moment, M_u: $\phi M_n \geq M_u$. Note that a steel ratio ρ smaller than ρ_{max} can be assumed in step 1, say $\rho = 0.6\rho_b$ or $\rho = 0.9\rho_{max}$, so that the final tension bars can be chosen to meet the given ρ_{max} limitation.
10. The strain at the bars level can be checked as follows:
$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 \geq 0.005$$

4.4.2 Assuming Two Rows of Tension Bars

In the case of two rows of bars, it can be assumed that $d = h - 3.5$ in. and $d_t = h - 2.5$ in. $= d + 1.0$ in.

Two approaches may be used to design the section.

1. One approach is to assume a strain at the level of the centroid of the tension steel equal to 0.005 or $\varepsilon_s = 0.005$ (at d level). In this case, the strain in the lower row of bars is greater than 0.005. $\varepsilon_t = (d_t - c/c)0.003 > 0.005$, which still meets the ACI Code limitation. For this case, follow the above steps 1 to 9. Example 4.6, solution 1 explains this approach.
2. A second approach is to assume a strain $\varepsilon_t = 0.005$ at the level of the lower row of bars, d_t. In this case, the stain at the level of the centroid of bars is less than 0.005: $\varepsilon_s = [(d_t - c)/c]0.003 < 0.005$, which is still acceptable. Example 4.6, solution 2 explains this approach. The solution can be summarized as follows:

a. Calculate $d_t = h - 2.5$ in., and then form the strain diagram and calculate c, the depth of the neutral axis

$$c = \left(\frac{0.003}{0.003 + \varepsilon_t}\right) d_t$$

For $\varepsilon_t = 0.005$,

$$c = \left(\frac{3}{8}\right) d_t \text{ and } a = \beta_1 c$$

b. Calculate the compression force in the concrete.

$$C_1 = 0.85 f'_c ab = T_1 = A_{s1} f_y$$

Determine A_{s1}. Calculate $M_{u1} = \phi A_{s1} f_y (d - a/2)$. $\rho_1 = A_{s1}/bd$, $\phi = 0.9$.

c. Calculate $M_{u2} = M_u - M_{u1}$; assume $d' = 2.5$ in.

d. Calculate A_{s2}: $M_{u2} = \phi A_{s2} f_y (d - d')$, $f'_c = f_y$, $\phi = 0.9$. Total $A_s = A_{s1} + A_{s2}$.

e. Check if compression steel yields similar to step 6 above in section 4.4.1.

Example 4.5

A beam section is limited to a width of $b = 10$ in. and a total depth of $h = 22$ in. and has to resist a factored moment of 226.5 K·ft. Calculate the required reinforcement. Given: $f'_c = 3$ ksi and $f_y = 50$ ksi.

Solution

1. Determine the design moment strength that is allowed for the section as singly reinforced based on tension-control conditions. This is done by starting with $\rho_{\max}$. For $f'_c = 3$ ksi and $f_y = 50$ ksi, and from Eqs. 3.18, 3.22, and 3.31,

$$\rho_b = 0.0275 \qquad \rho_{\max} = 0.01624 \qquad R_u = 614 \text{ psi}$$

$$M_u = R_u bd^2 \qquad b = 10 \text{ in.} \qquad d = 22 - 3.5 = 18.5 \text{ in.}$$

$$M_u = 226.5 \times 12 = 2718 \text{ K·in.}$$

(This calculation assumes two rows of steel, to be checked later.) $M_{u1} = 0.614 \times 10 \times (18.5)^2 = 2101$ K·in. = max ϕM_n, as singly reinforced. Design $M_u = 2718$ K·in. > 2101 K·in. Therefore, compression steel is needed to carry the difference.

2. Compute A_{s1}, M_{u1}, and M_{u2}:

$$A_{s1} = \rho_{\max} bd = 0.01624 \times 10 \times 18.5 = 3.0 \text{ in.}^2$$

$$M_{u1} = 2101 \text{ K·in.}$$

$$M_{u2} = M_u - M_{u1} = 2718 - 2102 = 617 \text{ K·in.}$$

3. Calculate A_{s2} and A'_s, the additional tension and compression steel due to M_{u2}. Assume $d' = 2.5$ in.; $M_{u2} = \phi\, A_{s2} f_y\, (d - d')$.

$$A_{s2} = \frac{M_{u2}}{\phi f_y (d - d')} = \frac{617}{0.9 \times 50(18.5 - 2.5)} = 0.86 \text{ in.}^2$$

Total tension steel is equal to A_s.

$$A_s = A_{s1} + A_{s2} = 3.0 + 0.86 = 3.86 \text{ in.}^2$$

The compression steel has $A'_s = 0.86$ in.^{2} (in A'_s yields).

4. Check if compression steel yields:

$$\varepsilon_y = \frac{f_y}{29{,}000} = \frac{50}{29{,}000} = 0.00172$$

Let $a = (A_{s1}f_y)/(0.85f_c'b) = (3.0 \times 50)/(0.85 \times 3 \times 10) = 5.88$ in.

$$c(\text{distance to neutral axis}) = \frac{a}{\beta_1} = \frac{5.88}{0.85} = 6.92 \text{ in.}$$

$$\varepsilon_s' = \text{strain in compression steel (from strain triangles)}$$

$$= 0.003 \times \left(\frac{5.88 - 2.5}{5.88}\right) = 0.00173 > \varepsilon_y = 0.001724$$

5. Check ε_t:

$$\rho_1 = \frac{3}{10 \times 18.5} = 0.016216$$

$$\frac{\rho_1}{\rho_b} = 0.5897 \qquad f_y = 50$$

From Eq. 3.24, $\varepsilon_{t_s} = 0.005$ is assumed at the centroid of the tension steel for ρ_{max} and R_u used. Calculate ε_t (at the lower row of bars):

$$d_t = 22 - 2.5 = 19.5 \text{ in.}$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right)0.003$$

$$= \left(\frac{19.5 - 6.92}{6.92}\right)0.003$$

$$= 0.00545 > 0.005$$

as expected.

6. Choose steel bars as follows: $A_s = 3.86\text{ in.}^2$ Choose five no. 8 bars ($A_s = 3.95\text{ in.}^2$) in two rows, as assumed. $A_s' = 0.86\text{ in.}^2$ Choose two no. 6 bars ($A_s' = 0.88\text{ in.}^2$).
7. Check actual d: Actual $d = 22 - (1.5 + 0.375 + 1.5) = 18.625$ in. It is equal approximately to the assumed depth. The final section is shown in Fig. 4.5.

Example 4.6

A beam section is limited to $b = 12$ in. and to a total depth of $h = 20$ in. and is subjected to a factored moment $M_u = 298.4$ K·ft. Determine the necessary reinforcement using $f_c' = 4$ ksi and $f_y = 60$ ksi. (Refer to Fig. 4.6.)

Solution 1: Two Solutions Are Presented

1. Determine the maximum moment capacity of the section as singly reinforced based on tension-controlled conditions. For $f_c' = 4$ ksi and $f_y = 60$ ksi, $\rho_{max} = 0.01806$ and $R_u = 820$ psi (Table 4.1). Assuming two rows of bars, $d = 20 - 3.5 = 16.5$ in.

$$\text{Max } M_{u1} = R_{u\,max}bd^2 = 0.82(12)(16.5)^2 = 2679 \text{ K·in.} = 223.25 \text{ K·ft.}$$

The design moment is $M_u = 298.4 \times 12 = 3581$ K·in. $> M_{u1}$; therefore, compression steel is needed.

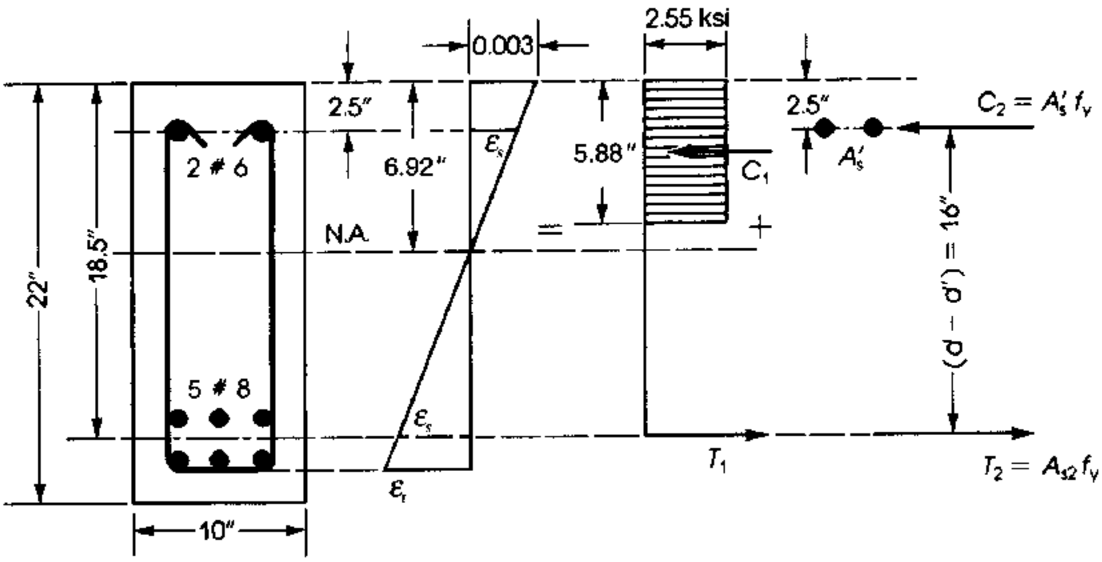


Figure 4.5 Example 4.5: doubly reinforced concrete section.

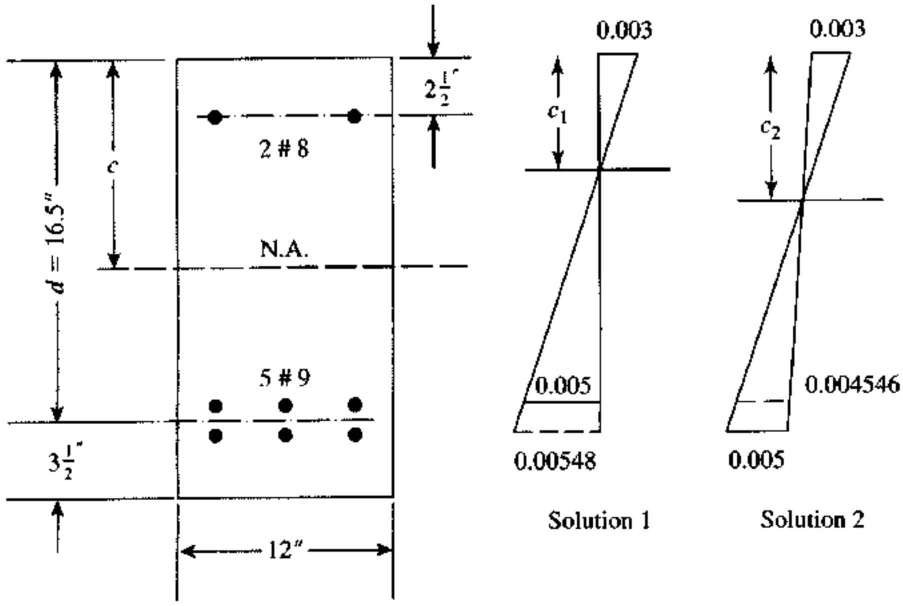


Figure 4.6 Example 4.6.

2. Calculate A_{s1}, M_{u2}, A_{s2}, and A_s.

$$A_{s1} = \rho_{\max} bd = 0.01806(12)(16.5) = 3.576 \text{ in.}^2$$

$$M_{u2} = M_u - M_{u1} = 3581 - 2679 = 902 \text{ K·in.}$$

$$M_{u2} = \phi A_{s2} f_y (d - d'), \quad \text{assume } d' = 2.5 \text{ in.}$$

$$902 = 0.9A_{s2}(60)(16.5 - 2.5), \; A_{s2} = 1.19 \text{ in.}^2$$

$$\text{Total } A_s = A_{s1} + A_{s2} = 3.576 + 1.19 = 4.77 \text{ in.}^2 \qquad \text{(five no. 9 bars)}$$

3. Check if compression steel yields by Eq. 3.46. Compression steel yields if

$$\rho - \rho' \geq K = 0.85\beta_1 \frac{f_c'}{f_y}\left(\frac{d'}{d}\right)\left(\frac{87}{87 - f_y}\right)$$

$$K = (0.85)^2\left(\frac{4}{60}\right)\left(\frac{2.5}{16.5}\right)\left(\frac{87}{27}\right) = 0.0235$$

$$\rho - \rho' = \frac{A_{s1}}{bd} = \frac{3.576}{(12)(16.5)} = 0.01806 \leq K$$

Therefore, compression steel does not yield: $f_s' < f_y$

4. Calculate f_s' : $f_s' = 87[(c - d')/c] \leq f_y$. Determine c from A_{s1}: $A_{s1} = 3.576\,\text{in.}^2$,

$$a = \frac{A_{s1} f_y}{0.85 f_c' b} = \frac{3.576 \times 60}{0.85 \times 4 \times 12} = 5.26 \text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{5.26}{0.85} = 6.19 \text{ in.}$$

$$f_s' = 87 \times \left(\frac{6.19 - 2.5}{6.19}\right) = 51.8 \text{ ksi} < 60 \text{ ksi}$$

5. Calculate A_s' from $M_{u2} = \phi A_s' f_s'(d - d')$:

$$902 = 0.9A_s'(51.8)(16.5 - 2.5)$$

Thus, $A_s' = 1.38$ in.^{2}, or calculate A_s' from $A_s' = A_{s2}(f_y/f_s') = 1.38$ in.^{2} (two no. 8 bars). Note that the condition $[\rho - \rho'(f_s'/f_y)] = (\rho - \rho') \leq \rho_{max}$ is already met.

$$\left(\rho - \rho' \frac{f_s'}{f_y}\right) = \frac{1}{bd}(A_s - A_{s2}) = \frac{3.576}{12 \times 16.5} = 0.01806$$

as assumed in the solution.

6. These calculations using ρ_{max} and R_u are based on a strain of 0.005 at the centroid of the tension steel.

$$\varepsilon_t(\text{at bottom row}) = \left(\frac{d_t - c}{c}\right)0.003$$

$$d_t = 20 - 2.5 = 17.5 \text{ in.} \qquad \varepsilon_t = \left(\frac{17.5 - 6.19}{6.19}\right)0.003 = 0.00548 > 0.005$$

as expected.

Solution 2

Assuming two rows of tension bars and a strain at the lower row, $\varepsilon_t = 0.005$, the solution will be as follows:

1. Calculate $d_t = 20 - 2.5 = 17.5$ in. From the strain diagram:

$$\frac{c}{d_t} = \frac{0.003}{0.003 + \varepsilon_t} = \frac{0.003}{0.008} = 0.375$$

$$c = 0.375(17.5) = 6.5625 \text{ in.} \qquad a = 0.85c = 5.578 \text{ in.}$$

2. The compression force in the concrete $= C_1 = 0.85\ f_c'ab$

$$C_1 = 0.85(4)(5.578)(12) = 227.6 \text{ K} = T_1 \text{ (as singly reinforced)}$$

$$A_{s1} = \frac{C_1}{f_y} = \frac{T_1}{f_y} = \frac{227.6}{60} = 3.793 \text{ in.}^2$$

$$d = 20 - 3.5 = 16.5 \text{ in.}$$

$$M_{u1} = \phi A_{s1} f_y \left(d - \frac{a}{2}\right) = 0.9(3.793)(60)\left(16.5 - \frac{5.578}{2}\right) = 2808 \text{ K·in.}$$

$$= 234 \text{ K} \cdot \text{ft}$$

$$R_{u1} = \frac{M_{u1}}{bd^2} = \frac{2808.3}{12(16.5)^2} = 0.86 \text{ ksi} = 860 \text{ psi}$$

$$\rho_1 = \frac{A_{s1}}{bd} = 0.01916$$

3. Since $M_u = 3581$ K·in. $> M_{u1}$, compression steel is needed.

$$M_{u2} = 3581 - 2808 = 773 \text{ K·in.}$$

$$M_{u2} = 0.9 A_{s2} f_y (d - d')$$

$$773 = 0.9 A_{s2}(60)(16.5 - 2.5) \qquad A_{s2} = 1.022 \text{ in.}^2$$

$$\text{Total } A_s = A_{s1} + A_{s2} = 3.793 + 1.022 = 4.815 \text{ in.}^2$$

Use five no. 9 bars.

4. Check if compression steel yields as in step 3 in the first solution.

$$K = 0.0235(\rho - \rho') = \rho_1 = 0.01916 < K$$

Compression steel does not yield.

$$f_s' = 87\left(\frac{c - d'}{c}\right) = \left(\frac{6.56 - 2.5}{6.56}\right) 87 = 53.84 \text{ ksi}$$

Calculate A_{s2}:

$$M_{u2} = \phi A_s' f_s' (d - d')$$

$$773 = 0.9 A_s'(53.84)(16.5 - 2.5) \qquad A_s' = 1.14 \text{ in.}^2$$

Use two no. 7 bars ($A_s' = 1.2$ in.^{2}).

5. Check the design moment strength.

$$A_s = 5.0 \text{ in.}^2 \qquad A_s' = 1.2 \text{ in.}^2 \qquad A_{s1} = (A_s - A_s') = 3.8 \text{ in.}^2$$

$$\phi M_n = \phi \left[A_{s1} f_y \left(d - \frac{a}{2}\right) + A_s' f_s' (d - d')\right]$$

$$= 0.9[3.8(60)(16.5 - 5.578/2) + 1.2(53.84)(16.5 - 2.5)]$$

$$= 3627.6 \text{ K·in.} = 302.3 \text{ K·ft.}$$

which is adequate. Note that the strain ε_s at the centroid level of the tension steel is less than 0.005.

$$\varepsilon_s = \left(\frac{d - c}{c}\right) 0.003 = \left(\frac{16.5 - 6.56}{6.56}\right) 0.003 = 0.004546$$

Both solutions are adequate.

Discussion

1. In the first solution, the net tensile strain, $\varepsilon_t = 0.005$, was assumed at the centroid of the tension steel. In this case ρ_{max} and $R_{u\,max}$ can be determined from Table 4.1 or tables in Appendix A. The strain in the lower row of bars will be always greater than 0.005, which meets the ACI Code requirement.
2. In the second solution, the strain limit, $\varepsilon_t = 0.005$, is assumed at the lower row. In this case, the strain at the centroid of the two rows of bars will be less than 0.005 and its value depends on the depth of the section. Moreover, ρ and R_u for this case are not known and their values depend on the effective depth d.
3. Comparing the two solutions, the neutral axis depth, c_1, in solution 1 is slightly smaller than c_2 for the second solution because of the strain limitations, producing a smaller A_{s1} and then higher A_{s2}. Total A_s will normally be very close. It is clear that solution 1 is easier to use because of the use of tables.
4. Note that solution 1 can have the same results as solution 2 by calculating A_{s1} as follows: $A_{s1} = \rho_{max}\, bd_t = 0.01806\ (12 \times 17.5) = 3.793\ \text{in}^2$, which is the same A_{s1} calculated in solution 2, producing $\varepsilon_t = 0.005$ at the lower row of bars.

4.5 DESIGN OF T-SECTIONS

In slab-beam-girder construction, the slab dimensions as well as the spacing and position of beams are established first. The next step is to design the supporting beams, namely, the dimensions of the web and the steel reinforcement. Referring to the analysis of T-section in the previous chapter, we can see that a large area of the compression flange, forming a part of the slab, is effective in resisting a great part or all of the compression force due to bending. If the section is designed on this basis, the depth of the web will be small; consequently, the moment arm is small, resulting in a large amount of tension steel, which is not favorable. Shear requirements should be met, and this usually requires quite a deep section.

In many cases web dimensions can be known based on the flexural design of the section at the support in a continuous beam. The section at the support is subjected to a negative moment, the slab being under tension and considered not effective, and the beam width is that of the web.

In the design of a T-section for a given factored moment, M_u, the flange thickness, t, and width, b, would have been already established from the design of the slab and the ACI Code limitations for the effective flange width, b, as given in Section 3.15. The web thickness, b_w, can be assumed to vary between 8 in. and 20 in., with a practical width of 12 to 16 in. Two more unknowns still need to be determined, d and A_s. Knowing that M_u, f'_c, and f_y are always given, two cases may develop as follows:

1. When d is given and we must calculate A_s,
   a. Check if the section acts as a rectangular or T-section by assuming $a = t$ and calculating the moment strength of the whole flange:

$$\phi M_{nf}(\text{flange}) = \phi(0.85\ f'_c)bt\left(d - \frac{t}{2}\right) \tag{4.7}$$

   If $M_u > \phi M_{nf}$, then $a > t$. If $M_u < \phi M_{nf}$, then $a < t$, and the section behaves as a rectangular section.

   b. If $a < t$, then calculate ρ using Eq. 4.2, and $A_s = \rho bd$. Check that $\rho_w \geq \rho_{min}$.

c. If $a > t$, determine A_{sf} for the overhanging portions of the flange, as explained in Section 3.15.4.

$$A_{sf} = 0.85 f'_c (b - b_w) t / f_y \tag{4.8}$$

$$M_{u2} = \phi A_{sf} f_y \left(d - \frac{t}{2}\right) \tag{4.9}$$

The moment resisted by the web is

$$M_{u1} = M_u - M_{u2}$$

Calculate ρ_1 using M_{u1}, b_w, and d in Eq. 4.2 and determine $A_{s1} = \rho_1 b_w d$.

$$\text{Total } A_s = A_{s1} + A_{sf}$$

Then check that $A_s \leq A_{s\,max}$, as explained in Section 3.15. Also check that $\rho_w = A_s/(b_w d) \geq \rho_{min}$.

d. If $a = t$, then $A_s = \phi(0.85 f'_c) bt / f_y$.

2. When d and A_s are not known, the design may proceed as follows:

a. Assume $a = t$ and calculate the amount of total steel, A_{sft}, needed to resist the compression force in the whole flange, bt.

$$A_{sft} = \frac{(0.85\ f'_c)bt}{f_y} \tag{4.10}$$

b. Calculate d based on A_{sft} and $a = t$ from the following equation:

$$M_u = \phi A_{sft} f_y \left(d - \frac{t}{2}\right) \tag{4.11}$$

If the depth, d, is acceptable, then $A_s = A_{sft}$ and $h = d + 2.5$ in. for one row of bars or $h = d + 3.5$ in. for two rows of bars.

c. If a new d_1 is adopted greater than the calculated d, then the section behaves as a rectangular section, and ρ can be calculated using Eq. 4.2; $A_s = \rho \text{bd} < A_{sft}$.

d. If a new d_2 is adopted that is smaller than the calculated d, then the section will act as a T-section, and the final A_s will be greater then A_{sft}. In this case, proceed as in step 1(c) to calculate A_s.

Example 4.7

The T-beam section shown in Fig. 4.7 has a web width, b_w, of 10 in., a flange width, b, of 40 in., a flange thickness of 4 in., and an effective depth, d, of 14.5 in. Determine the necessary reinforcement if the applied factored moment is 3350 K·in. Given: $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Check the position of the neutral axis; the section may be rectangular. Assume the depth of compression block a is 4 in.; that is, $a = t = 4$ in. Then

$$\phi M_n = \phi(0.85 f'_c)bt\left(d - \frac{t}{2}\right) = 4590 \text{ K·in.} > M_u = 3350 \text{ K·in.}$$

The design moment that the concrete flange can resist is greater than the factored applied moment. Therefore, the section behaves as a rectangular section.

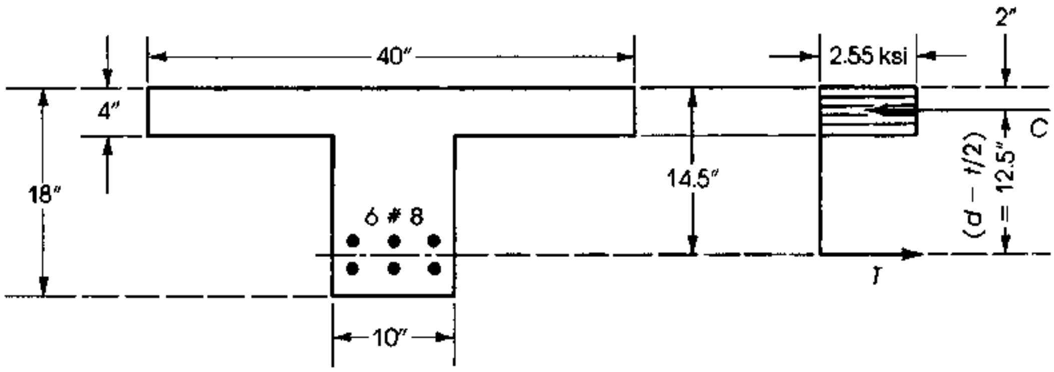


Figure 4.7 Example 4.7: T-section.

2. Determine the area of tension steel, considering a rectangular section, $b = 40$ in.

$$R_u = \phi M_n/(bd^2) = \frac{3{,}350{,}000}{40 \times 14.5^2} = 398 \text{ psi}$$

From Eq. 4.2 or from tables in Appendix A, for $R_u = 398$ psi, and $\rho = 0.00817$,

$$A_s = \rho bd = 0.00817 \times 40 \times 14.5 = 4.74 \text{ in.}^2$$

Use six no. 8 bars, $A_s = 4.74$ in.^{2} (in two rows).

3. Check that $\rho_w = A_s/b_w d \geq \rho_{\min}$; $\rho_w = 4.74/(10 \times 14.5) = 0.0327 > \rho_{\min} = 0.00333$. Note that A_s used is less than $A_{s\,\max}$ of 7.06 in.^{2} Calculated by Eq. 3.72.

Also, $a = 2.788$ in., $c = 3.28$ in., $d_t = 14.5$ in., and $\varepsilon_t = 0.003(d_t - c)/c = 0.01 > 0.005$, which is o.k.

Example 4.8

The floor system shown in Fig. 4.8 consists of 3-in. slabs supported by 14-ft-span beams spaced at 10 ft on centers. The beams have a web width, b_w, of 14 in. and an effective depth, d, of 18.5 in. Calculate the necessary reinforcement for a typical interior beam if the factored applied moment is 5080 K·in. Use $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Find the beam flange width: Flange width is the smallest of

$$b = 16t + b_w = 3 \times 16 + 12 = 60 \text{ in.}$$

$$b = \frac{\text{span}}{4} = \frac{14 \times 12}{4} = 42 \text{ in.}$$

Center-to-center of adjacent slabs is $10 \times 12 = 120$ in. Use $b = 42$ in.

2. Check the position of the neutral axis, assuming $a = t$.

$$\phi M_n \text{ (based on flange)} = \phi \times 0.85 f'_c bt \left(d - \frac{t}{2}\right)$$

$$= 0.9 \times 0.85 \times 3 \times 42 \times 3(18.5 - 1.5) = 4916 \text{ K·in.}$$

The applied moment is $M_u = 5080$ K·in. > 4916 K·in.; the beam acts as a T-section, so $a > t$.

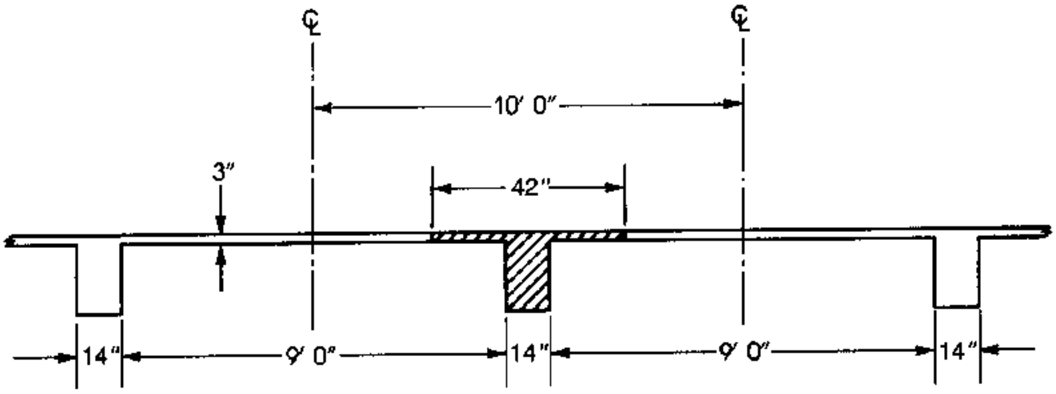


Figure 4.8 Example 4.8: effective flange width.

