This is because the exterior surface of the wall is normally exposed