3. Find the portion of the design moment taken by the overhanging portions of the flange (Fig. 4.9). First calculate the area of steel required to develop a tension force balancing the compressive force in the projecting portions of the flange:

$$A_{sf} = \frac{0.85 f'_c (b - b_w) t}{f_y} = \frac{0.85 \times 3 \times (42 - 14) \times 3}{60} = 3.57 \text{ in.}$$

$\phi M_n = M_{u1} + M_{u2}$, that is, the sum of the design moment of the web and the design moment of the flanges.

$$M_{u2} = \phi A_{sf} f_y \left(d - \frac{t}{2}\right)$$

$$= 0.9 \times 3.57 \times 60 \left(18.5 - \frac{3}{2}\right) = 3272 \text{ K·in.}$$

4. Calculate the design moment of the web (as a singly reinforced rectangular section):

$$M_{u1} = M_u - M_{u2} = 5080 - 3272 = 1808 \text{ K·in.}$$

$$R_u = \frac{M_{u1}}{(b_w d^2)} = \frac{1,808,000}{14 \times (18.5)^2} = 377 \text{ psi}$$

From Eq. 4.2 or the tables in Appendix A, for $R_u = 377$ psi, $\rho_1 = 0.0077$.

$$A_{s1} = \rho_1 b_w d = 0.0077(14)(18.5) = 1.99 \text{ in.}^2$$

Total $A_s = A_{sf} + A_{s1} = 3.57 + 1.99 = 5.56 \text{ in.}^2$ (Use six no. 9 bars in two rows.)

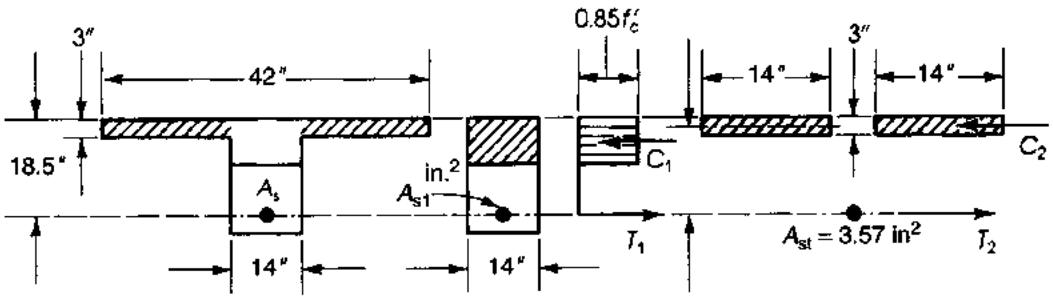


Figure 4.9 Analysis of Example 4.8.

5. Total $h = 18.5 + 3.5 = 22$ in. Calculate $A_{s\,max}$ for T-sections using Eq. 3.72:

$$\text{Max } A_s = 7.02 \text{ in.}^2 > 5.56 \text{ in.}^2$$

6. Check ε_t: $a = 1.99 \times 60/(0.85 \times 3 \times 14) = 3.34$ in., $c = 3.93$ in., $d_t = 19.5$ in. Then $\varepsilon_t = 0.003(d_t - c)/c = 0.0119 > 0.005$, tension-controlled section ($\phi = 0.9$).

Example 4.9

In a slab-beam system, the flange width was determined to be 48 in., the web width was $b_w = 16$ in., and the slab thickness was $t = 4$ in. (Fig. 4.10). Design a T-section to resist an external factored moment of $M_u = 812$ K·ft. Use $f_c' = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Because the effective depth is not given, let $a = t$ and calculate A_{sft} for the whole flange.

$$A_{sft} = \frac{0.85 f_c' b t}{f_y} = \frac{0.85(3)(48)(4)}{60} = 8.16 \text{ in.}^2$$

Let $M_u = \phi A_{sft} f_y (d - t/2)$ and calculate d:

$$812 \times 12 = 0.9(8.16)(60)\left(d - \frac{4}{2}\right) \qquad d = 24.1 \text{ in.}$$

Now, if an effective $d = 24.1$ in. is chosen, then $A_s = A_{sft} = 8.16$ in.^{2}

2. If a depth $d > 24.1$ in. is chosen, say 26.5 in., then $a < t$ and it is a rectangular analysis. The steel ratio can be calculated from Eq. 4.2 with $\rho = 0.00574$ and $A_s = \rho b d = 0.00574 \times 48 \times 26.5 = 7.3$ in.^{2} (six no. 10 bars in two rows, $A_s = 7.62$ in.^{2}).
3. If a depth $d < 24.1$ in. is chosen, say, 23.5 in., then $a > t$, and the section behaves as a T-section. Calculate

$$A_{sf} = 0.85 f_c' t (b - b_w)/f_y = 0.85(3)(4)(48 - 16)/60 = 5.44 \text{ in.}^2$$

$$M_{u2} = \phi A_{sf} f_y \left(d - \frac{t}{2}\right) = 0.9(5.44)(60)\left(23.5 - \frac{4}{2}\right) = 6316 \text{ K·in.}$$

$$M_{u1} = 812 \times 12 - 6316 = 3428 \text{ K·in.}$$

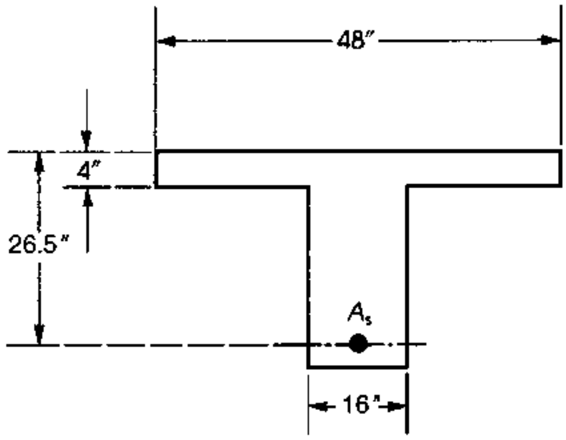


Figure 4.10 Example 4.9.

4. For the basic singly reinforced section, $b_w = 16$ in., $d = 23.5$ in., and $M_{u1} = 3428$ K·in., $R_u = 387$ psi. Calculate ρ_1 from Eq. 4.2 to get $\rho_1 = 0.0079$.

$$A_{s1} = \rho_1 b_w d = 0.0079(16)(23.5) = 2.97 \text{ in.}^2$$

$$\text{Total } A_s = A_{sf} + A_{s1} = 5.44 + 2.97 = 8.41 \text{ in.}^2 \text{(seven no. 10 bars in two rows,}$$
$$A_s = 8.89 \text{ in.}^2)$$

5. Check ε_t: $a = 2.97 \times 60/(.85 \times 3 \times 16) = 4.368$ in., $c = a/0.85 = 5.14$ in., $d_t = 24.5$ in., and $\varepsilon_t = 0.003\,(d_t - c)/c = 0.0113 > 0.005$, a tension-controlled section.
6. Calculate the total max A_s that can be used for the T-section by Eq. 3.72:

$$\text{Max } A_s =$$
$$= 0.0425[(b - bw)t + 0.319bwd] = 10.54 \text{ in.}^2$$
$$A_s \text{ (used)} \leq \max A_s$$

7. *Note*: If there are no restrictions on the total depth of the beam, it is a common practice to adopt the case when $a \leq t$ (step 2). This is because an increase in d produces a small increase in concrete in the web only while decreasing the quantity of A_s required.

4.6 ADDITIONAL EXAMPLES

The following design examples give some practical applications and combine structural analysis with concrete design of beams and frames.

Example 4.10

For the precast concrete I-section shown in Fig. 4.11, calculate the reinforcement needed to support a factored moment of 360 K·ft. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

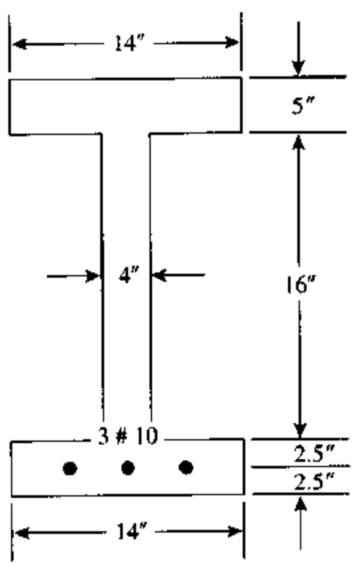


Figure 4.11 Example 4.10.

Solution

Determine if the force in the flange area 14 × 5 in. will be sufficient to resist a factored moment of 360 K·ft. Let $d = 23.5$ in. Force in flange $(C_c) = 0.85 \times f_c'$ (flange area) $= 0.85 \times 4 \times (14 \times 5) = 238$ K, located at 2.5 in. from the top fibers, and $a = 5$ in.

$$\phi M_n = 0.9C_c\left(d - \frac{a}{2}\right) = 0.9 \times 238\frac{(23.5 - 2.5)}{12} = 374.9 \text{ K·ft}$$

which is greater than the applied moment of 360 K·ft. Therefore, a is less than 5 in.

$$\phi M_n = \phi A_s f_y\left(d - \frac{a}{2}\right), \text{ where } a = \frac{A_s f_y}{(0.85 f_c' b)}$$

$$360 \times 12 = 0.9A_s(60)\left(\frac{23.5 - 60A_s}{(1.7 \times 4 \times 14)}\right)$$

Solve to get $A_s = 3.79 \text{ in.}^2$ Or use Eq. 4.2 to get $\rho = 0.01152$ and $A_s = 0.01152 \times 14 \times 23.5 = 3.79 \text{ in.}^2$ Use three no. 10 bars in one row, as shown in Fig. 4.11.

For similar T-sections or I-sections, it is better to adopt a section with a flange size to accommodate the compression force, C_c. In this case, a is less than or equal to the flange depth. The bottom flange is in tension and not effective.

Example 4.11

The simply supported beam shown in Fig. 4.12 carries a uniform service load of 2.8 K/ft (including self-weight) in addition to a service load of 1.6 K/ft. Also, the beam supports a concentrated dead load of 16 K and a concentrated live load of 7 K at C, 10 ft from support A.

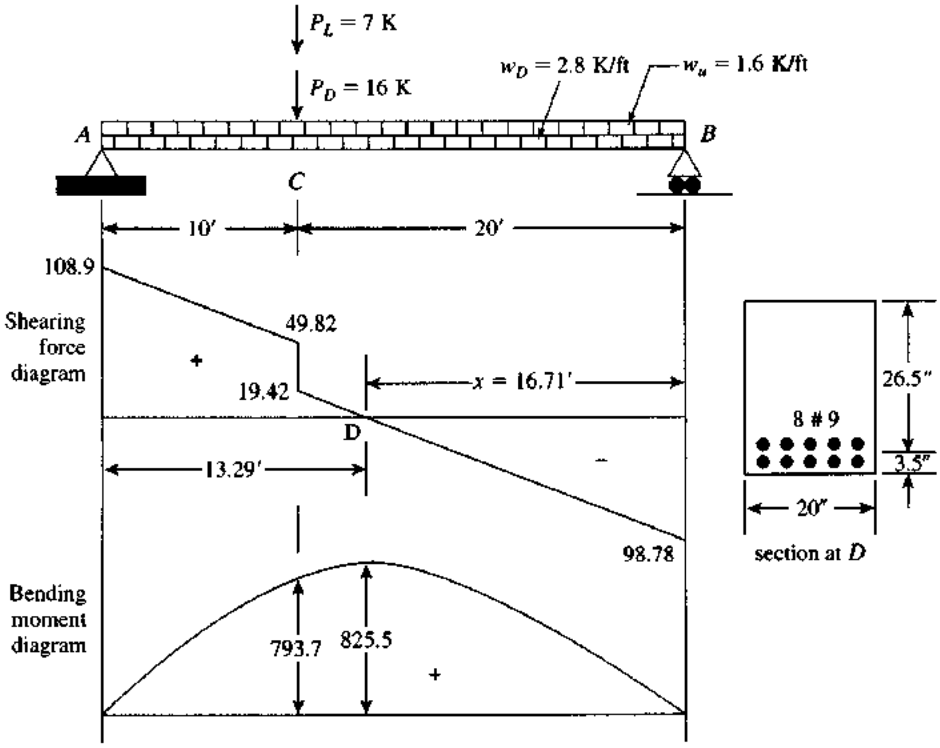


Figure 4.12 Example 4.11.

a. Determine the maximum factored moment and its location on the beam.

b. Design a rectangular section to carry the loads safely using a steel percentage of about 1.5%, $b = 20$ in., $f'_c = 4$ ksi, and $f_y = 60$ ksi.

Solution

a. Calculate the uniform factored load: $w_u = 1.2(2.8) + 1.6(1.6) = 5.91$ K/ft. Calculate the concentrated factored load: $P_u = 1.2(16) + 1.6(7) = 30.4$ K. Calculate the reaction at A by taking moments about B:

$$R_A = 5.91(30)\frac{(30/2)}{30} + \frac{30.4(20)}{30} = 108.92 \text{ K}$$

$$R_B = 5.91(30) + 30.4 - 108.92 = 98.78 \text{ K}$$

Maximum moment in the beam occurs at zero shear. Starting from B,

$$V = 0 = 98.78 - 5.91x \text{ and } x = 16.71 \text{ ft from } B \text{ at } D$$

$$M_u \text{ (at } D) = 98.72(16.71) - 5.91(16.71)\left(\frac{16.71}{2}\right) = 825.5 \text{ K·ft} \quad \text{(design moment)}$$

$$M_u \text{ (at } C) = 98.78(20) - 5.91(20)\left(\frac{20}{2}\right) = 793.6 \text{ K·ft}$$

b. Design of the section at D: For $f'_c = 4$ ksi, and $f_y = 60$ ksi, $\rho_{\text{max}} = 0.01806$ and $\rho_{\text{min}} = 0.00333$, and the design steel ratio of 1.5% is within the limits. For $\rho = 0.015$, $R_u = 700$ psi (from Table A.2) or from Eq. 3.22.

$$M_u = R_u bd^2 \quad \text{or} \quad 825.5 \times 12 = 0.7(20)d^2$$

Solve to get $d = 26.6$ in.

$$A_s = 0.015 \times 20 \times 26.6 = 7.98 \text{ in.}^2$$

Choose eight no. 9 bars in two rows (area = 8 in.^{2}), five in the lower row plus three in the upper row. Minimum b for five no. 9 bars in one row is 14 in. (Table A.7). Total depth $(h) = 26.6 + 3.5 = 30.1$ in. Use $h = 30$ in. Actual $d = 30 - 3.5 = 26.5$ in. Check the moment capacity of the section, $a = 8 \times 60/(0.85 \times 4 \times 20) = 7.06$ in.

$$\phi M_n = 0.9 \times 8 \times 60 \frac{\left(26.5 - \frac{7.06}{2}\right)}{12} = 826.9 \text{ K·ft}$$

which is greater than 825.5 K·ft. Check that $A_s = 8$ in.^{2} is less than $A_{s\,\text{max}}$.

$$A_{s\,\text{max}} = 0.01806 \times 20 \times 26.5 = 9.57 \text{ in.}^2$$

which exceeds 8 in.^{2} The final section is shown in Fig. 4.12.

Example 4.12

The two-hinged frame shown in Fig. 4.13 carries a uniform service dead load (including estimated self-weight) of 2.33 K/ft and a uniform service live load of 1.5 K/ft on frame beam BC. The moment at the corner B (or C) can be evaluated for this frame dimension, $M_b = M_c = -\,wL^2/18.4$ and the reaction at A or $D = wL/2$. A typical section of beam BC is shown, the column section is 16 × 21 in. It is required to

a. Draw the bending moment and shear diagrams for the frame $ABCD$ showing all critical values.

b. Design the beam BC for the factored moments, positive and negative, using $f'_c = 4$ ksi and $f_y = 60$ ksi. Show reinforcement details.

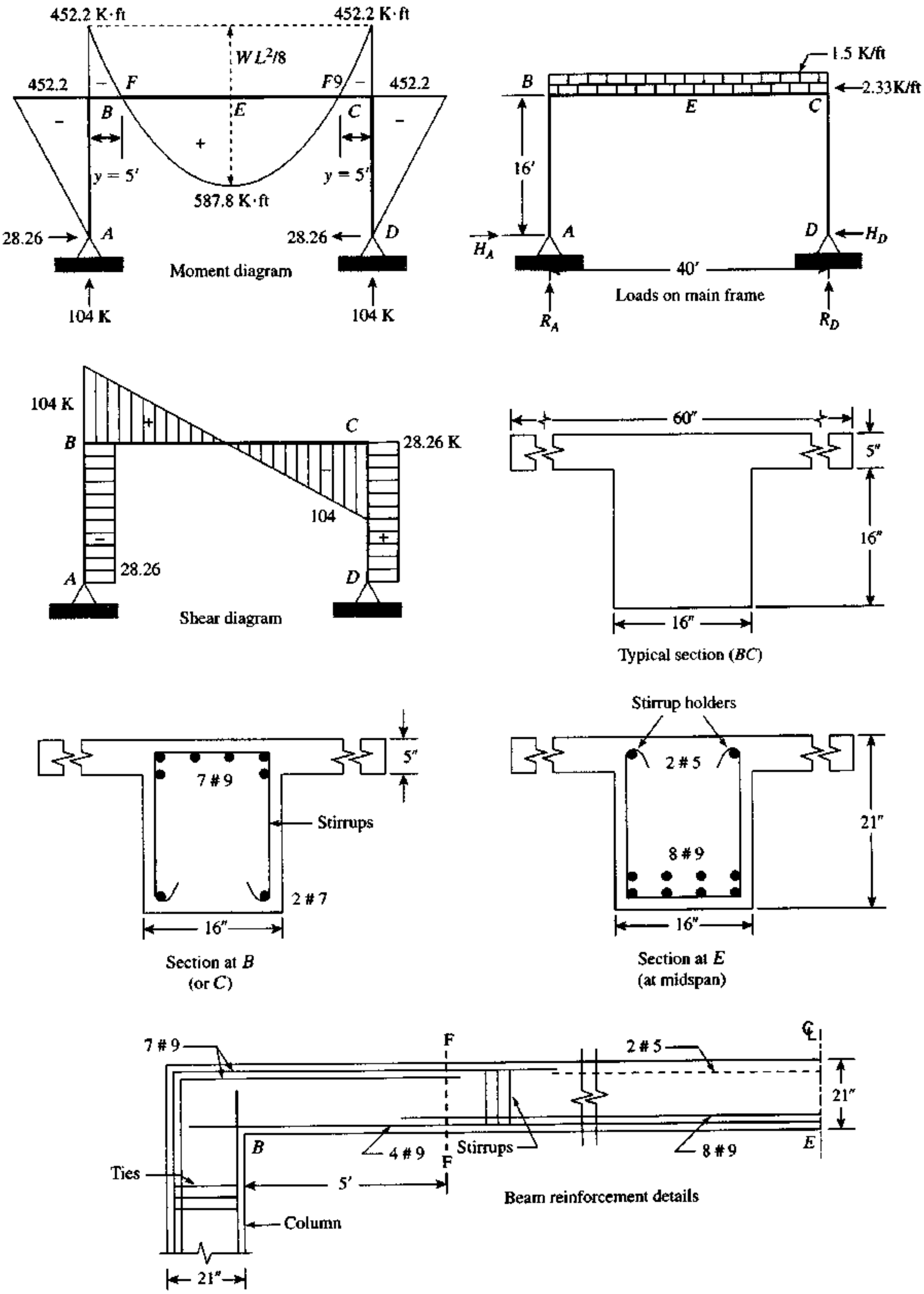


Figure 4.13 Example 4.12.

Solution

a. Calculate the forces acting on the frame using a computer program or the values mentioned previously. Factored load (w_u) = 1.2(2.33) + 1.6(1.5) = 5.2 K/ft. Because of symmetry, $M_B = M_C = -\ wL^2/18.4 = -\ 5.2(40)^2/18.4 = -452.2$ K·ft. Positive moment at midspan (E) $= w_u L^2/8 + M_B = 5.2(40)^2/8 - 452.2 = 587.8$ K·ft. Vertical reaction at $A = R_A = R_D = w_u L/2 = 5.2(40)/2 = 104$ K. Horizontal reaction at $A = H_A = M_B/h = 452.2/16 = 28.26$ K. The moment and shear diagrams are shown in Fig. 4.13.

Determine the location of zero moment at section F on beam BC by taking moments about F:

$$104(y) - 28.26(16) - 5.2(y)^2/2 = 0 \qquad y = 4.963 \text{ ft}, \qquad \text{say, 5 ft from joint } B$$

b. Design of beam BC:

1. Design of section E at midspan: $M_u = +\ 587.8$ K·ft. Assuming two rows of bars, $d = 21 - 3.5 = 17.5$ in. Calculate the moment capacity of the flange using $a = 5.0$ in.

$$\phi M_n(\text{flange}) = \phi(0.85 f_c')ab\left(d - \frac{5}{2}\right)$$

$$= 0.9(0.85 \times 4) \times (5 \times 60) \times \frac{(17.5 - 2.5)}{12} = 1147.5 \text{ K·ft}$$

which is greater than the applied moment; therefore, a is less than 5.0 in.

Assume $a = 2.0$ in. and calculate A_s.

$$M_u = \phi A_s f_y\left(d - \frac{a}{2}\right)$$

$$587.8 \times 12 = 0.9 \times 60A_s(17.5 - 1.0) \quad \text{and} \quad A_s = 7.92 \text{ in.}^2$$

Check assumed $a = A_s f_y/(0.85 f_c' b) = 7.92 \times 60/(0.85 \times 4 \times 60) = 2.33$ in. Revised $A_s = 587.8 \times 12/(0.9 \times 60 \times 16.33) = 7.99$ in.^{2} Check revised a: $a = 7.99 \times 2.33/7.92 = 2.35$ in., which is very close to 2.33 in.

Alternatively, Eq. 4.2 can be used to get ρ and A_s. Choose eight no. 9 bars in two rows (area = 8.0 in.^{2}), ($b_{\min}$ = 11.8 in.). Extend four no. 9 bars on both sides to the columns. The other four bars can terminate where they are not needed, beyond section F; see the longitudinal section in Fig. 4.13.

2. Design of section at B: $M_u = -452.2$ K·ft. The section acts as a rectangular section, $b = 16$ in. and $d = 17.5$ in. The main tension reinforcement lies in the flange.

$$\rho_{\max} = 0.01806 \quad \text{and} \quad R_{u\max} = 820 \text{ psi (Table 4.1)}$$

Check the maximum moment capacity of the section as singly reinforced.

$$\phi M_{n\max} = R_{u\max} bd^2 = 0.82(16)(17.5)^2/12 = 334.8 \text{ K·ft}$$

which is less than the applied moment. Compression steel is needed.

$$A_{s1} = 0.01806(16)(17.5) = 5.06 \text{ in.}^2$$

$$M_{u2} = 452.2 - 334.8 = 117.4 \text{ K·ft}$$

$$M_{u2} = \phi A_{s2} f_y(d - d'); \text{ assume } d' = 2.5 \text{ in.}$$

$$117.4 \times 12 = 0.9A_{s2}(60)(17.5 - 2.5) \quad \text{and} \quad A_{s2} = 1.74 \text{ in.}^2$$

Total tension steel = 5.06 + 1.74 = 6.8 in.^{2} Use seven no. 9 bars in two rows (area used = 7.0 in.^{2}, which is adequate). For compression steel, use two no. 9 bars (area = 2.0 in.^{2}), extended from the positive moment reinforcement to the column. Actually, four no. 9 bars are available; see the longitudinal section in Fig. 4.13.

The seven no. 9 bars must extend in the beam BC beyond section F into the compression zone, and also must extend into the column BA to resist the column moment of 452.2 K·ft without any splices at joints B or C.

Check if compression steel yields by using Eq. 3.49 or Table 3.4. K = 0.01552 (d'/d) = 0.1552(2.5)/(17.5) = 0.02217 > ρ_1 = 0.01806. Therefore, compression steel yields, and $f'_s = 60$ ksi as assumed.

Stirrups are shown in the beam to resist shear (refer to Chapter 8), and two no. 5 bars were placed at the top of the beam to hold the stirrups in position. Ties are used in the column to hold the vertical bars (refer to Chapter 10). To determine the extension of the development length of bars in beams or columns, refer to Chapter 7.

4.7 EXAMPLES USING SI UNITS

Example 4.13

Design a singly reinforced rectangular section to resist a factored moment of 280 kN·m using the maximum steel percentage for tension-controlled sections. Given: $f'_c = 20$ N/mm^2, $f_y = 400$ N/mm^2, and $b = 250$ mm.

Solution

$$\rho_b = (0.85)\beta_1 \left[\frac{f'_c}{f_y}\right]\left(\frac{600}{600+f_y}\right)$$

$$= 0.85 \times 0.85 \times \frac{20}{400} \times \left(\frac{600}{600+400}\right) = 0.0217$$

$$\rho_{\max} = \left(\frac{0.003 + \frac{f_y}{E_s}}{0.008}\right)\rho_b \qquad E_s = 200{,}000 \text{ MPa} \qquad \frac{f_y}{E_s} = 0.002$$

$$= 0.625\ \rho_b = 0.01356$$

$$\phi = 0.9$$

$$R_{u\max} = \phi\rho_{\max} f_y \left(1 - \frac{\rho_{\max} f_y}{1.7 f'_c}\right)$$

$$= 0.9 \times 0.01356 \times 400 \left(1 - \frac{0.01356 \times 400}{1.7 \times 20}\right) = 4.1 \text{ N/mm}^2\text{(MPa)}$$

$$M_u = R_u b d^2$$

$$d = \sqrt{\frac{M_u}{R_u b}} = \sqrt{\frac{280 \times 10^6}{4.1 \times 250}} = 523 \text{ mm}$$

$$A_s = \rho b d = 0.01356 \times 250 \times 523 = 1772 \text{ mm}^2 = 17.72 \text{ cm}^2$$

Choose four bars, 25 mm diameter, in two rows.

A_s provided = 4 × 4.9 = 19.6 cm^2. Total depth is

$$h = d + 25 \text{ mm} + 60 \text{ mm}$$

$$= 523 + 25 + 60 = 608 \text{ mm} \qquad \text{say, 610 mm (or 600 mm)}$$

Check minimum width:

$$b_{min} = 2D + 1S + 95 \text{ mm} = 3 \times 25 + 95 = 170 \text{ mm} < 250 \text{ mm}$$

Bars are placed in two rows.

Example 4.14

Calculate the required reinforcement for a beam that has a section of $b = 300$ mm and a total depth of $h = 600$ mm to resist $M_u = 696$ kN·m. Given: $f_c' = 30$ N/mm^2 and $f_y = 420$ N/mm^2.

Solution

1. Determine the design moment strength of the section using ρ_{max} (for tension-controlled section, $\phi = 0.9$):

$$\rho_b = (0.85)\beta_1 \left[\frac{f_c'}{f_y}\right]\left(\frac{600}{600 + f_y}\right)$$

$$= 0.85 \times 0.85 \times \frac{30}{420} \times \left(\frac{600}{600 + 1020}\right) = 0.0304$$

$$\rho_{max} = \left(\frac{0.003 + \frac{f_y}{E_s}}{0.008}\right) \rho_b = 0.6375\ \rho_b = 0.01938$$

$$R_{u\,max} = \phi \rho_{max} f_y \left(1 - \frac{\rho_{max} f_y}{1.7 f_c'}\right)$$

$$= 0.9 \times 0.01938 \times 420 \left(1 - \frac{0.01938 \times 420}{1.7 \times 30}\right) = 6.16 \text{ N/mm}^2\text{(MPa)}$$

$$d = h - 85 \text{ mm(assuming two rows of bars)}$$

$$= 600 - 85 = 515 \text{ mm}$$

$$\phi M_n = R_u b d^2 = 6.16 \times 300 \times (515)^2 \times 10^{-6} = 490 \text{ kN·m}$$

This is less than the external moment; therefore, compression reinforcement is needed.

2. Calculate A_{s1}, M_{u1}, and M_{u2}:

$$A_{s1} = \rho_{max} b d = 0.01938 \times 300 \times 515 = 2994 \text{ mm}^2$$

$$M_{u2} = M_u - M_{u1} = 696 - 490 = 206 \text{ kN·m}$$

3. Calculate A_{s2} and A_s' due to M_{u2}. Assume d' = 60 mm:

$$M_{u2} = \phi A_{s2} f_y (d - d')$$

$$206 \times 10^6 = 0.9 A_{s2} \times 420(515 - 60) \quad A_{s2} = 1198 \text{ mm}^2$$

Total tension steel is 2994 + 1198 = 4192 mm^2.

4. Compression steel yields if

$$(\rho - \rho') = \rho_1 \geq 0.85\beta_1 \times \frac{f_c'}{f_y} \times \frac{d'}{d} \times \frac{600}{600 - f_y} = K$$

$$K = (0.85)^2 \times \frac{30}{420} \times \frac{60}{515} \times \frac{600}{600 - 420} = 0.020$$

Because $(\rho - \rho') = \rho_{max} = 0.01938 < 0.020$, compression steel does not yield.

5. Calculate

$$a = \frac{A_s f_y}{0.85 f_c' b}$$

$$= \frac{2994(420)}{0.85 \times 30 \times 300} = 164.4 \text{ mm}$$

$$c = \frac{a}{0.85} = 193.4 \text{ mm} \qquad d' = 60 \text{ mm}$$

$$f_c' = 600\left(\frac{c - d'}{c}\right) = 414 \text{ N/mm}^2$$

$$A_s' = A_{s2}(420/414) = 1215 \text{ mm}^2$$

6. Choose steel bars as follows: For tension, choose six bars 30 mm in diameter (30 M). The A_s provided (4200 mm) is greater than A_s, as required. For compression steel, choose three bars 25 mm in diameter (25 M) (Table B.11).

$$A_s' = 1500 \text{ mm}^2 > 1215 \text{ mm}^2$$

SUMMARY

Sections 4.1–4.3 : Design of a Singly Reinforced Rectangular Section

Given: M_u (external factored moment), f_c' (compressive strength of concrete), and f_y (yield stress of steel).

Case 1.
When b, d, and A_s (or ρ) are *not* given:

1. Assume $\rho_{\min} \leq \rho \leq \rho_{\max}$. Choose $\rho_{\max}$ for a minimum concrete cross-section (smallest) or choose ρ between $\rho_{\max}/2$ and $\rho_b/2$ for larger sections. For example, if $f_y = 60$ ksi, you may choose

$$\rho = 1.2\% \qquad R_n = 618 \text{ psi} \quad \text{for } f_c' = 3 \text{ ksi}$$

$$\rho = 1.4\% \qquad R_n = 736 \text{ psi} \quad \text{for } f_c' = 4 \text{ ksi}$$

$$\rho = 1.4\% \qquad R_n = 757 \text{ psi} \quad \text{for } f_c' = 5 \text{ ksi}$$

For any other value of ρ, $R_n = \rho f_y[1 - (\rho f_y/1.7 f_c')]$, and $R_u = \phi R_n$.

2. Calculate $bd^2 = M_u/\phi R_n(\phi = 0.9)$, for tension-controlled sections.
3. Choose b and d. The ratio of d to b is approximately $1 \rightarrow 3$, or $d/b \approx 2.0$.
4. Calculate $A_s = \rho bd$; then choose bars to fit in b either in one row or two rows. (Check $b_{\min}$ from the tables.)
5. Calculate

$$h = d + 2.5 \text{ in. (for one row of bars)}$$

$$h = d + 3.5 \text{ in. (for two rows of bars)}$$

b and h must be to the nearest higher inch. *Note*: If h is increased, calculate new $d = h - 2.5$ (or 3.5) and recalculate A_s to get a smaller value.

Case 2.
When ρ is given, d, b, and A_s are required. Repeat steps (1) through (5) from Case 1.

Case 3.
When b and d (or h) are given, A_s is required.

1. Calculate $R_n = M_u/\phi bd^2$ ($\phi = 0.9$).
2. Calculate

$$\rho = \left(\frac{0.85f'_c}{f_y}\right)\left[1 - \sqrt{1 - \frac{2R_n}{0.85f'_c}}\right]$$

   (or get ρ from tables or Eq. 4.2).
3. Calculate $A_s = \rho bd$, choose bars, and check b_{min}.
4. Calculate h to the nearest higher inch (see note, Case 1 (step 5)).

Case 4.
When b and ρ are given, d and A_s are required.

1. Calculate

$$R_n = \rho f_y\left(1 - \frac{\rho f_y}{1.7f'_c}\right) \qquad R_u = \phi R_n(\phi = 0.9)$$

2. Calculate

$$d = \sqrt{\frac{M_u}{\phi R_n b}}$$

3. Calculate $A_s = \rho bd$, choose bars, and check b_{min}.
4. Calculate h to the nearest higher inch (see note, Case 1 (step 5)). *Note*: Equations that may be used to check the moment capacity of the section after the final section is chosen are

$$M_u = \phi M_n = \phi A_s f_y\left(d - \frac{A_s f_y}{1.7f'_c b}\right) = \phi A_s f_y d\left(1 - \frac{\rho f_y}{1.7f'_c}\right)$$

$$= \phi\rho f_y(bd^2)\left(1 - \frac{\rho f_y}{1.7f'_c}\right) = R_u bd^2$$

Section 4.4: Design of Rectangular Sections with Compression Steel

Given: M_u, b, d, d', f'_c, f_y, and $\phi = 0.9$.
Required: A_s and A'_s.

1. *General*
   a. Calculate ρ_{max} and ρ_{min} as singly reinforced from equations (or from tables).
   b. Calculate $R_{n\,max} = \rho_{max} f_y\left[1 - \left(\frac{\rho_{max} f_y}{1.7f'_c}\right)\right]$ (or use tables).
   c. Calculate the maximum capacity of the section as singly reinforced:

$$\phi M_n = \phi R_{n\,max} bd^2.$$

d. If $M_u > \phi M_n$, then compression steel is needed. If $M_u < \phi M_n$, it is a singly reinforced section.

2. If $M_u > \phi M_n$ and compression steel is needed,

a. Let $M_{u1} = \phi R_{n\,\max} bd^2$.

b. Calculate $A_{s1} = \rho_{\max} bd$ (basic section).

c. Calculate $M_{u2} = M_u - M_{u1}$ (for the steel section).

3. Calculate A_{s2} and A'_s as steel section.

a. $M_{u2} = \phi A_{s2} f_y (d - d')$.

b. Calculate total tension steel: $A_s = A_{s1} + A_{s2}$.

4. Calculate A'_s (compression steel area):

a. Calculate $a = (A_{s1} f_y / 0.85 f'_c b)$ and $c = a/\beta_1$.

b. Calculate $f'_s = 87[(c - d')/c] \le f_y$.

If $f'_s \ge f_y$, then $f'_s = f_y$ and $A'_s = A_{s2}$.

If $f'_s < f_y$, then $A'_s = A_{s2}\left(\frac{f_y}{f'_s}\right)$.

c. Check that total steel area $(A_s) \ge$ max A_s, or check $\varepsilon_t \ge 0.005$

$$A_s \le \left[\rho_{\max}(bd) + A'_s\left(\frac{f'_s}{f_y}\right)\right]$$

Section 4.5: Design of T-Sections

Given: M_u, f'_c, f_y, b, t, and b_w.
Required: A_s and d (if not given).
There are two cases:

Case 1.
When d and A_s (or ρ) are *not* given:

1. Let $a \le t$ (as singly reinforced rectangular section). If $a = t$ is assumed, then

$$M_u = (\text{total flange}) = \phi(0.85 f'_c)bt\left(d - \frac{t}{2}\right) = \phi A_s f_y\left(d - \frac{t}{2}\right)$$

Solve for d and then for A_s.

$$d = \frac{M_u}{\phi(0.85 f'_c)bt} + \frac{t}{2} A_s = \frac{M_u}{\phi f_y\left(d - \frac{t}{2}\right)}$$

2. If a is assumed to be less than t, then

$$d = \frac{M_u}{\phi(0.85 f'_c)ba} + \frac{a}{2} \text{ and } A_s = \frac{M_u}{\phi f_y\left(d - \frac{a}{2}\right)}$$

Case 2.
When d is given and A_s is required (one unknown):

1. Check if a is greater or less than t by considering the moment capacity of the flange (bt).

$$\text{(flange) } \phi M_n = \phi(0.85f'_c)bt\left(d - \frac{t}{2}\right)$$

If $\phi M_n > M_u$ (external), then $a < t$ (rectangular section).
If $\phi M_n < M_u$ (external), then $a > t$ (T-section).

2. If $a < t$, calculate $R_n = M_u/\phi bd^2$ and then calculate ρ (or determine ρ from tables or Eq. 4.2):

$$\rho = \frac{0.85f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_n}{0.85f'_c}}\right)$$

Then calculate $A_s = \rho bd$.

3. If $a > t$,

a. Calculate C_f and A_{sf}.

$$A_{sf} = 0.85f'_c t\frac{(b - b_w)}{f_y} = \frac{C_f}{f_y}\text{(flange)}$$

Then calculate M_{uf} (flange) $= \phi C_f(d - t/2)$.

b. Calculate M_{uw} (web) $= M_u - M_{uf}$. Calculate R_n (web) $= M_{uw}/(\phi b_w d^2)$; then find ρ_w (use the equation or tables). Calculate A_{sw} (web) $= \rho_w b_w d$.

c. Total $A_s = A_{sf}$ (flange) $+ A_{sw}$ (web). Total A_s must be less than or equal to $A_{s\,max}$ and greater than or equal to $A_{s\,min}$.

d. $\rho_w = \left(\frac{0.8f'_c}{f_y}\right)\left(1 - \sqrt{1 - \frac{2R_n}{0.85f'_c}}\right)$

e. Check that $\rho_w = A_s/b_w d \geq \rho_{min}$ (ρ_w = steel ratio in web) or $A_s > A_{s\,min}$, where $A_{s\,min} = \rho_{min}\,(b_w d)$. Check that $A_s \leq \max A_s$; or check $\varepsilon_t = (d_t - c)/c \geq 0.005$

PROBLEMS

4.1 Based on the information given in the accompanying table and for each assigned problem, design a singly reinforced concrete section to resist the factored moment shown in boldface. Use $f'_c = 4$ ksi and $f_y = 60$ ksi, and draw a detailed, neat section.

No.	M_u (K·ft)	b (in.)	d (in.)	ρ %
a	272.7	12	21.5	—
b	969.2	18	32.0	—
c	816.0	16	—	1.70
d	657.0	16	—	1.50
e	559.4	14	—	1.75
f	254.5	10	21.5	—
g	451.4	14	—	1.80
h	832.0	18	28.0	—
i	345.0	15	—	1.77

(continues)

No.	M_u (K·ft)	b (in.)	d (in.)	ρ %
j	510.0	0.5 d	—	ρ_{max}
k	720.0	—	2.5b	1.80
l	605.0	—	1.5b	1.80

For problems in SI units, 1 in. = 25.4 mm, 1 ksi = 6.9 MPa (N/mm_2), and 1 M_u (K·ft) = 1.356 kN·m.

4.2 Based on the information given in the following table and for each assigned problem, design a rectangular section with compression reinforcement to resist the factored moment shown. Use $f'_c =$ 4 ksi, $f_y = 60$ ksi, and $d' = 2.5$ in. Draw detailed, neat sections.

No.	M_u (K·ft)	b (in.)	d (in.)
a	554	14	20.5
b	790	16	24.5
c	448	12	18.5
d	520	12	20.5
e	765	16	20.5
f	855	18	22.0
g	555	16	18.5
h	300	12	16.5
i	400	16	16.5
j	280	12	16.5
k	290	14	14.5
l	400	14	17.5

For problems in SI units, 1 in. = 25.4 mm, 1 ksi = 6.9 MPa (N/mm2), and 1 M_u (K·ft) = 1.356 kN·m.

4.3 Based on the information given in the following table and for each assigned problem, calculate the tension steel and bars required to resist the factored moment shown. Use $f'_c = 3$ ksi and $f_y = 60$ ksi. Draw detailed, neat sections.

No.	M_u (K·ft)	b (in.)	b_w (in.)	t (in.)	d (in.)	Notes
a	394	48	14	3	18.5	
b	800	60	16	4	19.5	
c	250	44	15	3	15.0	
d	327	50	14	3	13.0	
e	577	54	16	4	18.5	
f	559	48	14	4	17.5	
g	388	44	12	3	16.0	
h	380	46	14	3	15.0	
i	537	60	16	3	16.5	
j	515	54	16	3	17.5	
k	361	44	15	3	15.0	
l	405	50	14	3	15.5	
m	378	44	16	3	—	Let $a = t$.
n	440	36	16	4	—	Let $a = t$.
o	567	48	12	3	—	Let $A_s = 6.0$ in^2.
p	507	46	14	3	—	Let $A_s = 7.0$ in^2.

For problems in SI units, 1 in. = 25.4 mm, 1 ksi = 6.9 MPa (N/mm2), and 1 1Mu (K·ft) = 1.356 kN·m.

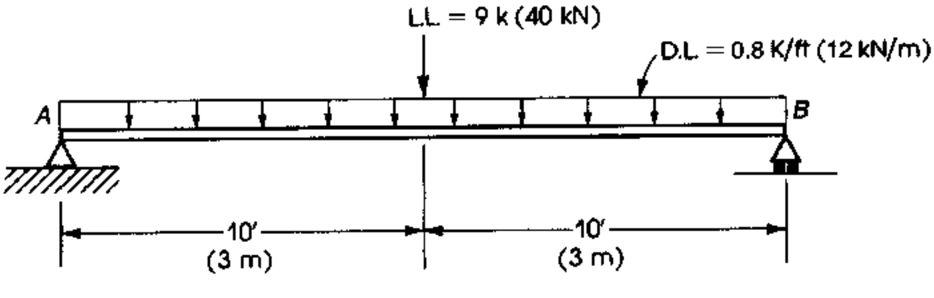


Figure 4.14 Problem 4.4.7.

4.4 Design a singly reinforced rectangular section to resist a factored moment of 232 K·ft (320 kN·m) if $f_c' = 4$ ksi (28 MPa), $f_y = 60$ ksi (420 MPa), and $b = 10$ in. (250 mm), using (a) ρ_{max}, (b) $\rho = 0.016$, and (c) $\rho = 0.012$.

4.5 Design a singly reinforced section to resist a factored moment of 186 K·ft (252 kN·m) if $b = 12$ in. (275 mm), $d = 20$ in. (500 mm), $f_c' = 3$ ksi (20 MPa), and $f_y = 40$ ksi (300 MPa).

4.6 Determine the reinforcement required for the section given in Problem 4.5 when $f_c' = 4$ ksi (30 MPa), and $f_y = 60$ ksi (400 MPa).

4.7 A simply supported beam has a 20-ft (6-m) span and carries a uniform dead load of 800 lb/ft (12 kN/m) and a concentrated live load at midspan of 9 kips (40 kN) (Fig. 4.14). Design the beam if $b = 12$ in. (300 mm), $f_c' = 4$ ksi (30 MPa), and $f_y = 60$ ksi (400 MPa). (Beam self-weight is not included in the dead load.)

4.8 A beam with a span of 24 ft (7.2 m) between supports has an overhanging extended part of 8 ft (2.4 m) on one side only. The beam carries a uniform dead load of 2.3 K/ft (30 kN/m) (including its own weight) and a uniform live load of 1.3 K/ft (18 kN/m) (Fig. 4.15). Design the smallest singly reinforced rectangular section to be used for the entire beam. Select steel for positive and negative moments. Use $f_c' = 4$ ksi (30 MPa), $f_y = 60$ ksi (400 MPa), and $b = 12$ in. (300 mm). (Determine the maximum positive and maximum negative moments by placing the live load once on the span and once on the overhanging part.)

4.9 Design a 15-ft (4.5-m) cantilever beam of uniform depth to carry a uniform dead load of 0.88 K/ft (12 kN/m) and a live load of 1.1 K/ft (15 kN/m). Assume a beam width $b = 14$ in. (350 mm), $f_c' = 4$ ksi (30 MPa), and $f_y = 60$ ksi (400 MPa).

4.10 10-ft (3-m) cantilever beam carries a uniform dead load of 1.50 K/ft (20 kN/m) (including its own weight) and a live load of 0.77 K/ft (10 kN/m) (Fig. 4.16). Design the beam using a variable depth. Draw all details of the beam and reinforcement. Given: $f_c' = 3$ ksi (20 MPa), $f_y = 40$ ksi (300 MPa), and $b = 12$ in. (300 mm). Assume h at the free end is 10 in. (250 mm).

4.11 Determine the necessary reinforcement for a concrete beam to resist an external factored moment of 290 K·ft (400 kN·m) if $b = 12$ in. (300 mm), $d = 19$ in. (475 mm), $d' = 2.5$ in. (65 mm), $f_c' = 3$ ksi (20 MPa), and $f_y = 60$ ksi (400 MPa).

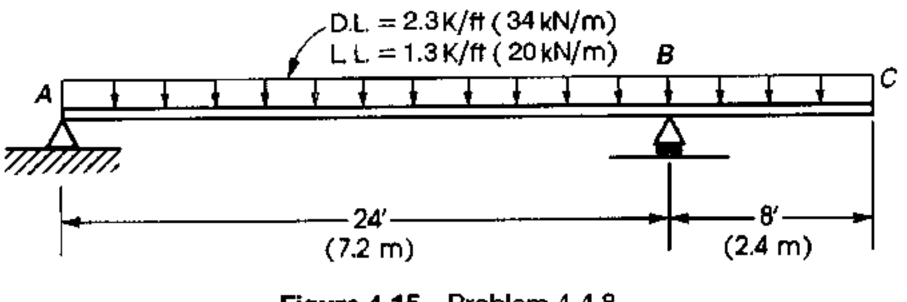


Figure 4.15 Problem 4.4.8.

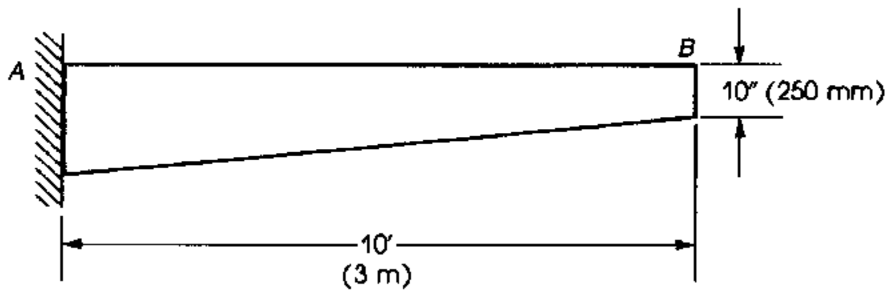


Figure 4.16 Problem 4.4.10.

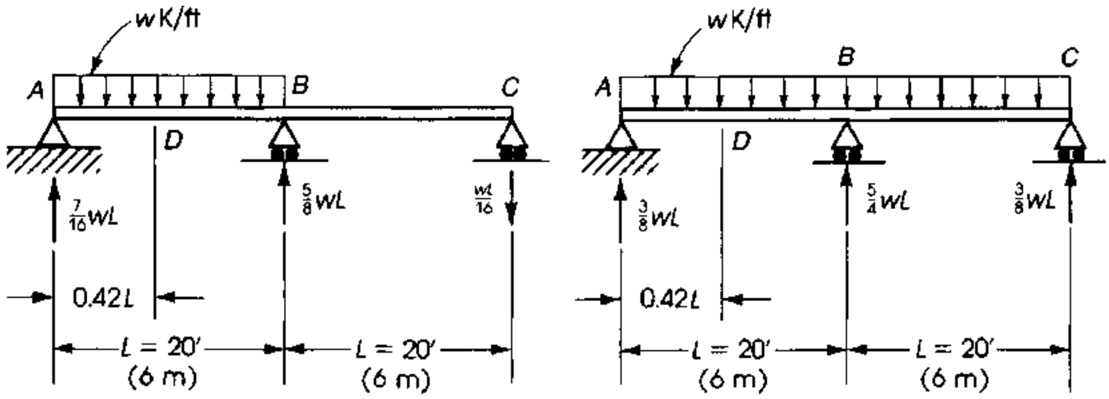


Figure 4.17 Problem 4.4.14.

4.12 Design a reinforced concrete section that can carry a factored moment of 260 K·ft (360 kN·m) as

a. Singly reinforced, $b = 10$ in. (250 mm)

b. Doubly reinforced, 25% of the moment to be resisted by compression steel, $b = 10$ in. (250 mm)

c. T-section, which has a flange thickness of 3 in. (75 mm), flange width of 20 in. (500 mm), and web width of 10 in. (250 mm)

$f'_c = 3$ ksi (20 MPa), and $f_y = 60$ ksi (400 MPa), for all problems.

Determine the quantities of concrete and steel designed per foot length (meter length) of beams and then determine the cost of each design if the price of the concrete equals \$50/yd^3 (67/m^3) and that of steel is \$0.30/lb (\$0.66/kg). Tabulate and compare results.

4.13 Determine the necessary reinforcement for a T-section that has a flange width of $b = 40$ in. (1000 mm), flange thickness of $t = 4$ in. (100 mm), and web width of $b_w = 10$ in. (250 mm) to carry a factored moment of 545 K·ft (750 kN·m). Given: $f'_c = 3$ ksi (20 MPa) and $f_y = 60$ ksi (400 MPa).

4.14 The two-span continuous beam shown in Fig. 4.17 is subjected to a uniform dead load of 2.6 K/ft (including its own weight) and a uniform live load of 3 K/ft. The reactions due to two different loadings are also shown. Calculate the maximum negative factored moment at the intermediate support B and the maximum positive factored moment within the span AB (at $0.42L$ from support A), design the critical section at B and D, and draw the reinforcement details for the entire beam ABC. Given: $L = 20$ ft, $b = 12$ in., $h = 24$ in. Use $d = 18$ in. for one row of bars and $d = 17$ in. for two rows. $f'_c = 4$ ksi, and $f_y = 60$ ksi.

4.15 The two-hinged frame shown in Fig. 4.18 carries a uniform dead load of 3.93 K/ft and a uniform live load of 2.4 K/ft on BC. The reactions at A and D can be evaluated as follows: $HA = HD = wL/9$ and $RA = RD = wL/2$, where w = uniform load on BC. A typical cross-section of the beam BC is also shown. It is required:

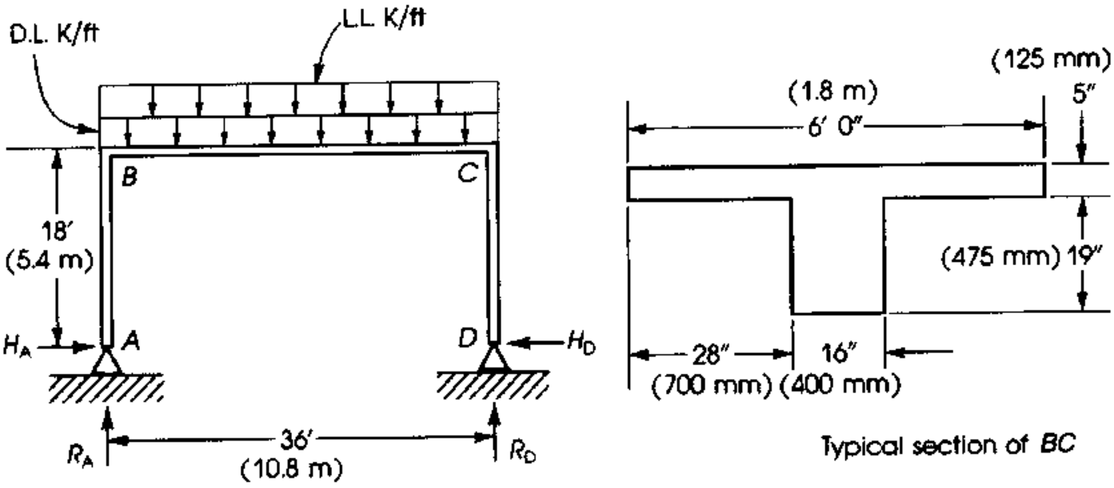


Figure 4.18 Problem 4.4.15.

a. Draw the bending moment diagram for the frame $ABCD$.

b. Design the beam BC for the applied factored moments (positive and negative).

c. Draw the reinforcement details of BC.

Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

CHAPTER 5

ALTERNATIVE DESIGN METHODS

Office building, Minneapolis, Minnesota.

5.1 INTRODUCTION

In the previous chapters, 3 and 4, the analysis and design of flexural reinforced concrete members were explained based on the provisions of the ACI Code 318-08. An alternative design approach is presented in Appendix B of the ACI Code according to the load factors given in Appendix C. This alternative design method was the basis of analysis and design in the ACI Code 318-99. It is to some extent similar to the method explained earlier except that it uses different load factors and strength reduction, ϕ. The basic analysis and design equations of the previous chapters will be used here. When Appendix B provisions are used in the design, they should replace all other corresponding provisions in the body of the Code.

5.2 LOAD FACTORS

If the required strength is denoted by U and those due to wind and seismic forces are W and E, respectively, then according to the ACI Code, Appendix C, the required strength U, shall be the most critical of the following:

1. In the case of dead, live, and wind loads,

$$U = 1.4D + 1.7L \quad (5.1a)$$

$$U = 0.75(1.4D + 1.7L) + (1.6W \text{ or } 1.0E) \quad (5.1b)$$

$$U = 0.9D + (1.6W \text{ or } 1.0E) \quad (5.1c)$$

2. When wind load, W, has not been reduced by a directionality factor, $1.3W$ can be used in place of $1.6W$. When seismic load is based on service forces, $1.4E$ can be used in place of $1.0E$.
3. In cases when earth pressure load, H, must be included in the design,

$$U = 1.4D + 1.7L + 1.7H \quad (5.2a)$$

Where dead load, D, and live load, L, reduce the effect of H, U shall be checked for

$$U = 0.9D + 1.7H \quad (5.2b)$$

For any combination of D, L, or H,

$$U = 1.4D + 1.7L$$

4. If weight and pressure loads from liquids, F, must be included in the design,

$$U = 1.4D + 1.7L + 1.4F \quad (5.3a)$$

Where dead load, D, and live load, L, reduce the effort of F,

$$U = 0.9D + 1.4F \quad (5.3b)$$

For any combination of D, L, or F,

$$U = 1.4D + 1.7L$$

The vertical pressure of liquids shall be considered as dead load with due regard to variation in liquid depth.
5. When impact effects are taken into account, they shall be included in the live load.
6. Where the structural effects, T, of differential settlement, creep, shrinkage, or temperature change may be significant, they shall be included with the most critical of

$$U = 0.75(1.4D + 1.4T + 1.7L) \quad (5.4a)$$

$$U = 1.4D + 1.4T \quad (5.4b)$$

Equation 5.1a is most generally used. The dead load factor is equal to 1.4, whereas the live load factor is equal to 1.7.

For applied concentrated dead and live loads, P_D and P_L, the factored concentrated load is $P_U = 1.4P_D + 1.7P_L$; also $M_U = 1.4M_D + 1.7M_L$, where M_D and M_L are the actual dead load and live load moments, respectively.

5.3 STRENGTH-REDUCTION FACTOR, ϕ

The nominal strength of a section is reduced by a factor ϕ to account for small adverse variations in material strengths, artisanry, dimensions, control, and degree of supervision. The factor ϕ constitutes a portion of the factor of safety, as discussed in Section 1.8.

The ACI Code, Section C.9.3 (Appendix C), specifies the following values to be used:

- Tension-controlled sections: $\phi = 0.90$
- Compression-controlled sections
  - Members with spiral reinforcement: $\phi = 0.75$
  - Other reinforced members: $\phi = 0.70$
- Shear and torsion: $\phi = 0.85$
- Bearing on concrete: $\phi = 0.70$
- Bending in plain concrete or in concrete with minimum reinforcement of $200/f_y$: $\phi = 0.65$

For sections that lie in the transition region between tension- and compression-controlled sections, ϕ may be increased linearly to 0.9.

Also, the strength eduction factor ϕ to be used for columns (or sections with $\varepsilon_t < 0.005$) may vary according to the following cases:

1. When $P_u = \phi P_n \geq 0.1 f'_c A_g$, then ϕ is 0.70 for tied columns and 0.75 for spirally reinforced columns. This case occurs generally when compression controls. A_g is the gross area of the concrete region.
2. Between values of $0.1 f'_c A_g$ or ϕP_b (whichever is smaller) and 0, P_u lies in the tension control zone and ϕ is larger than 0.7 (or 0.75). The ACI Code, Section C.9.3.2, specifies that for members in which f_y does not exceed 60 ksi, with symmetrical reinforcement and with the distance between compression and tension steel $(d - d')$ not less than $0.7h$ (h = total depth of section) and $d = h - d_s$, the value of ϕ is increased linearly to 0.9.

For this transition region, ϕ may be determined by linear interpolation between 0.7 (or 0.75) and 0.9. Figure 5.1 shows the variation of ϕ for grade 60 steel. The linear equations are as follows:

$$\phi = 0.57 + 67\varepsilon_t \qquad \text{(for tied sections)} \tag{5.5}$$

$$\phi = 0.65 + 50\varepsilon_t \qquad \text{(for spiral sections)} \tag{5.6}$$

Alternatively, ϕ in the transition region can be determined as a function of (d_t/c) for grade 60 steel as follows:

$$\phi = 0.37 + 0.20\left(\frac{d_t}{c}\right) \qquad \text{(for tied sections)} \tag{5.7}$$

$$\phi = 0.50 + 0.15\left(\frac{d_t}{c}\right) \qquad \text{(for spiral sections)} \tag{5.8}$$

where c is the depth of the neutral axis at nominal strength.

5.4 RECTANGULAR SECTIONS WITH TENSION REINFORCEMENT

From the analysis of rectangular singly reinforced section (Section 3.9), the following equations were derived, where f'_c and f_y are in ksi:

$$\rho_b = 0.85\beta_1 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right) \tag{3.18}$$

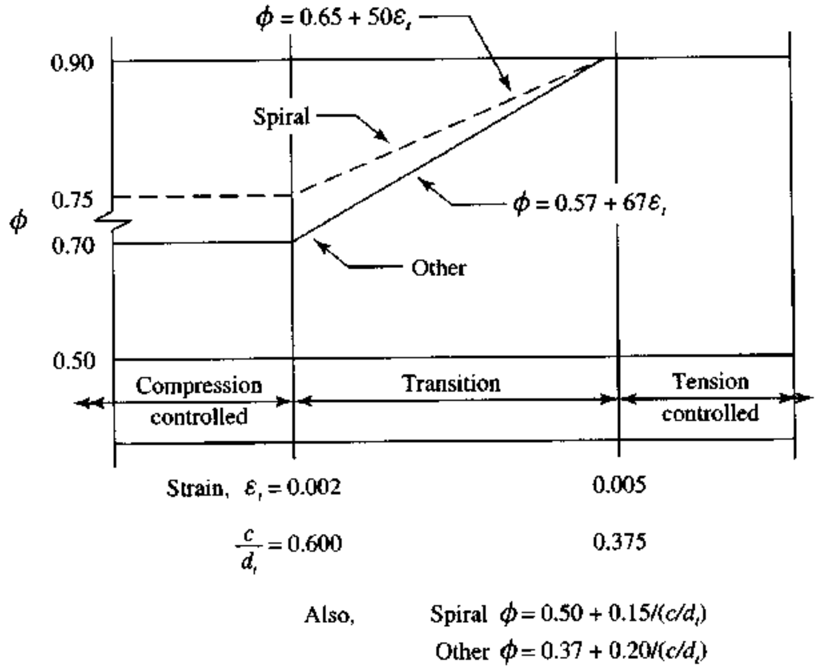


Figure 5.1 Variation of ϕ with the net tensile strain for grade 60 steel [1]. Courtesy of ACI 318-08.

If the maximum percentage of reinforcement is limited to $0.75\rho_b$, then

$$\rho_b = 0.75\rho_b = 0.6375\beta_1 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right) \tag{5.9}$$

It is to be noted that $\rho_{max} = 0.75\rho_b$ is greater than that of $0.634\rho_b$ as given earlier in Chapter 3, (Eq. 3.30 for $f_y = 60$ ksi).

For $f'_c \leq 4000$ psi,

$$\rho_{max} = 0.542 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right) \tag{5.10}$$

The value of β_1 is 0.85 when $f'_c \leq 4000$ psi (30 N/mm^2) and decreases by 0.05 for every increase of 1000 psi (7 N/mm^2) in concrete strength, or $\beta_1 = 0.85 - 0.05(f'_c - 4) \geq 0.65$.

The steel percentage of a balanced section, ρ_b, and the maximum allowable steel percentage, ρ_{max}, can be calculated for different values of f'_c and f_y, as shown in Table 5.1. Suggested design steel ratios for $\rho \leq \rho_{max}$ are also shown in Table 5.1.

The design moment equations were derived in the previous chapter in the following forms:

$$\phi M_n = M_u = R_u bd^2 \tag{3.21}$$

where

$$R_u = \phi \rho f_y \left(1 - \frac{\rho f_y}{1.7 f'_c}\right) = \phi R_n \tag{3.22}$$

Table 5.1 Suggested Design Steel Ratios ρ_s

f'_c (ksi)	f_y (ksi)	% ρ_b	% ρ_{max}	% ρ_s	Ratio ρ_s/ρ_b	Ratio ρ_s/ρ_{max}	R_u (psi)	$R_{u\,max}$ (psi)
3	40	3.71	2.78	1.4	0.38	0.50	450	783
	60	2.15	1.61	1.2	0.56	0.75	556	702
4	60	2.85	2.14	1.4	0.49	0.65	662	936
	75	2.07	1.55	1.2	0.58	0.77	702	867
5	60	3.36	2.52	1.4	0.42	0.56	681	1120
	75	2.44	1.83	1.2	0.49	0.66	722	1033

and $\phi = 0.9$. For tension-controlled sections, $\varepsilon_t \geq 0.005$:

$$\phi M_n = M_u = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b}\right) \tag{3.19a}$$

Also,

$$\phi M_n = M_u = \phi \rho f_y b d^2 \left(1 - \frac{\rho f_y}{1.7 f'_c}\right) \tag{3.20}$$

We can see that for a given factored moment and known f'_c and f_y, there are three unknowns in these equations: the width, b, the effective depth of the section, d, and the steel ratio, ρ. A unique solution is not possible unless values of two of these three unknowns are assumed. Usually ρ is assumed (using ρ_{max}, for instance), and b can also be assumed.

Based on the preceding discussion, the following cases may develop for a given M_u, f'_c and f_y:

1. If ρ is assumed, then R_u can be calculated from Eq. 3.19, giving $bd^2 = M_u/R_u$. The designer may use ρ up to ρ_{max}, which produces the minimum size of the singly reinforced concrete section. Using ρ_{min} will produce the maximum concrete section. If b is assumed in addition to ρ, then d can be determined as follows:

$$d = \sqrt{\frac{M_u}{R_u b}} \tag{5.11}$$

   If $d/b = 2$, then $d = \sqrt[3]{(2M_u/R_u)}$ and $b = d/2$, rounded to the nearest higher inch.

2. If b and d are given, the required reinforcement ratio, ρ, can be determined by rearranging Eq. 3.20 to obtain

$$\rho = \frac{0.85 f'_c}{f_y}\left[1 - \sqrt{1 - \frac{4M_u}{1.7\phi f'_c b d^2}}\right] \tag{5.12}$$

$$= \frac{0.85 f'_c}{f_y}\left[1 - \sqrt{1 - \frac{2R_n}{0.85 f'_c}}\right]$$

   and

$$A_s = \rho b d$$

where all units are in kips (or pounds) and inches. For example, if $M_u = 2440\,\text{K·in.}$, $b = 12\,\text{in.}$, $d = 18\,\text{in.}$, $f'_c = 3$ ksi, and $f_y = 60\,\text{ksi}$, then $\rho = 0.01389$ (from Eq. 5.22) and $A_s = \rho bd = 0.01389(12)(18) = 3.0\,\text{in}^2$. When b and d are given, it is better to check if compression steel is or is not required because of a small d. This can be achieved as follows:

a. Calculate $\rho_{\max}$ and $R_{u\max} = \phi\rho_{\max} f_y[1 - (\rho_{\max} f_y/1.7f'_c)]$.

b. Calculate $\phi M_{n\max} = R_{u\max} bd^2 =$ the maximum moment strength of a singly reinforced concrete section.

c. If $M_u < \phi M_{n\max}$, then no compression reinforcement is needed. Calculate ρ and A_s from the preceding equations.

d. If $M_u > \phi M_{n\max}$, then compression steel is needed.

3. If ρ and b are given, calculate R_u:

$$R_u = \phi\rho f_y\left(1 - \frac{\rho f_y}{1.7f'_c}\right)$$

The calculate d from Eq. 5.21:

$$d = \sqrt{\frac{M_u}{R_u b}} \quad \text{and} \quad A_s = \rho bd$$

Example 5.1

Find the necessary reinforcement for a given section 10 in. wide and 28 in. total depth (Fig. 5.2) if it is subjected to an external factored moment of 245 K·ft. Given: $f'_c = 4$ ksi and $f_y = 60\,\text{ksi}$.

Solution

1. Assuming one layer of no. 8 steel bars (to be checked later), $d = 28 - 2.5\,\text{in.} = 25.5\,\text{in.}$
2. Check if the section is adequate without compression reinforcement. Compare design moment strength of the section (using $\rho_{\max}$) with the design moment. For $f'_c = 4$ ksi and $f_y = 60\,\text{ksi}$, $\rho_{\max} = 0.02138$.

$$R_u = \phi\rho_{\max} f_y\left(1 - \frac{\rho_{\max} f_y}{1.7f'_c}\right) = 937\ \text{psi}$$

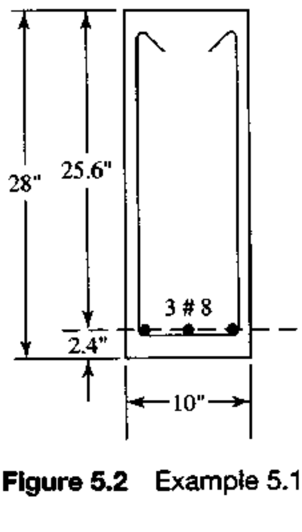


Figure 5.2 Example 5.1.

The design moment strength of a singly reinforced basic section is

$$\phi M_{n\,\max} = R_{u\,\max} bd^2 = 0.937(10)(25.5)^2$$

$$= 6093 \text{ K·in.} > 245 \times 12 = 2940 \text{ K·in.}$$

Therefore, $\rho < \rho_{\max}$ and the section is singly reinforced.

3. Calculate ρ from Eq. 5.12 to get $\rho = 0.009$. $A_s = \rho bd = 0.009(10)(25.5) = 2.30\,\text{in.}^2$ Use three no. 8 bars ($A_s = 2.35\,\text{in.}^2$) in one row, $b_{\min} < 10\,\text{in.}$ The final section is shown in Fig. 5.2.
4. Check ε_t:

$$a = \frac{2.35(60)}{0.85(4)(10)} = 4.15 \text{ in.}$$

$$c = \frac{a}{0.85} = 4.88 \text{ in.}$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = 0.0127 > 0.005 \quad \phi = 0.9$$

5.5 RECTANGULAR SECTIONS WITH COMPRESSION REINFORCEMENT

A singly reinforced section has a maximum design moment strength when $\rho_{\max}$ of steel is used. If the applied factored moment is greater than the internal moment strength, as in the case of a limited cross-section, a doubly reinforced section may be used, adding steel bars in both the compression and tension zones.

The procedure for designing a rectangular section with compressive steel when M_u, f'_c, b, d, and d' are given was summarized in Section 4.4. The only difference is that $\rho_{\max} = 0.75\,\rho_b$ is used in this design approach here.

$$\rho_{\max} = 0.6375\beta_1 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right) \tag{5.9}$$

Also, check that $\varepsilon_t \geq 0.005$ for $\phi = 0.9$.

Example 5.2

A beam section is limited to $b = 12\,\text{in.}$ and to a total depth of $h = 20\,\text{in.}$ and is subjected to a factored moment of $M_u = 330\,\text{K·ft}$. Determine the necessary reinforcement using $f'_c = 4$ ksi and $f_y = 60\,\text{ksi}$. (Refer to Fig. 5.3)

Solution

1. Determine the design moment strength of the section as singly reinforced. Assume $\rho = 0.018$. Therefore, $R_u = 818\,\text{psi}$ (Table A2). For two rows of bars, $d = 20 - 3.5 = 16.5\,\text{in.}$

$$M_{u1} = R_u bd^2 = 0.818(12)(16.5)^2 = 2672 \text{ K·in.}$$

The design moment is $M_u = 330 \times 12 = 3960\,\text{K·in.} > M_{u1}$; therefore, compression steel is needed.

2. Calculate A_{s1}, M_{u2}, A_{s2}, and total A_s.

$$A_{s1} = \rho bd = 0.018(12)(16.5) = 3.56 \text{ in.}$$

$$M_{u2} = M_u - M_{u1} = 3960 - 2672 = 1288 \text{ K·in.}$$

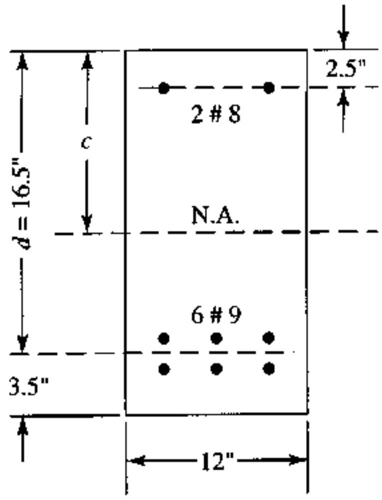


Figure 5.3 Example 5.2.

$$M_{u2} = \phi A_{s2} f_y (d - d'), \text{ assume } d' = 2.5 \text{ in.}$$

$$1288 = 0.9 A_{s2}(60)(16.5 - 2.5) \quad A_{s2} = 1.7 \text{ in.}^2$$

$$\text{Total } A_s = A_{s1} + A_{s2} = 3.56 + 1.7 = 5.26 \text{ in.}^2 \text{ (six no. 9 bars)}$$

3. Check if compression steel yields by Eq. 3.49. Compression steel yields if

$$\rho - \rho' \geq K = 0.85\beta_1 \frac{f_c'}{f_y}\left(\frac{d'}{d}\right)\left(\frac{87}{87 - f_y}\right)$$

$$K = (0.85)^2\left(\frac{4}{60}\right)\left(\frac{2.5}{16.5}\right)\left(\frac{87}{27}\right) = 0.0235$$

$$\rho - \rho' = \frac{A_{s1}}{bd} = \frac{3.56}{(12)(16.5)} = 0.018 \leq K$$

Therefore, compression steel does not yield: $f_s' < f_y$.

4. Calculate f_s' : $f_s' = 87[(c - d')/c] \leq f_y$. Determine c from A_{s1}: $A_{s1} = 3.56\,\text{in.}^2$

$$a = \frac{A_{s1} f_y}{0.85 f_c' b}$$

$$= \frac{3.56 \times 60}{0.85 \times 4 \times 12} = 5.24 \text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{5.24}{0.85} = 6.16 \text{ in.}$$

$$f_s' = 87 \times \left(\frac{6.16 - 2.5}{6.16}\right) = 52 \text{ ksi} < 60 \text{ ksi}$$

5. Calculate A_s' from $M_{u2} = \phi A_s' f_s'(d - d')$:

$$1288 = 0.9 A_s'(52)(16.5 - 2.5)$$

Thus, $A_s' = 1.97$ in.^{2}, or calculate A_s' from $A_s' = A_{s2}(f_y/f_s') = 1.97$ in.^{2} (two no. 9 bars).

6. Check

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003$$

$$d_t = h - 2.5 \text{ in.} = 17.5 \text{ in.}$$

$$\varepsilon_t = \left(\frac{17.5 - 6.16}{6.16}\right) 0.003$$

$$= 0.0055 > 0.005 \quad \phi = 0.9$$

or

$$\frac{c}{dt} = \frac{6.16}{17.5} = 0.352 < 0.375 \quad \text{(o.k.)}$$

7. Check final ϕM_n. $A_s = 6.0 \text{in.}^2$, $A'_s = 2.0 \text{ in.}^2$, $A_{s1} = 4.0 \text{in.}^2$, $a = 5.88$ in., and $c = 6.92$ in.

$$M_n = 4 \times 60\left(16.5 - \frac{5.88}{2}\right) + 2 \times 52(16.5 - 2.5) = 4710 \text{ K·in.}$$

Check ε_t, $d_t = 17.5$ in.

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = 0.0459 < 0.005$$

$$\phi = 0.57 + 67(0.0459) = 0.88$$

$$\phi M_n = 4145 \text{ K·in.} > M_u = 3960 \text{ K·in.}$$

5.6 DESIGN OF T-SECTIONS

In the design of a T-section for a given factored moment, M_u, the flange thickness t and width b would have been already established from the design of the slab and the ACI Code limitations for the effective flange width b, as given in Section 3.14. The web thickness, b_w, can be assumed to vary between 8 in. and 20 in., with a practical width of 12 to 16 in. Two more unknowns still need to be determined, d and A_s. The design procedure was summarized in Section 4.5.

Example 5.3

The T-beam section shown in Fig. 5.4 has a web width, b_w, of 10 in., a flange width, b, of 40 in., a flange thickness of 4 in., and an effective depth, d, of 14.5 in. Determine the necessary reinforcement if the applied factored moment is 3800 K·in. Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Check the position of the neutral axis; the section may be rectangular. Assume the depth of compression block a is 4 in.; that is, $a = t = 4$ in. Then

$$\phi M_n = \phi(0.85 f'_c) bt \left(d - \frac{t}{2}\right) = 6120 \text{ K·in.} > M_u = 3800 \text{ K·in.}$$

The design moment that the concrete flange can resist is greater than the factored moment applied. Therefore, the section behaves as a rectangular section.

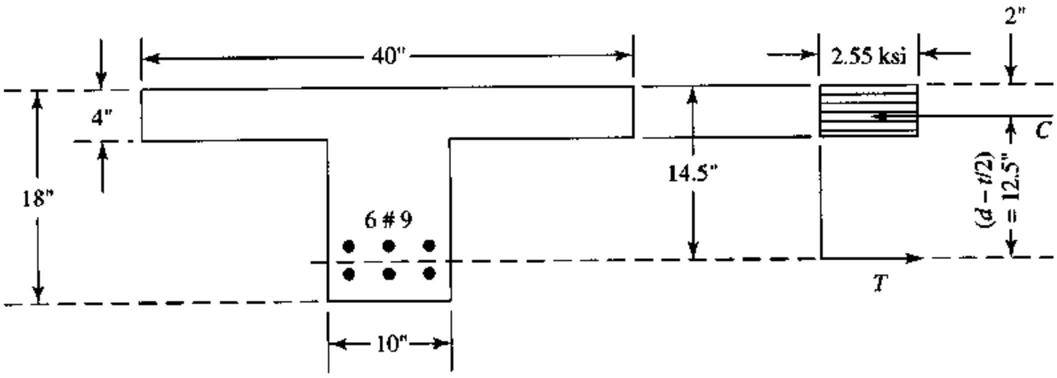


Figure 5.4 Example 5.3: T-section.

2. Determine the area of tension steel, considering a rectangular section, $b = 40$ in.

$$R_u = \frac{M_u}{(bd^2)} = \frac{3,800,000}{40 \times 14.5^2} = 452 \text{ psi}$$

From Eq. 5.22, for $R_u = 452$ psi, $\rho = 0.0091$.

$$A_s = \rho bd = 0.0091 \times 40 \times 14.5 = 5.28 \text{ in.}^2$$

Use six no. 9 bars, $A_s = 6.00 \text{ in.}^2$ (in two rows).

3. Check that $\rho_w = A_s/b_w d \geq \rho_{\min}$; $\rho_w = 5.28/(10 \times 14.5) = 0.0364 > \rho_{\min} = 0.00333$.
4. Check

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right)0.003 \qquad d_t = 14.5$$

$$a = \frac{5.28(60)}{0.85 \times 4 \times 40} = 2.33 \text{ in.} \qquad c = 2.74 \text{ in.}$$

$$\varepsilon_t = 0.0129 > 0.005 \qquad \phi = 0.9$$

Note that other examples will be similar to those in Chapters 3 and 4.

5.7 STRUT AND TIE METHOD

5.7.1 Introduction

The ACI Code, Appendix A, introduces an alternative approach to the method explained earlier in Chapter 3, called the strut and tie models. This alternative method can be applied effectively in regions of discontinuity in the structural member, such as support areas, zones of load application, or areas with sudden change in the geometrical dimensions as brackets and portal frames. In these regions, the plane sections do not remain plane after bending (as was assumed in Chapter 3, Section 3.2), and they are called D-regions (Fig. 5.5*a*). The other regions of a standard beam, the basic beam theory, and a linear strain relationship apply. These regions are called B-regions (Fig. 5.5*a*).

The discontinuity in the stress distribution in region D (due to geometry or loading condition), based on St. Venants principles, indicates that the stresses due to axial load and bending approach a linear distribution at a distance approximately equal to the height of the member, h,

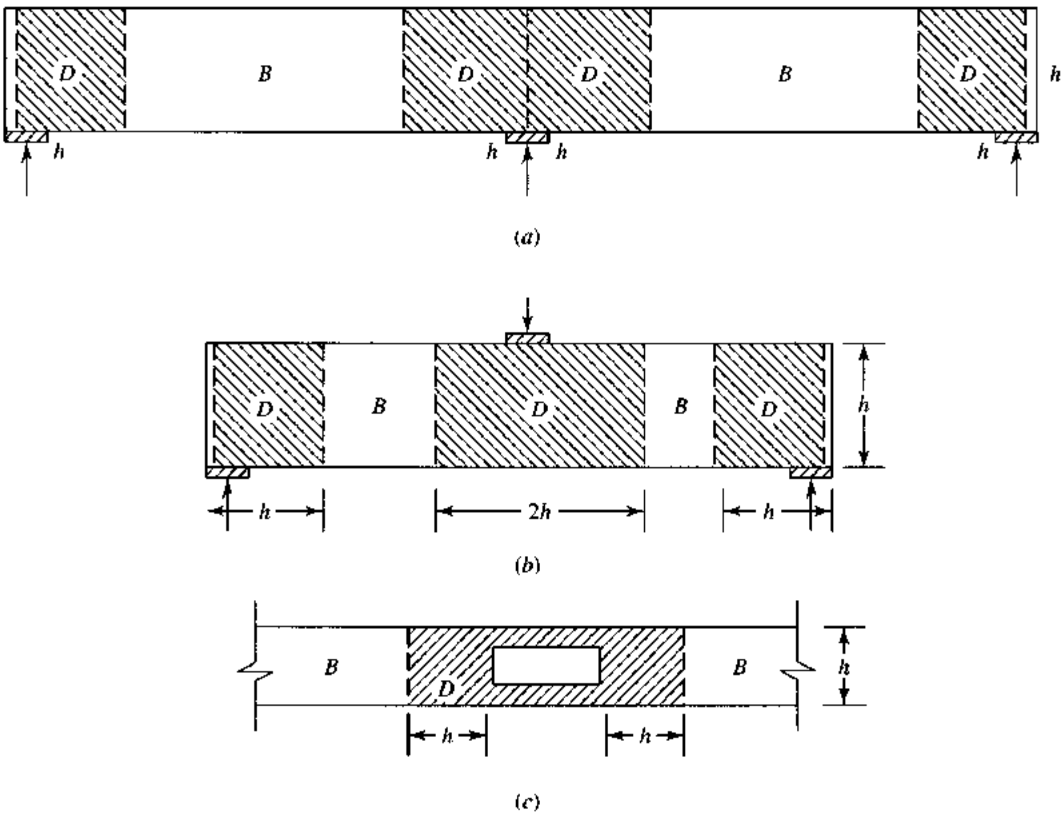


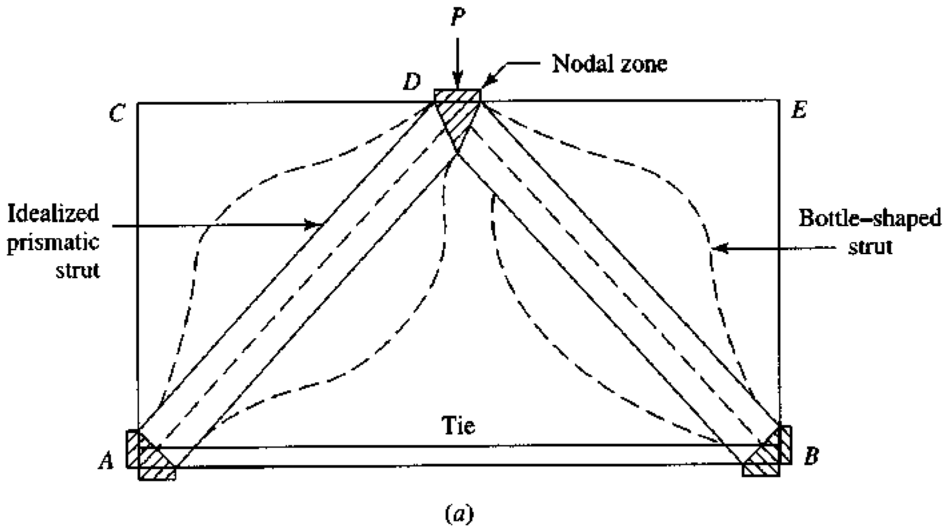
Figure 5.5 D- and B-regions in beam. (*a*) Continuous beam, (*b*) beam with concentrated load, (*c*) beam with an opening [1]. Courtesy of ACI 318-05.

away from the discontinuity (Fig. 5.5*b* and *c*) [1]. If two D-regions overlap or meet, they can be considered a single D-region. The maximum length to depth ratio would be equal to two, producing a minimum angle of 26.5° (tan $\frac{1}{2}$) between the strut and tie (or approximately 25°).

In a strut and tie model (Fig. 5.6), the point where the three forces meet at joint D is called a node, and the volume of concrete around a node is called a nodal zone. Forces at a node can vary between different combinations of compression and tensile forces, C-C-C, C-C-T, C-T-T, or T-T-T (Fig. 5.7). Figure 5.8 shows typical nodal zones for different load applications, while Fig. 5.9 shows extended nodal zones for one or more layers of reinforcing bars [6].

5.7.2 Strut and Tie Models

A strut and tie model can be represented by an idealized truss model with forces acting at the different nodes. Now consider the steel truss shown in Fig. 5.10. Due to symmetry, the reactions at A and B are equal, $R_A = R_B$ 20 K, and from the equilibrium of joints A and D, the tensile force in $AB = 20$ K, while the compressive force in AD or $BD = 28.3$ K. Member AB is considered a tie, while AD and BD act as struts. The forces in the other members are equal to 0. Comparing this truss with concrete beam in Fig. 5.6*a*, it can be seen that most of the areas ACD and BED and below the nodal zone D are not effective and act as fillers. The forces in the struts, for this loading condition, are greater than the force in the tie. In this case, adequate concrete areas are available to act as idealized struts (Fig. 5.6*a*). Steel reinforcement is needed to act as a tie for AB. Proper anchoring of the ties are essential for a safe design and should be anchored in a nodal zone.



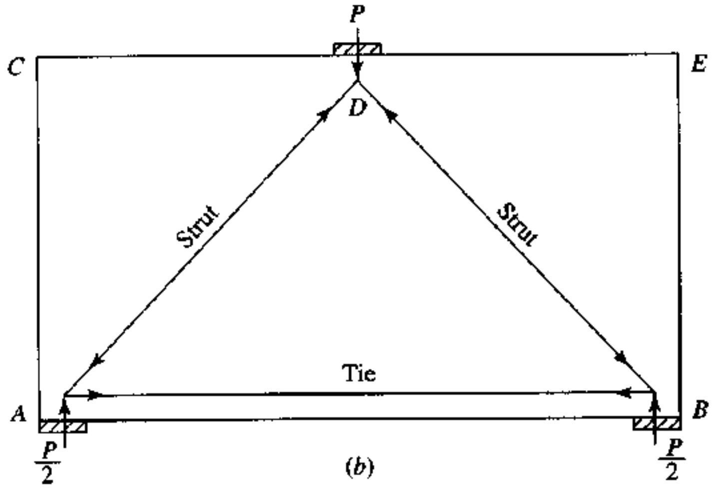


Figure 5.6 (*a*) Strut and tie model, (*b*) idealized model [1]. Courtesy of ACI 318-05.

5.7.3 ACI Design Procedure

Based on the ACI Code, Section A.2, the design of a D-region includes the following steps [1]:

1. Define and isolate each region.
2. Determine the resultant forces acting on each D-region boundary.
3. Select a truss model to transfer the resultant forces across the D-region. The axes of the struts and ties should coincide, approximately, with the compression and tension fields.
4. Determine the effective widths of the struts and nodal zones based on the concrete and steel strengths and the truss model chosen.
5. Check serviceability conditions according to the ACI Code requirements. Deflections of deep beams can be estimated using an elastic analysis. Crack control conditions of the ACI Code, Section 10.6.4, should be checked assuming the tie is encased in a prism of concrete according to RA.4.2.

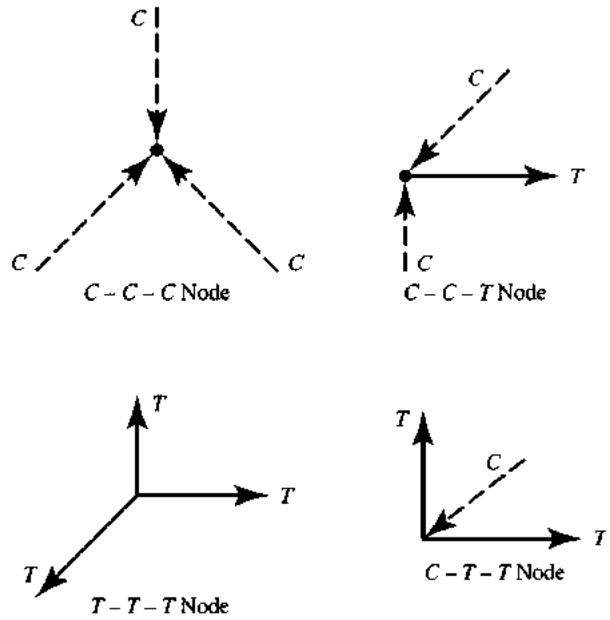


Figure 5.7 Classification of nodes.

5.7.4 Design Requirements

The design requirements for struts and ties can be summarized as follows:

1. Design of struts, ties, and nodal zones:

$$\phi F_n \geq F_u \tag{5.13}$$

where

F_u = force in a strut, tie, or nodal zone due to factored loads
F_n = nominal strength of a strut, tie, or nodal zone
ϕ = 0.75 for both struts and ties

2. Strength of struts: The nominal compressive strengths of a strut without longitudinal reinforcement, F_{ns}, shall be the smaller value of F_{ns} at the two ends of the strut such that:

$$F_{ns} = f_{ce} A_{cs} \tag{5.14}$$

where

A_{cs} = cross-sectional area at one end of a strut
f_{ce} = the smaller effective compressive strength of concrete in a strut or nodal zone

$$f_{ce} = 0.85 \beta_s f'_c \tag{5.15}$$

where

a. β_s = 1.0 for a strut of uniform cross section (ACI A.3.2.1)
b. β_s = 0.4 for struts in tension members, or the tension flanges of members (ACI A.3.2.3)

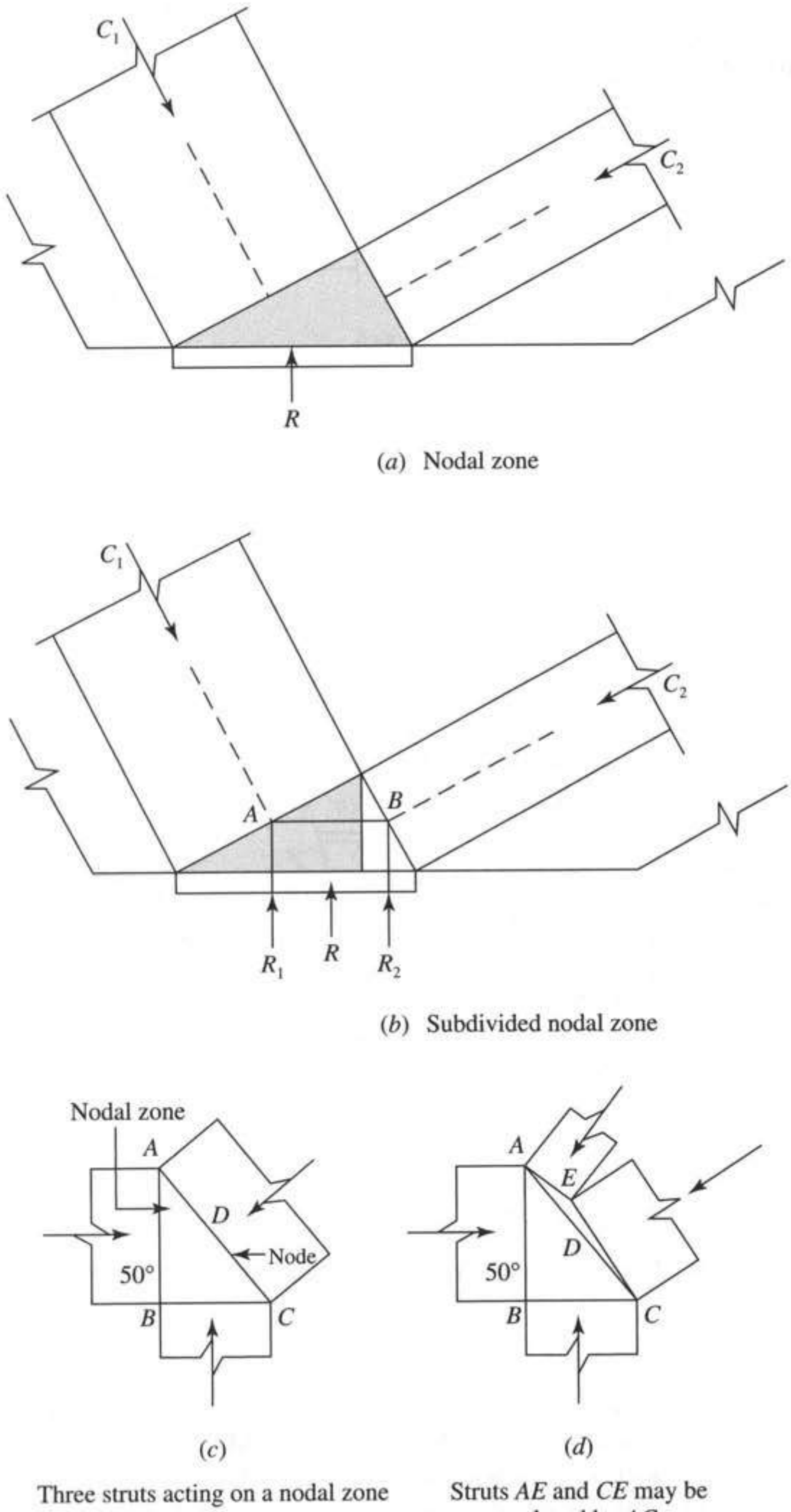


Figure 5.8 Nodal zones [1]. (*a*, *b*) Subdivision of nodal zone, (*c*, *d*) resolution of forces on a nodal zone.

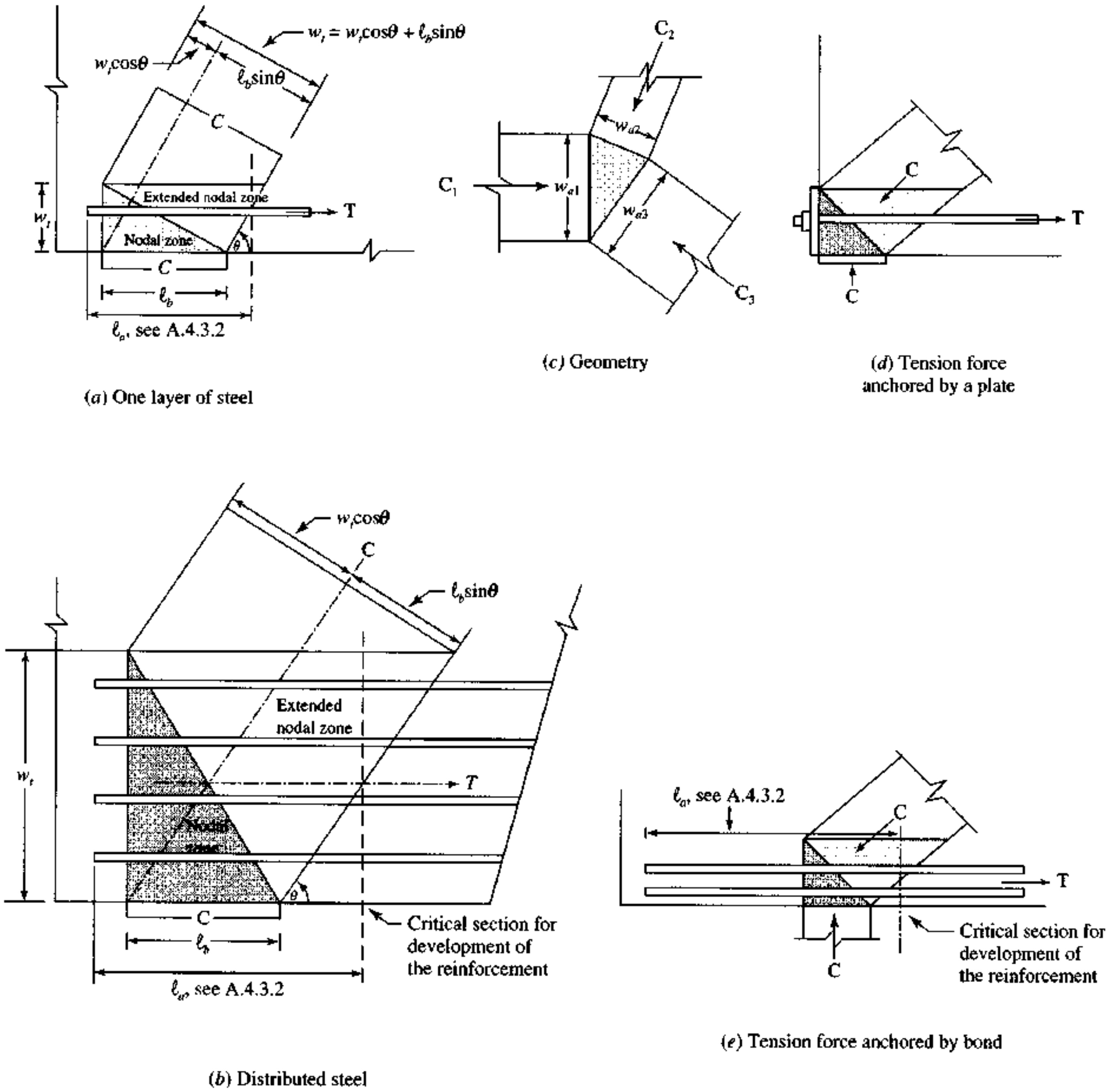


Figure 5.9 Extended nodal zones and hydrostatic nodes [1]. Courtesy of ACI 318-05.

c. For struts located such that the width of the midsection of the strut is larger than the width of the nodes (bottle-shaped struts) (ACI 3.2.2):
$\beta_s = 0.75$ with reinforcement satisfying ACI A.3.3
$\beta_s = 0.6\lambda$ without reinforcement satisfying ACI A.3.3

d. $\beta_s = 0.6\lambda$ for all other cases (ACI 3.2.4)

$$\lambda = 1.00 \text{ normal-weight concrete}$$
$$= 0.85 \text{ sand lightweight concrete}$$
$$= 0.75 \text{ for all other lightweight concrete}$$

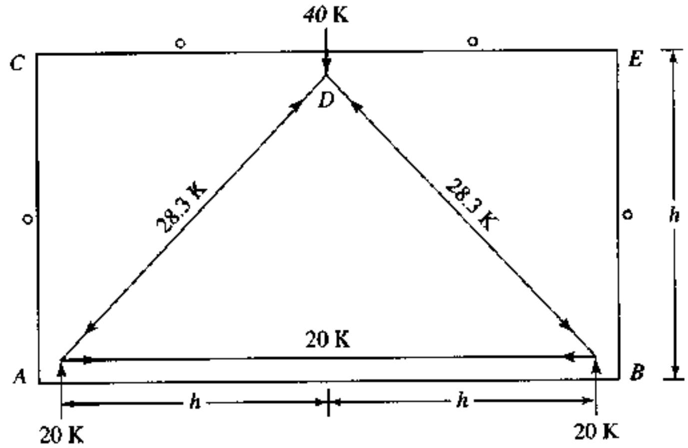


Figure 5.10 Example of a steel truss.

Linear interpolation between 0.75 and 0.85 shall be permitted, on the basis of volumetric fraction, when a portion of the lightweight fine aggregate is replaced with normal-weight fine aggregate. Linear interpolation between 0.85 and 1.0 shall be permitted for concrete containing normal-weight fine aggregate and a blend of light- and normal-weight coarse aggregate.

3. Reinforcement crossing struts (Fig. 5.11): For $f_c' \leq 6$ ksi, the value $\beta_s = 0.75$ can be used if the axis of the strut is crossed by layers of bars such that

$$\sum \frac{A_{si}}{b_s s_i} \sin \gamma_i \geq 0.003 \tag{5.16}$$

where

A_{si} = total area of surface reinforcement at a spacing s_i in the ith layer crossing a strut with reinforcement at an angle α_i to the axis of the strut

s_i = spacing of reinforcement in the ith layer crossing a strut at an angle α_i to the axis of the strut member

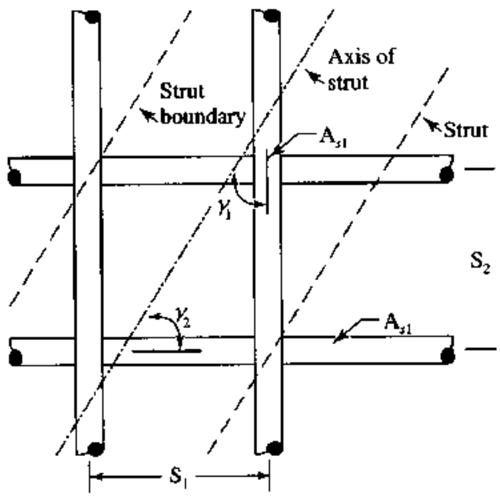


Figure 5.11 Reinforcing bars crossing a strut [1]. Courtesy of ACI 318-08.

b_s = width of member

α_i = angle between the axis of the strut and the bars in ith layer of bars crossing the strut

4. Compression reinforcement in struts: Compression reinforcement can be used to increase the strength of a strut such that

$$F_{ns} = f_{ce}A_{cs} + A'_s f'_s \tag{5.17}$$

where

F_{ns} = strength of a longitudinal reinforced strut

A'_s = area of the compression reinforcement in a strut

f'_s = stress in A'_s ($f'_s = f_y$ for grades 40 to 60)

5. Strength of ties: The nominal strength of a tie, F_{nt} is:

$$F_{nt} = A_{ts}f_y + A_{tp}(f_{se} + \Delta f_p) \tag{5.18}$$

where

A_{ts} = area of nonprestressed reinforcement in the tie

A_{tp} = area of prestressing reinforcement

f_{se} = effective stress after losses in prestressed reinforcement

Δf_p = increase in prestressing stress due to factored loads

A_{tp} = 0 for nonprestressed members

$$(f_{se} + \Delta f_p) \le f_{py} \tag{5.19}$$

It is permitted to take $\Delta f_p = 60$ ksi for bonded prestressed reinforcement or 10 ksi for unbonded prestressed reinforcement. Also, a practical upper limit of the tie width, $w_{t,max}$ can be taken as follows:

$$w_{t,max} = F_{nt}/(f_{ce}b_s) \tag{5.20}$$

6. Strength of nodal zones: The nominal compression strength of a nodal zone, F_{nn}, shall be

$$F_{nn} = f_{ce}A_{nz} \tag{5.21}$$

where A_{nz} = the area of the face of the nodal zone or a section through a nodal zone perpendicular to the resultant force on the section.

7. Confinement in nodal zones: Unless confining reinforcement is provided within the nodal zone and its effect is supported by tests and analysis, the calculated effective compressive stress on a face of a nodal zone due to the strut and tie forces should not exceed the following:

$$f_{ce} = 0.85\beta_n f'_c \tag{5.22}$$

where

β_n = 1.0 in nodal zones bounded by struts or bearing areas, or both, C-C-C node.

β_n = 0.80 in nodal zones anchoring one tie, C-C-T node.

β_n = 0.60 in nodal zones anchoring two or more ties, C-T-T node.

The application of the strut and tie method to a deep beam is given in Example 8.6, Section 8.11.

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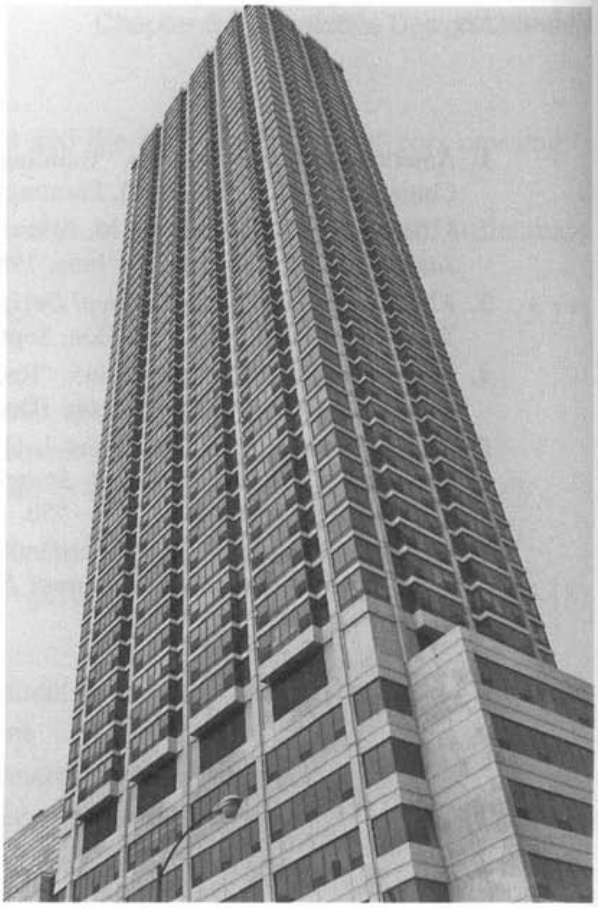
CHAPTER 6

DEFLECTION AND CONTROL OF CRACKING



High-rise building, Chicago, Illinois.

6.1 DEFLECTION OF STRUCTURAL CONCRETE MEMBERS

Flexural concrete members must be designed for safety and serviceability. The members will be safe if they are designed according to the ACI Code equations and limitations. Consequently, as explained in previous chapters, the size of each member is determined as well as the reinforcement required to maintain an internal moment capacity equal to or greater than that of the external moment. Once the final dimensions are determined, the beam must be checked for serviceability: cracks and deflection. Adequate stiffness of the member is necessary to prevent excessive cracks and deflection.

The use of the ACI Code provisions, taking into consideration the nonlinear relationship between stress and strain in concrete, has resulted in smaller sections than those designed by the elastic theory. The ACI Code, Section 9.4, recognizes the use of steel up to a yield strength of 80 ksi (560 MPa) and the use of high-strength concrete. The use of high-strength steel and concrete results in smaller sections and a reduction in the stiffness of the flexural member and consequently increases its deflection.

The permissible deflection is governed by many factors, such as the type of the building, the appearance of the structure, the presence of plastered ceilings and partitions, the damage expected due to excessive deflection, and the type and magnitude of live load.

The ACI Code, Section 9.5, specifies minimum thickness for one-way flexural members and one-way slabs, as shown in Table 6.1 in this chapter. The values are for members not supporting or attached to partitions or other constructions likely to be damaged by large deflections.

Table 6.1 Minimum Thickness of Beams and One-Way Slabs (L = Span Length)

Member	Yield Strength f_y (ksi)	Simply Supported	One End Continuous	Both Ends Continuous	Cantilever
Solid one-way slabs	40	$L/25$	$L/30$	$L/35$	$L/12.5$
	50	$L/22$	$L/27$	$L/31$	$L/11$
	60*	$L/20$	$L/24$	$L/28$	$L/10$
Beams or ribbed one-way slabs	40	$L/20$	$L/23$	$L/26$	$L/10$
	50	$L/18$	$L/20.5$	$L/23.5$	$L/9$
	60*	$L/16$	$L/18.5$	$L/21$	$L/8$

*Values reported in ACI Table 9.5(a).

The minimum thicknesses indicated in Table 6.1 are used for members made of normal-weight concrete, and for steel reinforcement with yield strengths as mentioned in the table. The values are modified for cases of lightweight concrete or a steel yield strength different from 60 ksi as follows:

- For lightweight concrete having unit weights in the range of 90 to 115 pcf, the values in the tables for $f_y = 60$ ksi (420 MPa) shall be multiplied by the greater of $(1.65 - 0.005 W_c)$ but not less than 1.09, where W_c is the unit weight of concrete in pounds per cubic foot.
- For yield strength of steel different from 60 ksi (420 MPa), the values in the tables for 60 ksi shall be multiplied by $(0.4 + f_y/100)$, where f_y is in ksi.

6.2 INSTANTANEOUS DEFLECTION

The deflection of structural members is due mainly to the dead load plus a fraction of or all the live load. The deflection that occurs immediately upon the application of the load is called the *immediate*, or *instantaneous, deflection*. Under sustained loads, the deflection increases appreciably with time. Various methods are available for computing deflections in statically determinate and indeterminate structures. The instantaneous deflection calculations are based on the elastic behavior of the flexural members. The elastic deflection, Δ, is a function of the load, W, span, L, moment of inertia, I, and the modulus of elasticity of the material, E:

$$\Delta = f\left(\frac{WL}{EI}\right) = \alpha\left(\frac{WL^3}{EI}\right) = K\left(\frac{ML^2}{EI}\right) \tag{6.1}$$

where W = total load on the span and α and K are coefficients that depend on the degree of fixity at the supports, the variation of moment of inertia along the span, and the distribution of load. For example, the maximum deflection on a uniformly loaded simply supported beam is

$$\Delta = \frac{5WL^3}{384EI} = \frac{5wL^4}{384EI} \tag{6.2}$$

where W = the total load on the span = wL (uniform load per unit length × span). Deflections of beams with different loadings and different end conditions as a function of the load, span, and EI are given in Appendix C and in books of structural analysis.

Because W and L are known, the problem is to calculate the modulus of elasticity, E, and the moment of inertia, I, of the concrete member or the flexural stiffness of the member EI.

6.2.1 Modulus of Elasticity

The ACI Code, Section 8.5,specifies that the modulus of elasticity of concrete, E_c, may be taken as

$$E_c = 33W_c^{1.5}\sqrt{f'_c}\text{ psi} \tag{6.3}$$

for values of W_c between 90 and 160 pcf. For normal-weight concrete ($W_c = 145$ pcf),

$$E_c = 57{,}400\sqrt{f'_c}\text{ psi} \quad (\text{or } 57{,}000\sqrt{f'_c})$$

The modulus of elasticity is usually determined by the short-term loading of a concrete cylinder. In actual members, creep due to sustained loading, at least for the dead load, affects the modulus on the compression side of the member. For the tension side, the modulus in tension is assumed to be the same as in compression when the stress magnitude is low. At high stresses the modulus decreases appreciably. Furthermore, the modulus varies along the span due to the variation of moments and shear forces.

6.2.2 Modular Ratio

The modular ratio, $n = Es/E_c$, which is used in the transformed area concept was explained in Section 2.10. It may be used to the nearest whole number but may not be less than 6. For example,

$$\text{when } f'_c = 2500\text{ psi (17.5 MPa)}, \quad n = 10$$

$$\text{when } f'_c = 3000\text{ psi (20 MPa)}, \quad n = 9$$

$$\text{when } f'_c = 4000\text{ psi (30 MPa)}, \quad n = 8$$

$$\text{when } f'_c = 5000\text{ psi (17.5 MPa)}, \quad n = 7$$

For normal-weight concrete, n may be taken as $500/\sqrt{f'_c}$, (psi units).

6.2.3 Cracking Moment

The behavior of a simply supported structural concrete beam loaded to failure was explained in Section 3.3. At a low load, a small bending moment develops, and the stress at the extreme tension fibers will be less than the modulus of rupture of concrete, $f_r = 7.5\lambda\sqrt{f'_c}$. If the load is increased until the tensile stress reaches an average stress of the modulus of rupture, f_r, cracks will develop. If the tensile stress is higher than f_r, the section will crack, and a cracked section case will develop. This means that there are three cases to be considered:

1. When the tensile stress, f_t, is less than f_r, the whole-uncracked section is considered to calculate the properties of the section. In this case, the gross moment of inertia, I_g, is used: $I_g = bh^3/12$, where bh = the whole concrete section.
2. When the tensile stress, f_t, is equal to the modulus of rupture, $f_r = 7.5\lambda\sqrt{f'_c}$, a crack may start to develop, and the moment that causes this stress is called the cracking moment. Using the flexural formula;

$$f_r = M_{cr}\frac{c}{I_g} \quad \text{or} \quad M_{cr} = f_r \cdot \frac{I_g}{c} \tag{6.4}$$

where $f_r = 7.5\lambda\sqrt{f'_c}$, I_g = the gross moment of inertia, and c = the distance from the neutral axis to the extreme tension fibers. For example, for a rectangular section, $I_g = bh^3/12$ and $c = h/2$, and where

λ is a modification factor for type of concrete (ACI 8.6.1)
$\lambda = 1.0$ Normal-weight concrete
$\lambda = 0.85$ Sand-lightweight concrete
$\lambda = 0.75$ For all-lightweight concrete

Linear interpolation shall be permitted between 0.85 and 1.0 on the basis of volumetric fractions, for concrete containing normal-weight fine aggregate and a blend of lightweight and normal-weight coarse aggregate.

3. When the applied external moment exceeds the cracking moment, M_{cr}, a cracked section case is developed, and the concrete in the tension zone is neglected. A transformed cracked section is used to calculate the cracking moment of inertia, I_{cr}, using the concrete area in compression and the transformed steel area nA_s.

Example 6.1

A rectangular concrete section is reinforced with three no. 9 bars in one row and has a width of 12 in., a total depth of 25 in., and $d = 22.5$. (Fig. 6.1. Calculate the modulus of rupture, f_r, the gross moment of inertia, I_g, and the cracking moment, M_{cr}. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. The modulus of rupture is $f_r = 7.5\lambda\sqrt{f'c} = 7.5 \times 1 \times \sqrt{4000} = 474$ psi. ($\lambda = 1$ normal-weight concrete)
2. The gross moment of inertia for a rectangular section is

$$bh^3/12 = \frac{12(25)^3}{12} = 15{,}625 \text{ in.}^4$$

3. The cracking moment is $M_{cr} = f_r \cdot I_g/c$

$$f_r = 474 \text{ psi} \qquad I_g = 15{,}625 \text{ in.}^4 \qquad c = h/2 = 12.5 \text{ in.}$$

Therefore, $M_{cr} = 474 \times 15{,}625/(12.5 \times 1000) = 592.5$ K·in. $= 49.38$ K·ft

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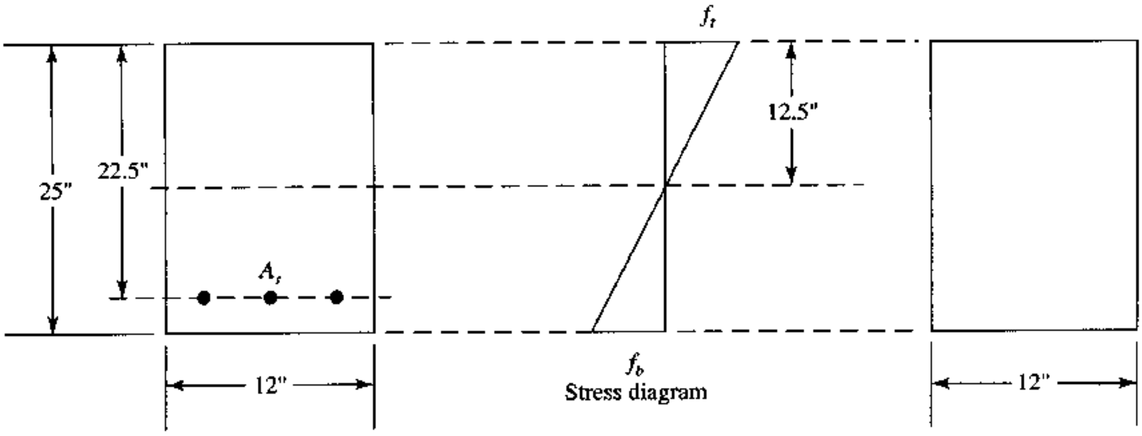


Figure 6.1 Example 6.1.

6.2.4 Moment of Inertia

The moment of inertia, in addition to the modulus of elasticity, determines the stiffness of the flexural member. Under small loads, the produced maximum moment will be small, and the tension stresses at the extreme tension fibers will be less than the modulus of rupture of concrete; in this case, the gross transformed cracked section will be effective in providing the rigidity. At working loads or higher, flexural tension cracks are formed. At the cracked section, the position of the neutral axis is high, whereas at sections midway between cracks along the beam, the position of the neutral axis is lower (nearer to the tension steel). In both locations only the transformed cracked sections are effective in determining the stiffness of the member; therefore, the effective moment of inertia varies considerably along the span. At maximum bending moment, the concrete is cracked, and its portion in the tension zone is neglected in the calculations of moment of inertia. Near the points of inflection the stresses are low, and the entire section may be uncracked. For this situation and in the case of beams with variable depth, exact solutions are complicated.

Figure 6.2*a* shows the load–deflection curve of a concrete beam tested to failure. The beam is a simply supported 17-ft span and loaded by two concentrated loads 5 ft apart, symmetrical about the centerline. The beam was subjected to two cycles of loading: In the first (curve *cy* 1), the load–deflection curve was a straight line up to a load $P = 1.7$ K when cracks started to occur in the beam. Line *a* represents the load–deflection relationship using a moment of inertia for the uncracked transformed section. It can be seen that the actual deflection of the beam under loads less than the cracking load, based on a homogeneous uncracked section, is very close to the calculated deflection (line *a*). Curve *cy* 1 represents the actual deflection curve when the load is increased to about one-half the ultimate load. The slope of the curve, at any level of load, is less than the slope of line *a* because cracks developed, and the cracked part of the concrete section reduces the stiffness of the beam. The load was then released, and a residual deflection was observed at midspan. Once cracks developed, the assumption of uncracked section behavior under small loads did not hold.

In the second cycle of loading, the deflection (curve *c*) increased at a rate greater than that of line *a*, because the resistance of the concrete tension fibers was lost. When the load was increased, the load–deflection relationship was represented by curve *cy* 2. If the load in the first cycle is increased up to the ultimate load, curve *cy* 1 will take the path *cy* 2 at about 0.6 of the ultimate load. Curve *c* represents the actual behavior of the beam for any additional loading or unloading cycles.

Line *b* represents the load–deflection relationship based on a cracked transformed section; it can be seen that the deflection calculated on that basis differs from the actual deflection. Figure 6.2*c* shows the variation of the beam stiffness EI with an increase in moment. ACI Code, Section 9.5, presents an equation to determine the effective moment of inertia used in calculating deflection in flexural members. The effective moment of inertia given by the ACI Code (Eq. 9.8) is based on the expression proposed by Branson [3] and calculated as follows:

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \leq I_g \tag{6.5}$$

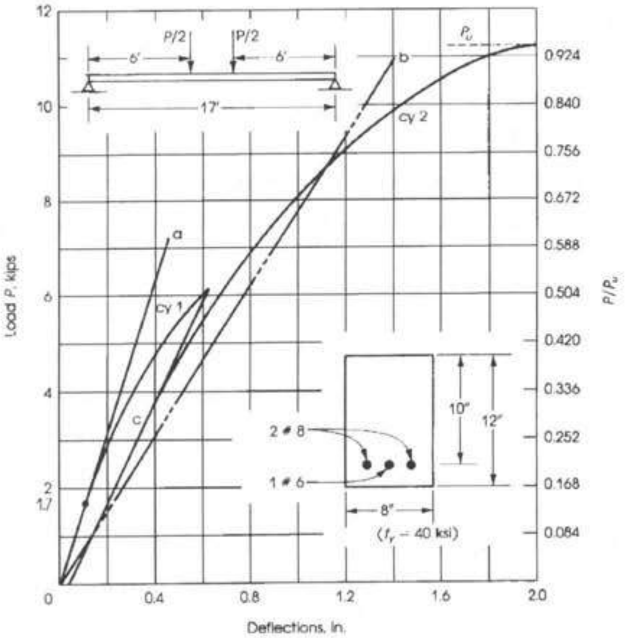
where

I_e = effective moment of inertia

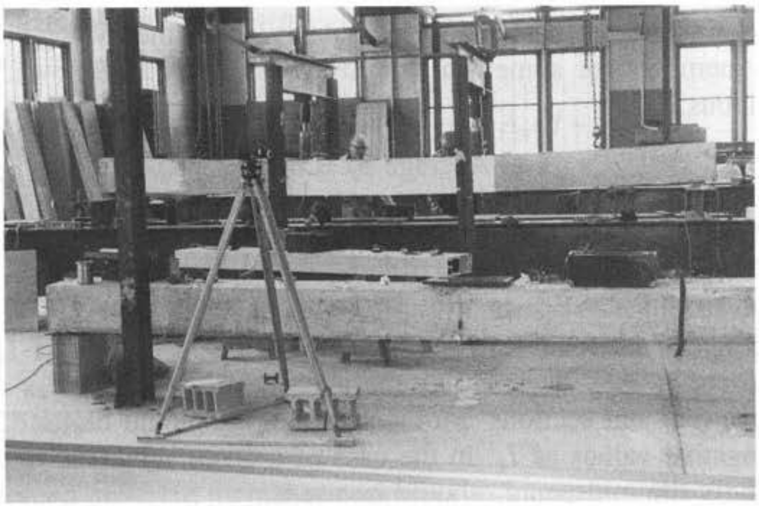
$$M_{cr} = \text{cracking moment, } \left(\frac{f_r I_g}{Y_t}\right) \tag{6.6}$$

f_r = modulus of rupture of concrete

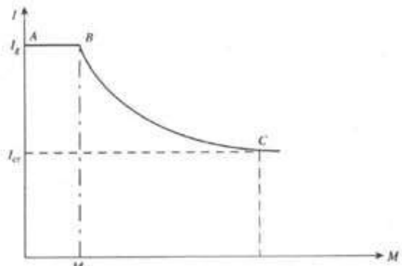
$$= 7.5\lambda\sqrt{f'_c} \text{ psi, } (0.623\lambda\sqrt{f'_c} \text{ MPa}) \tag{6.7}$$



(a)



(b)



(c)

Figure 6.2 (*a*) Experimental and theoretical load—deflection curves for a beam of the section and load illustrated, (*b*) deflection of a reinforced concrete beam, and (*c*) variation of beam moment of inertia, I, with an increase in moment (E = constant). BC is a transition curve between I_g and I_{cr}.

M_a = maximum unfactored moment in member at stage for which deflection is being computed

I_g = moment of inertia of gross concrete section about the centroidal axis, neglecting the reinforcement

I_{cr} = moment of inertia of cracked transformed section

Y_t = distance from centroidal axis of cross-section, neglecting steel, to the tension face

The following limitations are specified by the code:

1. For continuous spans, the effective moment of inertia may be taken as the average of the moment of inertia of the critical positive and negative moment sections.
2. For prismatic members, I_e may be taken as the value obtained from Eq. 6.5 at midspan for simple and continuous spans and at the support section for cantilevers (ACI Code, Section 9.5.2).

   Note that I_e, as computed by Eq. 6.5, provides a transition between the upper and lower bounds of the gross moment of inertia, I_g, and the cracked moment of inertia, I_{cr}, as a function of the level of M_{cr}/M_a. Heavily reinforced concrete members may have an effective moment of inertia, I_e, very close to that of a cracked section, I_{cr}, whereas flanged members may have an effective moment of inertia close to the gross moment of inertia, I_g.
3. For continuous beams, an approximate value of the average I_e for prismatic or nonprismatic members for somewhat improved results is as follows: For beams with both ends continuous,

$$\text{Average } I_e = 0.70 I_m + 0.15(I_{e1} + I_{e2}) \tag{6.8}$$

For beams with one end continuous,

$$\text{Average } I_e = 0.85 I_m + 0.15(I_{con}) \tag{6.9}$$

where I_m = midspan I_e, I_{e1}, I_{e2} = I_e at beam ends, and I_{con} = I_e at the continuous end. Also, I_e may be taken as the average value of the I_es at the critical positive- and negative-moment sections. Moment envelopes should be used in computing both positive and negative values of I_e. In the case of a beam subjected to a single heavy concentrated load, only the midspan I_e should be used.

6.2.5 Properties of Sections

To determine the moment of inertia of the gross and cracked sections, it is necessary to calculate the distance from the compression fibers to the neutral axis (x or kd).

1. Gross moment of inertia, I_g (neglect all steel in the section)
   a. For a rectangular section of width b and a total depth h, $I_g = bh^3/12$.
   b. For a T-section, flange width b, web width b_w, and flange thickness t, calculate y, the distance to the centroidal axis from top of flange:

$$y = \frac{\left(\dfrac{bt^2}{2}\right) + b_w(h-t)\left[\dfrac{(h-t)}{2+t}\right]}{bt + b_w(h-t)} \tag{6.10}$$

Then calculate I_g:

$$I_g = \left[\frac{bt^3}{12} + bt\left(y - \frac{t}{2}\right)^2\right] + \left[b_w\frac{(y-t)^3}{3}\right] + \left[b_w\frac{(h-y)}{3}\right] \tag{6.10a}$$

2. Cracked moment of inertia, I_{cr}: Let x = the distance of the neutral axis from the extreme compression fibers ($x = kd$).

a. Rectangular section with tension steel, A_s, only

i. Calculate x from the following equation:

$$\frac{bx^2}{2} - nA_s(d - x) = 0 \tag{6.11}$$

ii. Calculate $I_{cr} = bx^3/3 + nA_s\ (d - x)^2$ (6.11a)

b. Rectangular section with tension steel A_s and compression steel A'_s

i. Calculate x:

$$\frac{bx^2}{2} + (n-1)A'_s(x - d') - nA_s(d - x) = 0 \tag{6.12}$$

ii. Calculate $I_{cr} = (bx^3/3) + (n-1)A'_s(x - d')^2 + nA_s(d - x)^2$. (6.12a)

c. T-sections with tension steel A_s

i. Calculate x : $bt\left(x - \frac{t}{2}\right) + b_w\frac{(x-t)^2}{2} - nA_s(d - x) = 0$ (6.13)

ii. Calculate I_{cr}:

$$I_{cr} = \left[\frac{bt^3}{12} + bt\left(x - \frac{t}{2}\right)^2\right] + \left[b_w\frac{(x-t)^3}{3}\right] + nA_s(d - x)^2 \tag{6.13a}$$

6.3 LONG-TIME DEFLECTION

Deflection of reinforced concrete members continues to increase under sustained load, although more slowly with time. Shrinkage and creep are the cause of this additional deflection, which is called long-time deflection [1]. It is influenced mainly by temperature, humidity, age at time of loading, curing, quantity of compression reinforcement, and magnitude of the sustained load. The ACI Code, Section 9.5.2.5, suggests that unless values are obtained by a more comprehensive analysis, the additional long-term deflection for both normal and lightweight concrete flexural members shall be obtained by multiplying the immediate deflection by the factor

$$\lambda_\Delta = \frac{\zeta}{1 + 50\rho'} \tag{6.14}$$

where

λ_Δ = multiplier for additional deflection due to long-term effect.

ρ' = A'_s/bd for the section at midspan of a simply supported or continuous beam or at the support of a cantilever beam

ζ = time-dependent factor for sustained loads that may be taken as shown in Table 6.2.

Table 6.2 Multipliers for Long-time Deflections

Period (months)	1	3	6	12	24	36	48	60 &over
ζ	0.5	1.0	1.2	1.4	1.7	1.8	1.9	2.0

The factor λ_Δ is used to compute deflection caused by the dead load and the portion of the live load that will be sustained for a sufficient period to cause significant time-dependent deflections. The factor λ_Δ is a function of the material property, represented by ζ, and the section property, represented by $(1 + 50\rho')$. In Eq. 6.14, the effect of compression reinforcement is related to the area of concrete rather than the ratio of compression to tension steel.

The ACI Code Commentary, Section 9.5.2.5, presents a curve to estimate ζ for periods less than 60 months. These values are estimated as shown in Table 6.2.

The total deflection is equal to the immediate deflection plus the additional long-time deflection. For instance, the total additional long-time deflection of a flexural beam with $\rho' = 0.01$ at a 5-year period is equal to λ_Δ times the immediate deflection, where $\lambda_\Delta = 2/(1 + 50 \times 0.01) = 1.33$.

6.4 ALLOWABLE DEFLECTION

Deflection shall not exceed the following values according to the ACI Code, Section 9.5:

- $L/180$ for immediate deflection due to live load for flat roofs not supporting elements that are likely to be damaged
- $L/360$ for immediate deflection due to live load for floors not supporting elements likely to be damaged
- $L/480$ for the part of the total deflection that occurs after attachment of elements, that is, the sum of the long-time deflection due to all sustained loads and the immediate deflection due to any additional live load, for floors or roofs supporting elements likely to be damaged
- $L/240$ for the part of the total deflection occurring after elements are attached, for floors or roofs not supporting elements likely to be damaged

6.5 DEFLECTION DUE TO COMBINATIONS OF LOADS

If a beam is subjected to different types of loads (uniform, nonuniform, or concentrated loads) or subjected to end moments, the deflection may be calculated for each type of loading or force applied on the beam separately and the total deflection calculated by superposition. This means that all separate deflections are added up algebraically to get the total deflection. The deflections of beams under individual loads are shown in Table 6.3.

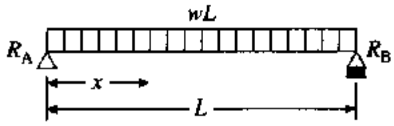
Example 6.2

Calculate the instantaneous midspan deflection for the simply supported beam shown in Fig. 6.3, which carries a uniform dead load of 0.4 K/ft and a live load of 0.6 K/ft in addition to a concentrated dead load of 5 kips at midspan. Given: $f'_c = 4$ ksi normal-weight concrete, $f_y = 60$ ksi, $b = 13$ in., $d = 21$ in., and total depth $= 25$ in. ($n = 8$).

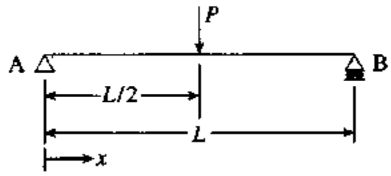
Table 6.3 Deflection of Beams

$$\Delta_{max} = \frac{5}{384} \times \frac{WL^3}{EI} \quad \text{(at center)}$$

$$W = \text{total load} = wL$$



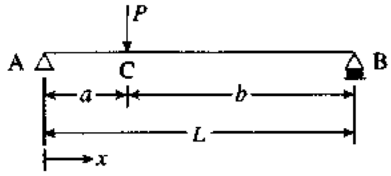
$$\Delta_{max} = \frac{PL^3}{48EI} \quad \text{(at midspan)}$$



$$\Delta_C = \frac{Pa^2b^2}{3EIL} \quad \text{(at point load)}$$

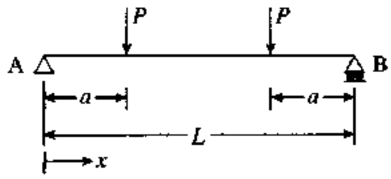
$$\Delta_{max} = \frac{PL^3}{48EI}\left[\frac{3a}{L} - 4\left(\frac{a}{L}\right)^3\right] \quad \text{(when } a \geq b\text{)}$$

$$\text{at } x = \sqrt{a(b + L)/3}$$



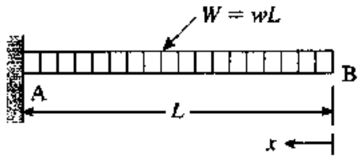
$$\Delta_{max} = \frac{PL^3}{6EI}\left[\frac{3a}{4L} - \left(\frac{a}{L}\right)^3\right] \quad \text{(at midspan)}$$

$$\Delta_{max} = \frac{23PL^3}{648EI} \quad \text{(at midspan) when } a = L/3$$



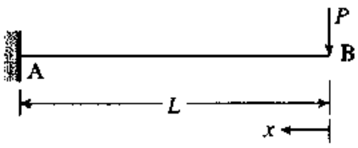
$$\Delta_B \max = \frac{WL^3}{8EI} \quad (W = wL)$$

$$\Delta_x = \frac{w}{24EIL}(x^4 - 4L^3x + 3L^4)$$



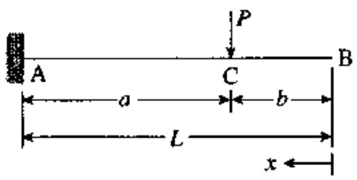
$$\Delta_B \max = \frac{PL^3}{3EI}$$

$$\Delta_x = \frac{P}{6EI}(2L^3 - 3L^2x + x^3)$$



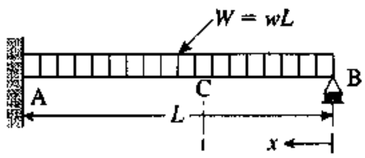
$$\Delta_C = Pa^3/3EI$$

$$\Delta_B \max = \frac{Pa^3}{3EI}\left(1 + \frac{3b}{2a}\right) \quad \text{(at free end)}$$

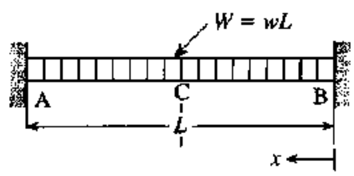


$$\Delta_{max} = \frac{WL^3}{185EI}$$

at a distance $x = 0.4215L$ (from support B)



$$\Delta_{max} = \frac{WL^3}{384EI} \quad \text{(at midspan)}$$



$$\Delta_1 = \frac{5wL^4}{384E_cI_e}$$

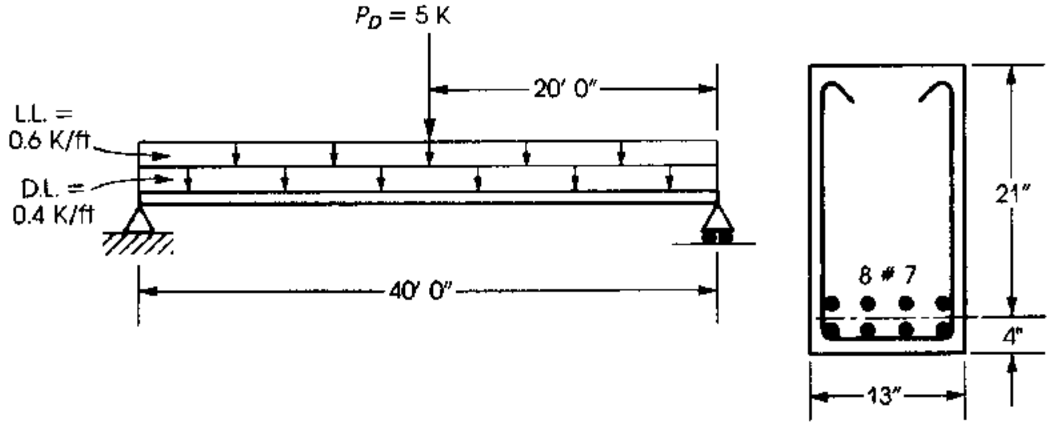


Figure 6.3 Example 6.2.

Solution

1. Check minimum depth according to the ACI Code, Table 6.1.

$$\text{Minimum total depth} = \frac{L}{16} = \frac{40 \times 12}{16} = 30 \text{ in.}$$

The total thickness used in 25 in. < 30 in.; therefore, deflection must be checked.

2. The deflection at midspan due to a distributed load is

$$\Delta_1 = \frac{5wL^4}{384E_cI_e}$$

The deflection at midspan due to a concentrated load is

$$\Delta_2 = \frac{PL^3}{48E_cI_e}$$

Because w, P, and L are known, we must determine the modulus of elasticity, E_c, and the effective moment of inertia, I_e.

3. The modulus of elasticity of concrete is

$$E_c = 57{,}400\sqrt{f'_c} = 57{,}400\sqrt{4000} = 3.63 \times 10^6 \text{psi}$$

4. The effective moment of inertia is equal to

$$\Delta_1 = \frac{5wL^4}{384E_cI_e}$$

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \leq I_g$$

Determine values of all terms on the right-hand side:

$$M_a = \frac{wL^2}{8} + \frac{PL}{4} = \frac{(0.6 + 0.4)}{8}(40)^2 \times 12 + \frac{5 \times 40}{4} \times 12 = 3000 \text{ K·in.}$$

$$I_g = \frac{bh^3}{12} = \frac{13(25)^3}{12} = 16{,}927 \text{ in.}^4$$

$$M_{cr} = \frac{f_r I_g}{Y_t} \qquad Y_t = \frac{h}{2} = 12.5 \text{ in.} \qquad f_r = 7.5\lambda\sqrt{f'_c} = 474 \text{ psi} \quad \lambda = 1 \text{ (normal-weight)}$$

$$M_{cr} = \frac{0.474 \times 16{,}927}{12.5} = 642 \text{ K-in.}$$

The moment of inertia of the cracked transformed area, I_{cr}, is calculated as follows: Determine the position of the neutral axis for a cracked section by equating the moments of the transformed area about the neutral axis to 0, letting $x = kd$ = distance to the neutral axis:

$$\frac{bx^2}{2} - nA_s(d - x) = 0 \quad n = \frac{E_s}{E_c} = 8.0 \quad A_s = 4.8 \text{ in.}^2$$

$$\frac{13}{2}x^2 - (8)(4.8)(21 - x) = 0$$

$$x^2 + 5.9x - 124 = 0 \quad x = 8.8 \text{ in.}$$

$$I_{cr} = \frac{bx^3}{3} + nA_s(d - x)^2 = \frac{13(8.8)^3}{3} + 38.4(21 - 8.8)^2 = 8660 \text{ in.}^4$$

With all terms calculated,

$$I_e = \left(\frac{642}{3000}\right)^3 \times 16{,}927 + \left[1 - \left(\frac{642}{3000}\right)^3\right] \times 8660 = 8740 \text{ in.}^4$$

5. Calculate the deflections from the different loads:

$$\Delta_1(\text{due to distributed load}) = \frac{5wL^4}{384 E_c I_e}$$

$$\Delta_1 = \left(\frac{5}{384}\right) \times \left(\frac{1000}{12}\right) \times \frac{(40 \times 12)^4}{3.63 \times 10^6 \times 8740} = 1.82 \text{ in.}$$

$$\Delta_2(\text{due to concentrated load}) = \frac{PL^3}{48 E_c I_e}$$

$$\Delta_2 = \frac{5000 \times (40 \times 12)^3}{48 \times 3.63 \times 10^6 \times 8740} = 0.36 \text{ in.}$$

Total immediate deflection = $\Delta_1 + \Delta_2 = 1.82 + 0.36 = 2.18$ in.

6. Compare the calculated values with the allowable deflection: The immediate deflection due to a uniform live load of 0.6 K/ft is equal to 0.6(1.82) = 1.09 in. If the member is part of a floor construction not supporting or attached to partitions or other elements likely to be damaged by large deflection, the allowable immediate deflection due to live load is equal to

$$\frac{L}{360} = \frac{40 \times 12}{360} = 1.33 \text{ in.} > 1.09 \text{ in.}$$

If the member is part of a flat roof and similar to the preceding, the allowable immediate deflection due to live load is $L/180 = 2.67$ in. Both allowable values are greater than the actual deflection of 1.09 in. due to the uniform applied live load.

Example 6.3

Determine the long-time deflection of the beam in Example 6.2 if the time-dependent factor equals 2.0.

Solution

1. The sustained load causing long-time deflection is that due to dead load, consisting of a distributed uniform dead load of 0.4 K/ft and a concentrated dead load of 5 K at midspan.

$$\text{Deflection due to uniform load} = 0.4 \times 1.82 = 0.728 \text{ in.}$$

Deflection is a linear function of load, w, all other values (L, E_c, I_e) being the same.

$$\text{Deflection due to concentrated load} = 0.36 \text{ in.}$$

$$\text{Total immediate deflection due to sustained loads} = 0.728 + 0.36 = 1.088 \text{ in.}$$

2. For additional long-time deflection, the immediate deflection is multiplied by the factor λ_Δ:

$$\lambda_\Delta = \frac{\zeta}{1 + 50\rho'} = \frac{2}{1 + 0}$$

In this problem, $A'_s = 0$; therefore, $\lambda_\Delta = 2.0$.

$$\text{Additional long-time deflection} = 2 \times 1.088 = 2.176 \text{ in.}$$

3. Total long-time deflection is the immediate deflection plus additional long-time deflection: 2.18 + 2.176 = 4.356 in.
4. Deflection due to dead load plus additional long-time deflection due to shrinkage and creep is 1.088 + 2.176 = 3.264 in.

Example 6.4

Calculate the instantaneous and 1-year long-time deflection at the free end of the cantilever beam shown in Fig. 6.4. The beam has a 20-ft span and carries a uniform dead load of 0.4 K/ft, a uniform live load of 0.4 K/ft, a concentrated dead load, P_D, of 3 K at the free end, and a concentrated live load, P_L, of 4 K placed at 10 ft from the fixed end. Given: $f'_c = 4$ ksi, $f_y = 60$ ksi, $b = 12$ in., $d = 21.5$ in., and total depth of section = 25 in. (Tension steel is six no. 8 bars and compression steel is two no. 8 bars.). Assume normal-weight concrete.

Solution

1. Minimum depth = $L/8 = \frac{20}{8} = 2.5$ ft = 30 in., which is greater than the 25 in. used. Therefore, deflection must be checked. The maximum deflection of a cantilever beam is at the free end. The deflection at the free end is as follows.

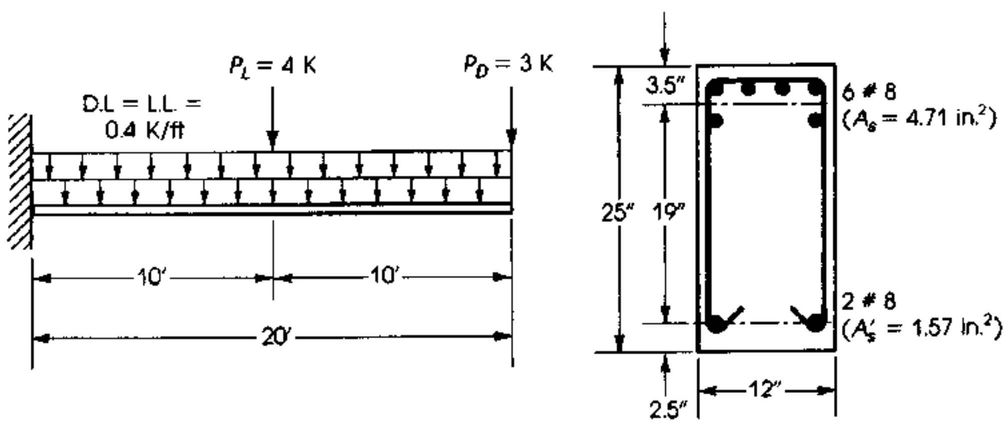


Figure 6.4 Example 6.4.

Deflection due to distributed load:

$$\Delta_1 = \frac{wL^4}{8EI}$$

Deflection due to a concentrated dead load at the free end:

$$\Delta_2 = \frac{P_D L^3}{3EI}$$

Deflection due to concentrated live load at $a = 10$ ft from the fixed end is maximum at the free end:

$$\Delta_3 = \frac{P_L(a)^2}{6EI}(3L - a) \quad \text{or} \quad \frac{Pa^3}{3EI}\left(1 + \frac{3b}{2a}\right)$$

2. The modulus of elasticity of normal-weight concrete is

$$E_c = 57{,}400\sqrt{f'_c} = 57{,}400\sqrt{4000} = 3.63 \times 10^6 \text{ psi}$$

3. Maximum moment at the fixed end is

$$M_a = \frac{wL^2}{2} + P_D \times 20 + P_L \times 10$$

$$= \frac{(0.4 + 0.4)(400)}{2} + 3 \times 20 + 4 \times 10 = 260 \text{ K·ft}$$

4. I_g = gross moment of inertia (concrete only)

$$= \frac{bh^3}{12} = \frac{12 \times (25)^3}{12} = 15{,}625 \text{ in.}^4$$

5. $$M_{cr} = \frac{f_r I_g}{Y_t} = \frac{((7.5)(1)\sqrt{4000}) \times 15{,}625}{\frac{25}{2}} = 592.9 \text{ K·in.} = 49.40 \text{ K·ft}$$

6. Determine the position of the neutral axis; then determine the moment of inertia of the cracked transformed section. Take moments of areas about the neutral axis and equate them to 0. Use $n = 8$ to calculate the transformed area of A_s and use $(n - 1) = 7$ to calculate the transformed area of A'_s. Let $kd = x$.

$$b\frac{(x^2)}{2} + (n-1)A'_s(x - d') - nA_s(d - x) = 0$$

For this section, $x = 8.44$ in.

$$I_{cr} = \frac{b}{3}x^3 + (n-1)A'_s(x - d')^2 + nA_s(d - x)^2 = 9220 \text{ in.}^4$$

7. Effective moment of inertia is

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \le I_g$$

$$= \left(\frac{49.40}{260}\right)^3 \times 15{,}625 + \left[1 - \left(\frac{49.40}{260}\right)^3\right] \times 9220 = 9264 \text{ in.}^4$$

8. Determine the components of the deflection:

$$\Delta_1 \text{ (due to uniform load of 0.8K/ft)} = \frac{800}{12} \times \frac{(20 \times 12)^4}{8 \times 3.63 \times 10^6 \times 9264} = 0.82 \text{ in.}$$

$$\Delta_1 \text{ (due to dead load)} = 0.82 \times \frac{0.4}{0.8} = 0.41 \text{ in.}$$

$$\Delta_2 \text{ (due to concentrated dead load) at free end} = \frac{3000(20 \times 12)^3}{3 \times 3.63 \times 10^6 \times 9264} = 0.41 \text{ in.}$$

$$\Delta_3 \text{ (due to concentrated live load at 10 ft from fixed end)} = \frac{4000(10 \times 12)^2 \times (3 \times 20 \times 12 - 10 \times 12)}{6 \times 3.63 \times 10^6 \times 9264} = 0.17 \text{ in.}$$

The total immediate deflection is

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 = 0.82 + 0.41 + 0.17 = 1.40 \text{ in.}$$

9. For additional long-time deflection, the immediated deflection is multiplied by the factor λ_Δ. For a 1-year period, $\zeta = 1.4$.

$$\rho' = \frac{A'_s}{bd} = \frac{1.57}{12 \times 21.5} = 0.0061$$

$$\lambda_\Delta = \frac{1.4}{1 + 50 \times 0.0061} = 1.073$$

Total immediate deflection Δ_s due to sustained load (here only the dead load of 0.4 K/ft and $P_D = 3$ K at free end): $\Delta_s = (0.41 + 0.41) = 0.82$ in. Additional long-time deflection = 1.073 $\times$ 0.82 = 0.88 in.

10. Total long-time deflection is the immediate deflection plus long-time deflection due to shrinkage and creep.
Total $\Delta = 1.40 + 0.88 = 2.28$ in.

Example 6.5

Calculate the instantaneous midspan deflection of beam AB in Fig. 6.5, which has a span of 32 ft. The beam is continuous over several supports of different span lengths. The absolute bending moment diagram and cross-sections of the beam at midspan and supports are also shown. The beam carries a uniform dead load of 4.2 K/ft and a live load of 3.6 K/ft. Given: $f'_c = 3$ ksi normal-weight concrete, $f_y = 60$ ksi, and $n = 9.2$.

Moment at midspan:	$M_D = 192$ K·ft	$M_{(D+L)} = 480$ K·ft
Moment at left support A :	$M_D = 179$ K·ft	$M_{(D+L)} = 420$ K·ft
Moment at right support B :	$M_D = 216$ K·ft	$M_{(D+L)} = 542$ K·ft

Solution

1. The beam AB is subjected to a positive moment that causes a deflection downward at midspan and negative moments at the two ends, causing a deflection upward at midspan. As was explained earlier, the deflection is a function of the effective moment of inertia, I_e. In a continuous beam, the value of I_e to be used is the average value for the positive and negative moment regions. Therefore, three sections will be considered: the section at midspan and the sections at the two supports.
2. Calculate I_e: For the gross area of all sections, $kd = 13.5$ in. and $I_g = 114{,}300$ in.^{4} Also, $f_r = 7.5\lambda\sqrt{f'_c} = 410$ psi and $E_c = 57{,}400\sqrt{f'_c} = 3.15 \times 10^6$ for all sections. The values of kd, I_{cr}, and M_{cr} for each cracked section, I_e for dead load only (using M_a of dead load), and

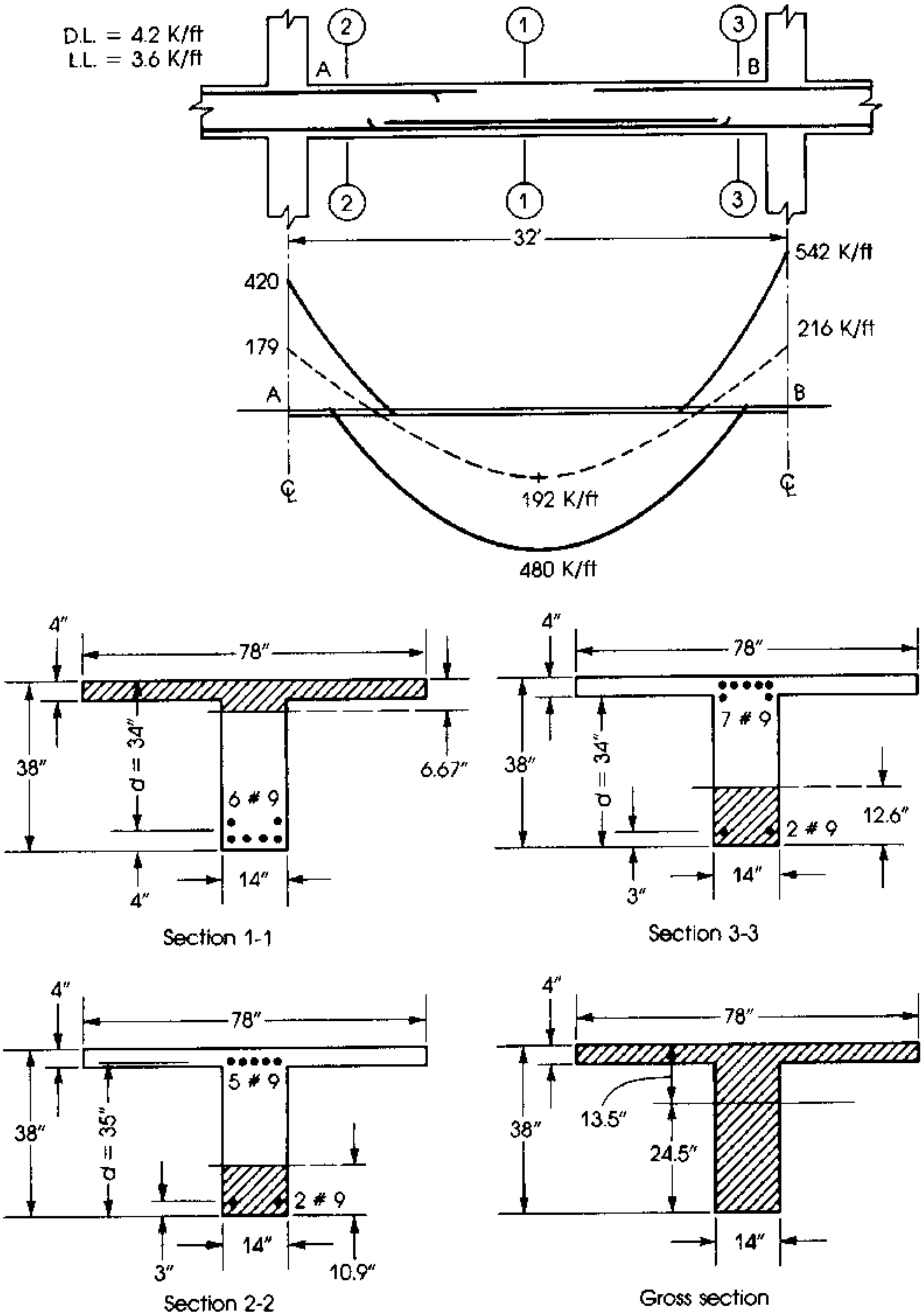


Figure 6.5 Example 6.5: deflection of a continuous beam.

I_e for dead and live loads (using M_a for dead and live loads) are calculated and tabulated as follows.

Section	*kd* (in.)	I_{cr} (in.^{4})	M_{cr} (K·ft)	I_e (in.^{4}) (Dead load)	I_e (in.^{4}) ($D + L$)
Midspan	6.67	48,550	159.4	86,160	50,960
Support *A*	10.9	34,930	289.3	114,300	60,880
Support *B*	12.6	44,860	289.3	114,300	55,415

Note that when the beam is subjected to dead load only and the ratio M_{cr}/M_a is greater than 1.0, I_e is equal to I_g.

3. Calculate average I_e from Eq. 6.8:

$$I_{e1}(\text{average}) = 0.7(50{,}960) + 0.15(60{,}880 + 55{,}415)$$
$$= 53{,}116 \text{ in.}^4$$

For dead and live loads,

$$\text{Average } I_e \text{ for end sections} = \tfrac{1}{2}(60{,}880 + 55{,}415)$$
$$= 58{,}150 \text{ in.}^4$$
$$I_{e2} \text{ (average)} = \tfrac{1}{2}(50{,}960 + 58{,}150) = 54{,}550 \text{ in.}^4$$

For dead loads only,

$$\text{Average } I_e \text{ for end sections} = 114{,}300 \text{ in.}^4$$
$$I_{e3} \text{ (average)} = \tfrac{1}{2}(86{,}160 + 114{,}300) = 100{,}230 \text{ in.}^4$$

4. Calculate immediate deflection at midspan:

$$\Delta_1 \text{ (due to uniform load)} = \frac{5wL^4}{384EI_e} \quad \text{(downward)}$$
$$\Delta_2 \text{ (due to a moment at } A, M_A) - \frac{M_A L^2}{16EI_e} \quad \text{(upward)}$$
$$\Delta_3 \text{ (due to a moment at } B, M_B) = -\frac{M_B L^2}{16EI_e} \quad \text{(upward)}$$
$$\text{Total deflection } \Delta = \Delta_1 - \Delta_2 - \Delta_3$$

The dead-load deflection for a uniform dead load of 4.2 K/ft, taking M_A(D.L.) = 179 K·ft, M_B (D.L.) = 216 K·ft, and I_{e3} = 100,230 in.^{4} and then substituting in the preceding equations, is

$$\Delta = 0.314 - 0.063 - 0.075 = 0.176 \text{ in.} \quad \text{(downward)}$$

The deflection due to combined dead and live loads is found by taking dead plus live load = 7.8 K/ft, M_A = 420 K·ft, M_B = 542 K·ft, and I_{e2} = 54,550 in.^{4}:

$$\Delta = 1.071 - 0.270 - 0.349 = 0.452 \text{ in.} \quad \text{(downward)}$$

The immediate deflection due to live load only is 0.542 − 0.176 = 0.276 in. (downward). If the limiting permissible deflection is $L/480 = (32 \times 12)/480 = 0.8$ in., then the section is adequate.

There are a few points to mention about the results.

a. If I_e of the midspan section only is used ($I_e = 50{,}960\,\text{in.}^4$) then the deflection due to dead plus live loads is calculated by multiplying the obtained value in step 4 by the ratio of the two I_e:

$$\Delta\ (\text{dead} + \text{live}) = 0.452 \times \left(\frac{54{,}550}{50{,}960}\right) = 0.484\ \text{in.}$$

The difference is small, about 7% on the conservative side.

b. If I_e 1 (average) is used ($I_{e1} = 53{,}116\,\text{in.}^4$), then Δ (dead + live) = 0.471 in. The difference is small, about 4% on the conservative side.

c. It is believed that it is more convenient to use I_e at midspan section unless a more rigorous solution is required.

6.6 CRACKS IN FLEXURAL MEMBERS

The study of crack formation, behavior of cracks under increasing load, and control of cracking is necessary for proper design of reinforced concrete structures. In flexural members, cracks develop under working loads, and because concrete is weak in tension, reinforcement is placed in the cracked tension zone to resist the tension force produced by the external loads.

Flexural cracks develop when the stress at the extreme tension fibers exceeds the modulus of rupture of concrete. With the use of high-strength reinforcing bars, excessive cracking may develop in reinforced concrete members. The use of high-tensile steel has many advantages, yet the development of undesirable cracks seems to be inevitable. Wide cracks may allow corrosion of the reinforcement or leakage of water structures and may spoil the appearance of the structure.

A crack is formed in concrete when a narrow opening of indefinite dimension has developed in the concrete beam as the result of internal tensile stresses. These internal stresses may be due to one or more of the following:

- External forces such as direct axial tension, shear, flexure, or torsion
- Shrinkage
- Creep
- Internal expansion resulting from a change of properties of the concrete constituents

In general, cracks may be divided into two main types: secondary cracks and main cracks.

6.6.1 Secondary Cracks

Secondary cracks, very small cracks that develop in the first stage of cracking, are produced by the internal expansion and contraction of the concrete constituents and by low flexural tension stresses due to the self-weight of the member and any other dead loads. There are three types of secondary cracks.

Shrinkage cracks. *Shrinkage cracks* are important cracks, because they affect the pattern of cracking that is produced by loads in flexural members. When they develop, they form a weak path in the concrete. When load is applied, cracks start to appear at the weakest sections, such as along the reinforcing bars. The number of cracks formed is limited by the amount of shrinkage in concrete and the presence of restraints. Shrinkage cracks are difficult to control.

Secondary flexural cracks. Usually *secondary flexural cracks* are widely spaced, and one crack does not influence the formation of others [8]. They are expected to occur under low loads, such as dead loads. When a load is applied gradually on a simple beam, tensile stress develops at the bottom fibers, and when it exceeds the flexural tensile stress of concrete, cracks start to develop. They widen gradually and extend toward the neutral axis. It is difficult to predict the sections at which secondary cracks start because concrete is not a homogeneous, isotropic material.

Salinger [9] and Billing [10] estimated the steel stress just before cracking to be from about 6000 to 7000 psi (42 to 49 MPa). An initial crack width of the order of 0.001 in. (0.025 mm) is expected at the extreme concrete tensile fibers. Once cracks are formed, the tensile stress of concrete at the cracked section decreases to 0, and the steel bars take all the tensile force. At this moment, some slip occurs between the steel bars and the concrete due to the differential elongation of concrete and steel and extends to a section where the concrete and steel strains are equal. Figure 6.6 shows the typical stress distribution between cracks in a member under axial tension.

Corrosion secondary cracks. *Corrosion secondary cracks* form when moisture containing deleterious agents such as sodium chloride, carbon dioxide, and dissolved oxygen penetrates the concrete surface, corroding the steel reinforcement [11]. The oxide compounds formed by deterioration of steel bars occupy a larger volume than the steel and exert mechanical pressure that perpetuates extensive cracking [12,13]. This type of cracking may be severe enough to result in eventual failure of the structure. The failure of a roof in Muskegan, Michigan, in 1955 due to the corrosion of steel bars was reported by Shermer [13]. The extensive cracking and spalling of concrete in the San Mateo–Hayward Bridge in California within 7 years was reported by Stratful [12]. Corrosion cracking may be forestalled by using proper construction methods and high-quality concrete. More details are discussed by Evans [14] and Mozer and others [15].

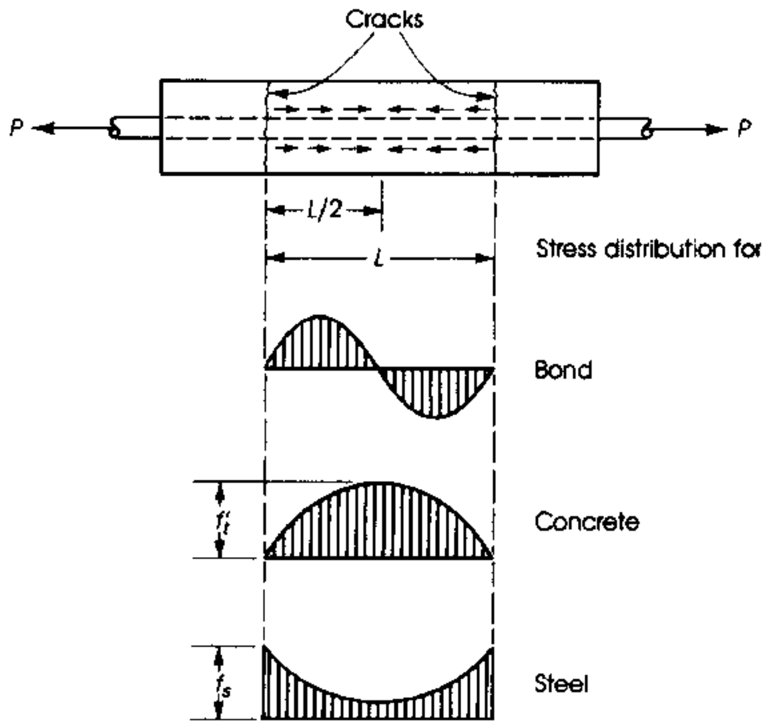


Figure 6.6 Typical stress distribution between cracks.

6.6.2 Main Cracks

Main cracks develop at a later stage than secondary cracks. They are caused by the difference in strains in steel and concrete at the section considered. The behavior of main cracks changes at two different stages. At low tensile stresses in steel bars, the number of cracks increases, whereas the widths of cracks remain small; as tensile stresses are increased, an equilibrium stage is reached. When stresses are further increased, the second stage of cracking develops, and crack widths increase without any significant increase in the number of cracks. Usually one or two cracks start to widen more than the others, forming critical cracks (Fig. 6.7).

Main cracks in beams and axially tensioned members have been studied by many investigators; prediction of the width of cracks and crack control were among the problems studied. These are discussed here, along with the requirements of the ACI Code.

Crack width. *Crack width* and *crack spacing*, according to existing crack theories, depend on many factors, which include steel percentage, its distribution in the concrete section, steel flexural stress at service load, concrete cover, and properties of the concrete constituents. Different equations for predicting the width and spacing of cracks in reinforced concrete members were presented at the Symposium on Bond and Crack Formation in Reinforced Concrete in Stockholm, Sweden, in 1957. Chi and Kirstein [16] presented equations for the crack width and spacing as a function of an effective area of concrete around the steel bar: A concrete circular area of diameter equal to four times the diameter of the bar was used to calculate crack width. Other equations were presented over the next decade [17–23].

Gergely and Lutz [23] presented the following formula for the limiting crack width:

$$W = 0.076\beta f_s \sqrt[3]{Ad_c} \times 10^{-6} \text{ (in.)} \tag{6.15}$$

where β, A, and f_s are as defined previously and d_c = thickness of concrete cover measured from the extreme tension fiber to the center of the closest bar. The value of β can be taken to be approximately equal to 1.2 for beams and 1.35 for slabs. Note that f_s is in psi and W is in inches.

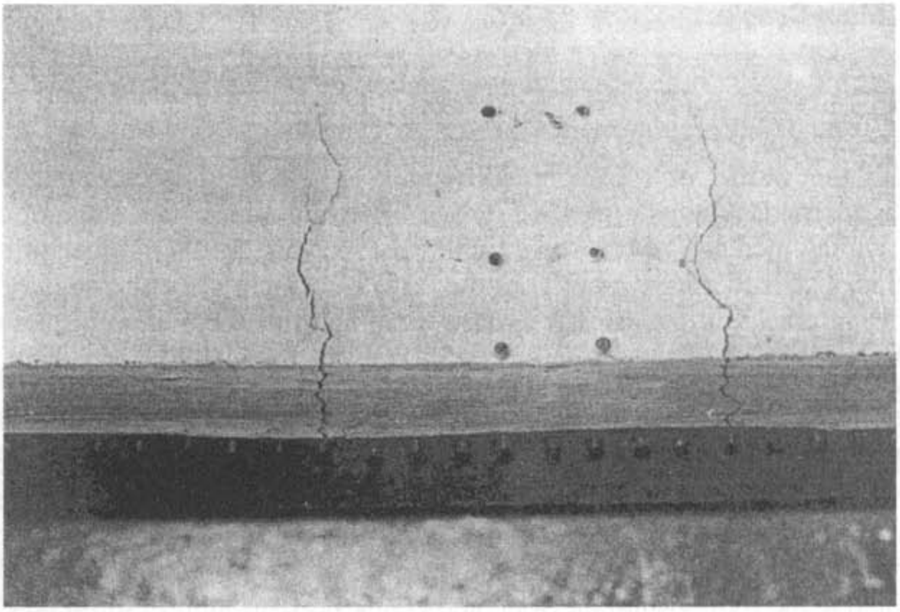
The mean ratio of maximum crack width to average crack width was found to vary between 1.5 and 2.0, as reported by many investigators. An average value of 1.75 may be used.

In SI units (mm and MPa), Eq. 6.15 is

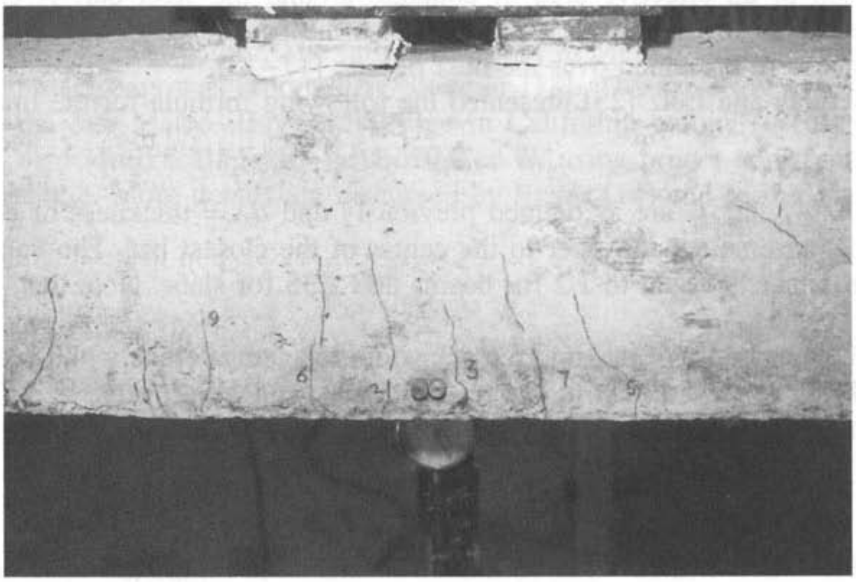
$$W = 11.0\beta f_s \sqrt[3]{Ad_c} \times 10^{-6} \tag{6.16}$$

Tolerable crack width. The formation of cracks in reinforced concrete members is unavoidable. Hairline cracks occur even in carefully designed and constructed structures. Cracks are usually measured at the face of the concrete, but actually they are related to crack width at the steel level, where corrosion is expected. The permissible crack width is also influenced by aesthetic and appearance requirements. The naked eye can detect a crack about 0.006 in. (0.15 mm) wide, depending on the surface texture of concrete. Different values for permissible crack width at the steel level have been suggested by many investigators, ranging from 0.010 to 0.016 in. (0.25–0.40 mm) for interior members and from 0.006 to 0.010 in. (0.15–0.25 mm) for exterior exposed members. A limiting crack width of 0.016 in. (0.40 mm) for interior members and 0.013 in. (0.32 mm) for exterior members under dry conditions can be tolerated.

Crack control. *Control* grows in importance with the use of high-strength steel in reinforced concrete members, as larger cracks develop under working loads because of the high allowable stresses. Control of cracking depends on the permissible crack width: It is always preferable to



(*a*)



(*b*)

Figure 6.7 (*a*) Main cracks in a reinforced concrete beam. (*b*) Spacing of cracks in a reinforced concrete beam.

have a large number of fine cracks rather than a small number of large cracks. Secondary cracks are minimized by controlling the total amount of cement paste, water–cement ratio, permeability of aggregate and concrete, rate of curing, shrinkage, and end-restraint conditions.

The factors involved in controlling main cracks are the reinforcement stress, the bond characteristics of reinforcement, the distribution of reinforcement, the diameter of the steel bars used, the steel percentage, the concrete cover, and the properties of concrete constituents. Any improvement in these factors will help in reducing the width of cracks.

6.7 ACI CODE REQUIREMENTS

To control cracks in reinforced concrete members, the ACI Code, Chapter 10, specifies the following:

1. Only deformed bars are permitted as main reinforcement.
2. Tension reinforcement should be well distributed in the zones of maximum tension (Section 10.6.3).
3. When the flange of the section is under tension, part of the main reinforcement should be distributed over the effective flange width or one-tenth of the span, whichever is smaller. Some longitudinal reinforcement has to be provided in the outer portion of the flange (Section 10.6.6).
4. The design yield strength of reinforcement should not exceed 80 ksi (560 MPa) (Section 9.4).
5. The maximum spacing s of reinforcement closest to a concrete surface in tension in reinforced concrete beams and one-way slabs is limited to

$$s\ (\text{in.}) = \left[15\left(\frac{40}{f_s}\right) - 2.5C_c\right] \tag{6.17}$$

but not greater than 12 $(40/f_s)$, where

f_s = calculated stress (ksi) in reinforcement at service load computed as the unfactored moment divided by the product of steel area and the internal moment arm, $f_s = M/(A_s jd)$. (Alternatively, $f_s = \frac{2}{3} f_y$ may be used; an approximate value of $jd = 0.87d$ may be used.)

C_c = clear cover from the nearest surface in tension to the surface of the flexural tension reinforcement (in.).

s = center to center spacing of flexural tension reinforcement nearest to the extreme concrete tension face (in.).

The preceding limitations are applicable to reinforced concrete beams and one-way slabs subject to normal environmental condition and do not apply to structures subjected to aggressive exposure. The spacing limitation just given is independent of the bar size, which may lead to the use of smaller bar sizes to satisfy the spacing criteria. For the case of concrete beams reinforced with grade 60 steel bars and $C_c = 2$ in., clear cover to the tension face, the maximum spacing is calculated as follows: Assume $f_s = 2/3\ f_y = (2/3) \times 60 = 40$ ksi and $s = 15\left(\frac{40}{40}\right) - 2.5 \times 2 = 10$ in. (controls), which is less than $12(40/40) = 12$ in.

6. In SI units, Eq. 6.17 becomes

$$s\ (\text{mm}) = 105{,}000/f_s - 2.5C_c \tag{6.18}$$

but not greater than 300 $(280/f_s)$, where f_s is in MPa and C_c is in mm. For example, if bars with a clear cover equal to 50 mm are used, then the maximum spacing, s, is calculated as follows:

$$s = (105{,}000/280) - 2.5 \times 50 = 250 \text{ mm (controls)},$$

which is less than 300(280/280) = 300 mm in this example. This is assuming that $f_s = \frac{2}{3} \times 420 = 280$ MPa.

7. In the previous Codes, control of cracking was based on a factor Z defined as follows:

$$Z = f_s\sqrt[3]{Ad_c} \leq 175 \text{ K/in.} \quad (31 \text{ kN/mm}) \quad \text{for interior members}$$

$$Z \leq 140 \text{ K/in.} \ (26 \text{ kN/mm}) \quad \text{for exterior members.} \tag{6.19}$$

where f_s = flexural stress at service load (ksi) and may be taken as 0.6 f_y. A and d_c are the effective tension area of concrete and thickness of concrete cover, respectively. This expression is based on Eq. 6.15 assuming a limiting crack width of 0.016 in. for interior members and 0.013 in. for exterior members. It encouraged a decrease in the reinforcement cover to achieve a smaller Z, while unfortunately it penalized structures with concrete cover that exceeded 2 in.

8. *Skin reinforcement*: For relatively deep girders, with a total depth, h, equal to or greater than 36 in. (900 mm), light reinforcement should be added near the vertical faces in the tension zone to control cracking in the web above the main reinforcement. The ACI Code, Section 10.6.7, referred to this additional steel as skin reinforcement. The skin reinforcement should be uniformly distributed along both side faces of the member for a distance $h/2$ from the tension face.

The spacing S between the longitudinal bars or wires of the skin reinforcement shall be as provided in Eq. 6.17 where C_c is the least distance from the skin reinforcement to the side face.

Referring to Figure 6.8, if b = 16 in., h = 40 in., fy = 60 ksi and choosing no. 3 bars spaced at 6.0 in. as skin reinforcement (3 spaces on each side), then the height covered = 3 × 6 + 2.5 = 20.5 in., which is greater than $h/2$ = 40/2 = 20 in.

Checking the spacing S by Eq. 6.18 and assuming f_s = 2/3, f_y = 2/3 × 60 = 40 ksi, and C_c = 2 in., then S = 15(40/40) − 2.5 × 2 = 10 in., which is less than 12(40/40) = 12 in. The spacing used is adequate. Note that C_c = 1.5 in. may be used for the skin reinforcement concrete cover.

It is recommended to use smaller spacing to control the propagation of tensile cracks along the side of the tension zone with the first side bar to be placed at 4 to 6 in. from the main tensile steel.

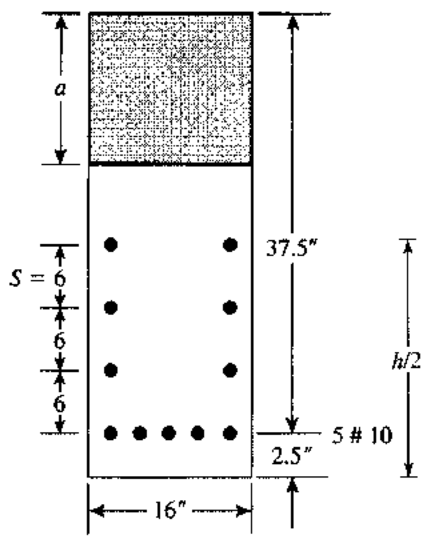


Figure 6.8 Skin reinforcement (6 no. 3 bars).

Example 6.6

The sections of a simply supported beam are shown in Fig. 6.9.

a. Check if the bar arrangement satisfies the ACI Code requirements.
b. Determine the expected crack width.
c. Check the Z-factor based on Eq. 6.19.

Given: $f_c' = 4$ ksi, $f_y = 60$ ksi, and no. 3 stirrups.

Solution

1. Fig. 6.9, section a:
   a. For three no. 8 bars, $A_s = 2.35\,\text{in.}^2$, clear cover, C_c, $= 2.5 - 8/16 = 2.0$ in. Assume $f_s = \frac{2}{3} f_y = 2/3 \times 60 = 40$ ksi. Maximum spacing s $= 600/40 - 2.5 \times 2 = 10$ in., which is less than 12(40/40) = 12 in. Spacing provided = 0.5(12 − 2.5 − 2.5) = 3.5 in., center to center of bars, which is less than 10 in.
   b. For this section, $d_c = 2.5$ in. The effective tension area of concrete for one bar is

$$A = 12(2 \times 2.5)/3 = 20\ \text{in.}^2$$

Estimated crack width using Eq. 6.16 is

$$W = 0.076(1.2)(36{,}000)\sqrt[3]{20 \times 2.5} \times 10^{-6} = 0.0121\ \text{in.}$$

This is assuming $\beta = 1.2$ for beams and $f_s = 36$ ksi. The crack width above is less than 0.016 in. and 0.013 in. for interior and exterior members.

2. Fig. 6.9, section b:
   a. Calculations of spacing of bars are similar to those in section a.
   b. For this section, $d_c = 2.5$ in., and the steel bars are placed in two layers. The centroid of the steel bars is 3.5 in. from the bottom fibers. The effective tension concrete area is $A = 12(2 \times 3.5)/6 = 14\,\text{in.}^2$

$$W = .076 \times 1.2 \times 36{,}000\sqrt[3]{14 \times 2.5} \times 10^{-6} = 0.0107\ \text{in.}$$

which is adequate.

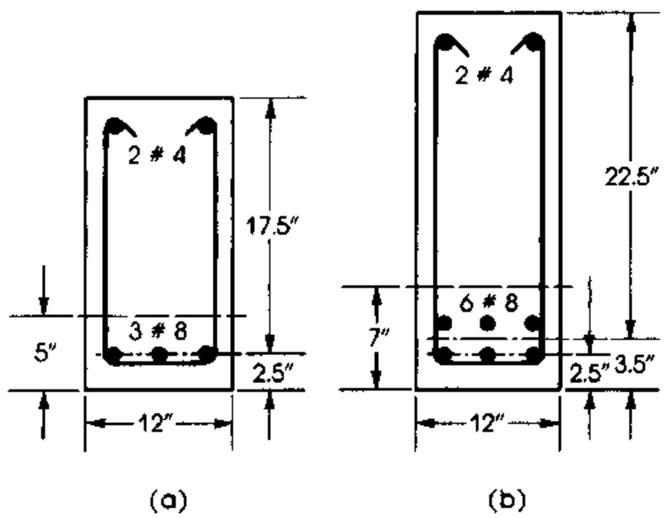


Figure 6.9 Two sections for Example 6.6.

Discussion

It can be seen that the spacing, s, in Eq. 6.17 is a function of the stress in the tension bars or, indirectly, is a function of the strain in the tension steel, $f_s = E_s \times \varepsilon_s$, and E_s for steel is equal to 29,000 ksi. The spacing also depends on the concrete cover, C_c. An increase in the concrete cover will reduce the limited spacing s, which is independent on the bar size used in the section.

In this example, the expected crack width was calculated by Eq. 6.17 to give the student or the engineer a physical feeling for the crack width and crack control requirement. The crack width is usually measured in beams when tested in the laboratory or else in actual structures under loading when serious cracks develop in beams or slabs and testing is needed. If the crack width measured before and after loading is greater than the yield strain of steel, then the main reinforcement is in the plastic range and ineffective. Sheets with lines of different thickness representing crack widths are available in the market for easy comparisons with actual crack widths. In addition to the steel stress and the concrete cover, W depends on a third factor, A, representing the tension area of concrete surrounding one bar in tension.

Example 6.7

Design a simply supported beam with a span of 24 ft to carry a uniform dead load of 1.5 K/ft and a live load of 1.18 K/ft. Choose adequate bars; then check their spacing arrangement to satisfy the ACI Code. Given: $b = 16$ in., $f'_c = 4$ ksi, $f_y = 60$ ksi, a steel percentage = 0.8%, and a clear concrete cover of 2 in.

Solution

1. For a steel percentage of 0.8%, $R_u = 400$ psi $= 0.4$ ksi ($\phi = 0.9$). The external factored moment is $M_u = w_u \times L^2/8$, and $w_u = 1.2(1.5) + 1.6(1.18) = 3.69$ K/ft.

$$M_u = 3.69(24)^2/8 = 265.68 \text{ K·ft} = 3188.2 \text{ K·in.}$$

$$M_u = R_u \cdot bd^2 \qquad d = 22.32 \qquad A_s = 0.008 \times 16 \times 22.32 = 2.86 \text{ in.}^2$$

Choose three no. 9 bars (area = 3.0 in.^{2}) in one row, and a total depth of $h = 25.0$ in. Actual $d = 25 - 2 - 9/16 = 22.44$ in. (Fig. 6.10).

2. Check spacing of bars using Eq. 6.18. Calculate the service load and moment: $w = 1.5 + 1.18 = 2.68$ K/ft.

$$M = 2.68(24)^2/8 = 193 \text{ K·ft} = 2315 \text{ K·in.}$$

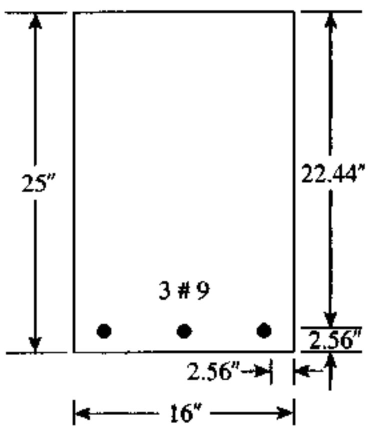


Figure 6.10 Example 6.7.

3. Calculate the neutral axis depth kd and the moment arm jd (Eq. 6.12).

$$b(kd)^2/2 - nA_s(d - kd) = 0 \qquad n = 8 \qquad A_s = 3.0 \qquad d = 22.44 \text{ in.}$$

$$kd = 6.85 \text{ in.} \qquad jd = d - kd/3 = 20.16 \text{ in.} \qquad j = 20.16/22.44 = 0.898$$

Note that an approximate value of $j = 0.87$ may be used if kd is not calculated.

4. Calculate the stress f_s:

$$M = A_s \cdot f_s \cdot jd \quad 2315 = 3(f_s)(20.16) \quad f_s = 38.3 \text{ ksi}$$

5. Calculate the spacing s by Eq. 6.18:

$$s = 600/38.3 - 2.5 \times 2 = 10.7 \text{ in. (controls)}$$

which is less than 12(40/40) = 12.0 in. Spacing provided, = 0.5 (16 − 2.56 − 2.56) = 5.44 in., which is less than 10.7 in.

Example 6.8: SI Units

Design a simply supported beam of 7.2-m span to carry a uniform dead load of 22.2 kN/m and a live load of 17 kN/m. Choose adequate bars, and check their spacing arrangement to satisfy the ACI Code.

Given: $b = 400$ mm, $f_c' = 30$ MPa, $f_y = 400$ MPa, a steel percentage of 0.8%, and a clear concrete cover of 50 mm.

Solution

1. For a steel percentage of 0.008 and from Eq. 3.22, $R_u = 2.7$ MPa. Factored load $w_u = 1.2(22.2) + 1.6(17) = 53.8$ kN/m. $M_u = w_u \cdot L^2/8 = 53.8(7.2)^2/8 = 348.6$ kN·m. $M_u = R_u \cdot bd^2$, or $348.6 \times 10^6 = 2.7 \times 400d^2$ then $d = 568$ mm $A_s = \rho bd = 0.008 \times 400 \times 568 = 1818\ \text{mm}^2$. Choose four bars, 25 mm (no. 25 M), $A_s = 2040\ \text{mm}^2$, in one row ($b_{\min} = 220$ mm). Let $h = 650$ mm, the actual $d = 650 - 50 - 25/2 = 587.5$ mm, say 585 mm. Final section: $b = 400$ mm, $h = 650$ mm, with four no. 25 mm bars (Fig. 6.11).
2. Check spacing of bars using Eq. 6.17. Calculate the service load moment, $w = 22.2 + 17 = 39.2$ kN/m.

$$M = 39.2(7.2)^2/8 = 254 \text{ kN·m}$$

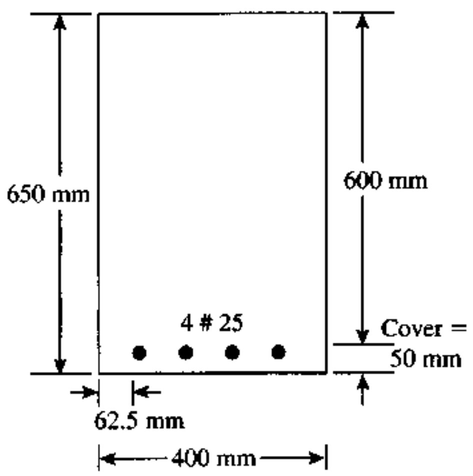


Figure 6.11 Example 6.8.

Calculate kd and jd as in the previous example. Alternatively, use a moment arm, $jd = 0.87d = 0.87(585) = 509$ mm and $f_s = M/(A_s \cdot jd) = 254(10)^6/(2040 \times 585) = 213$ MPa. From Eq. 6.19, maximum s = (105,000/213) − 2.5(50) = 368 mm (controls), which is less than $300(280/f_s) = 300(280)/213 = 394$ mm. Note that if $f_s = 0.6\ f_y = 0.6(400) = 240$ MPa is used, then maximum s = 312 mm. It is preferable to calculate f_s from the moment equation to reflect the actual stress in the bars. Spacing provided = (1/3)(400 − 50 − 25) = 92 mm, which is adequate.

SUMMARY

Sections 6.1–6.2

1. Deflection $\Delta = \alpha(WL^3/EI) = 5WL^3/384EI = 5\ wL^4/384EI$ for a simply supported beam subjected to a uniform total load of $W = wL$.

$$E_c = 33w^{1.5}\sqrt{f'_c} = 57{,}400\ f'_c \text{ psi}$$

for normal-weight concrete.

2. Effective moment of inertia is

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \leq I_g$$

$$M_{cr} = f_r \times \frac{I_g}{y_t} \quad \text{and} \quad f_r = 7.5\lambda\sqrt{f'_c} \tag{6.5}$$

Section 6.3

The deflection of reinforced concrete members continues to increase under sustained load. Additional long-time deflection = ζ_Δ × instantaneous deflection:

$$\zeta_\Delta = \frac{\zeta}{1 + 50\rho'} \tag{6.14}$$

ζ = 1.0, 1.2, 1.4, and 2.0 for periods of 3, 6, 12, and 60 months, respectively.

Sections 6.4–6.5

1. The allowable deflection varies between $L/180$ and $L/480$.
2. Deflections for different types of loads may be calculated for each type of loading separately and then added algebraically to obtain the total deflection.

Section 6.6

1. Cracks are classified as secondary cracks (shrinkage, corrosion, or secondary flexural cracks) and main cracks.
2. Maximum crack width is

$$W = 0.076\beta f_s \sqrt[3]{Ad_c} \times 10^{-6} \text{ (in.)} \tag{6.15}$$

Approximate values for β, f_s, and d_c are $\beta = 1.2$ for beams and 1.35 for slabs, $d_c =$ 2.5 in., and $f_s = (2/3)f_y$.

3. The limiting crack width is 0.016 in. for interior members and 0.013 in. for exterior members.

Section 6.7

The maximum spacing s of bars closest to a concrete surface in tension is limited to

$$s = 600/f_s - 2.5C_c \tag{6.17}$$

but not more than $12(40/f_s)$. Note that f_s may be taken as $2/3 f_y$.

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PROBLEMS

6.1 Determine the instantaneous and long-time deflection of a 20-ft-span simply supported beam for each of the following load conditions. Assume that 10% of the live loads are sustained and the dead loads include the self-weight of the beams. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, $d' = 2.5$ in., and a time limit of 5 years. Refer to Fig. 6.12.

No.	b (in.)	d (in.)	h (in.)	A_s (in.^{2})	A'_s (in.^{2})	W_D (K/ft)	W_L (K/ft)	P_D (K)	P_L (K)
a	14	17.5	20	5 no. 9	—	2.2	1.8	—	—
b	20	27.5	30	6 no. 10	—	7.0	3.6	—	—
c	12	19.5	23	6 no. 8	—	3.0	1.5	—	—
d	18	20.5	24	6 no. 10	2 no. 9	6.0	2.0	—	—
e	16	22.5	26	6 no. 11	2 no. 10	5.0	3.2	12	10
f	14	20.5	24	8 no. 9	2 no. 9	3.8	2.8	8	6

h-d = 2.5 in. indicates one row of bars, whereas h-d = 3.5 in. indicates two rows of bars. Concentrated loads are placed at midspan.

6.2 Determine the instantaneous and long-term deflection of the free end of a 12-ft-span cantilever beam for each of the following load conditions. Assume that only dead loads are sustained, and the dead loads include the self-weight of the beams. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and a time limit of more than 5 years. Refer to Fig. 6.13.

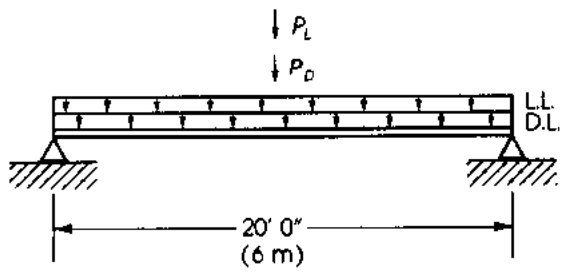


Figure 6.12 Problem 6.1.

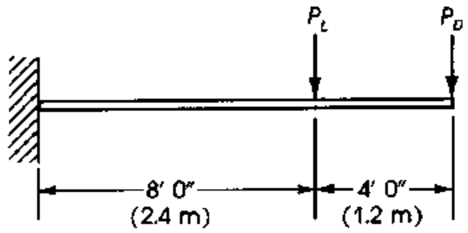


Figure 6.13 Problem 6.2.

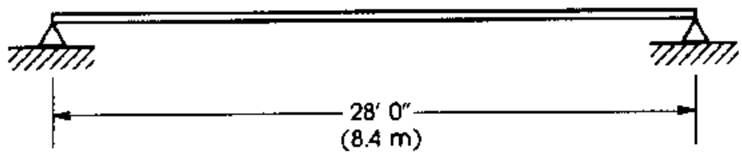


Figure 6.14 Problem 6.3: Dead load = 2 K/ft (30 kN/m) and live load = 1.33 K/ft (20 kN/m).

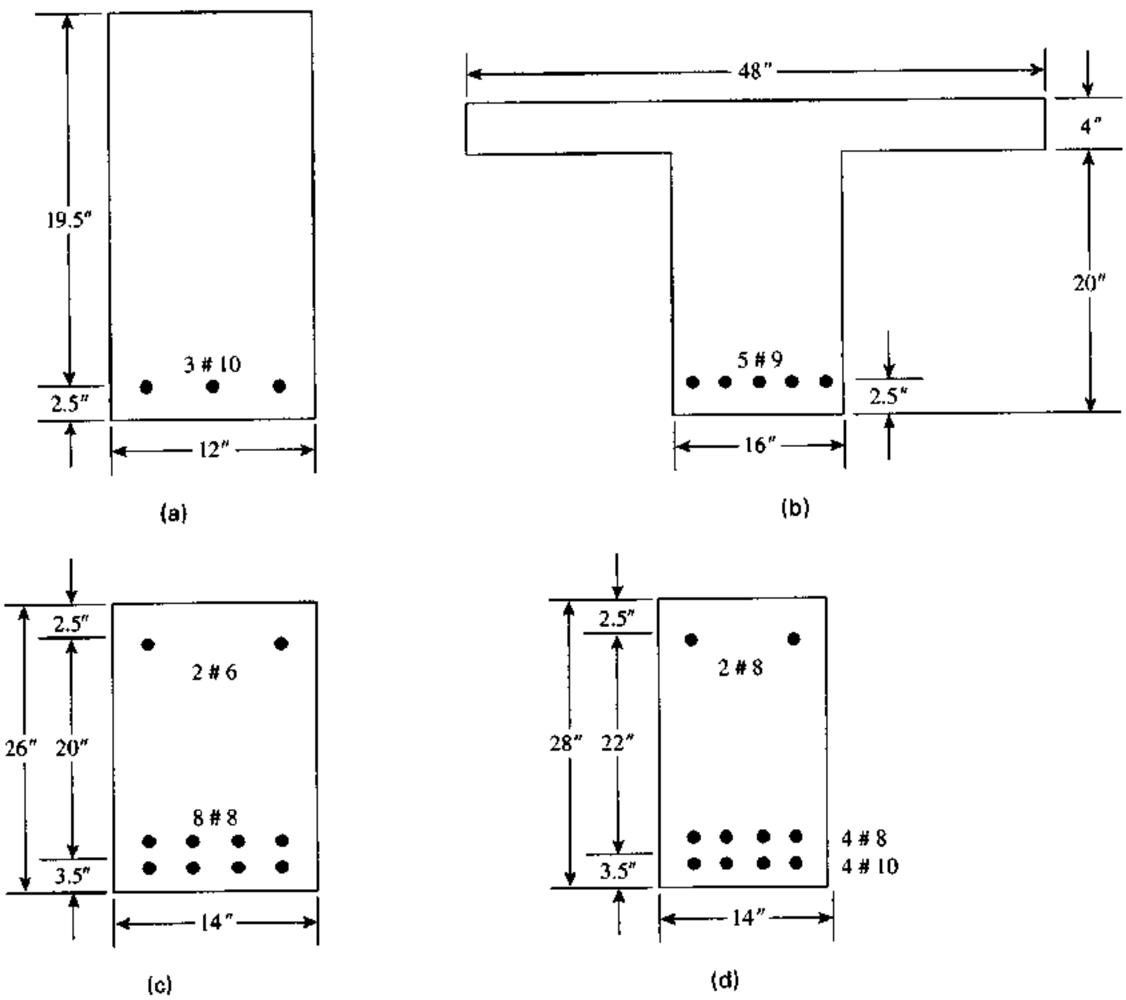


Figure 6.15 Problem 6.5.

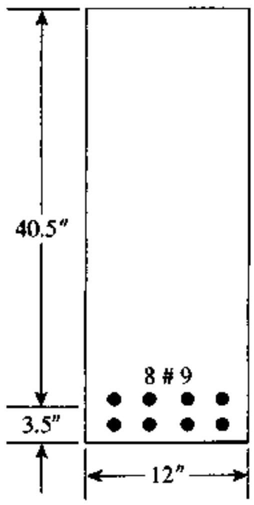


Figure 6.16 Problem 6.6 (skin reinforcement).

No.	b (in.)	d (in.)	h (in.)	A_s (in.^{2})	A'_s (in.^{2})	W_D (K/ft)	W_L (K/ft)	P_D (K)	P_L (K)
a	15	20.5	24	8 no. 9	2 no. 9	3.5	2.0	—	—
b	18	22.5	26	6 no. 10	—	2.0	1.5	7.4	5.0
c	12	19.5	23	8 no. 8	2 no. 8	2.4	1.6	—	—
d	14	20.5	24	8 no. 9	2 no. 9	3.0	1.1	5.5	4.0

h-d = 2.5 in. indicates one row of bars, whereas h-d = 3.5 in. indicates two rows of bars. Concentrated loads are placed as shown

6.3 A 28-ft simply supported beam carries a uniform dead load of 2 K/ft (including self-weight) and a live load of 1.33 K/ft. Design the critical section at midspan using the maximum steel ratio allowed by the ACI Code and then calculate the instantaneous deflection. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and $b = 12$ in. See Fig. 6.14.

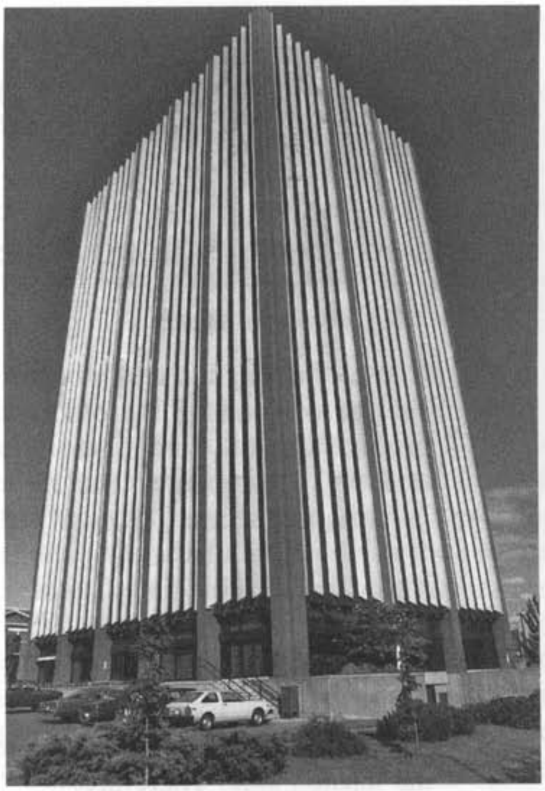
6.4 Design the beam in Problem 6.3asdoubly reinforced, considering that compression steel resists 20% of the maximum bending moment. Then calculate the maximum instantaneous deflection.

6.5 The four cross-sections shown in Fig. 6.15 belong to four different beams with $f'_c = 4$ ksi and $f_y = 60$ ksi. Check the spacing of the bars in each section according to the ACI Code requirement using $f_s = 0.6 f_y$. Then calculate the tolerable crack width, W.

6.6 Determine the necessary skin reinforcement for the beam section shown in Fig. 6.16. Then choose adequate bars and spacings. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

CHAPTER 7

DEVELOPMENT LENGTH OF REINFORCING BARS



Reinforced concrete columns supporting an office building, Toronto, Canada.

7.1 INTRODUCTION

The joint behavior of steel and concrete in a reinforced concrete member is based on the fact that a bond is maintained between the two materials after the concrete hardens. If a straight bar of round section is embedded in concrete, a considerable force is required to pull the bar out of the concrete. If the embedded length of the bar is long enough, the steel bar may yield, leaving some length of the bar in the concrete. The bonding force depends on the friction between steel and concrete. It is influenced mainly by the roughness of the steel surface area, the concrete mix, shrinkage, and the cover of concrete. Deformed bars give a better bond than plain bars. Rich mixes have greater adhesion than weak mixes. An increase in the concrete cover will improve the ultimate bond stress of a steel bar [2].

In general, the bond strength is influenced by the following factors:

1. Yield strength of reinforcing bars, f_y. Longer development length is needed with higher f_y.
2. Quality of concrete and its compressive strength, f'_c. An increase in f'_c reduces the required development length of reinforcing bars.
3. Bar size, spacing, and location in the concrete section. Horizontal bars placed with more than 12 in. of concrete below them have lower bond strength due to the fact that concrete shrinks and settles during the hardening process. Also, wide spacings of bars improve the bond strength, giving adequate effective concrete area around each bar.
4. Concrete cover to reinforcing bars. A small cover may cause the cracking and spalling of the concrete cover.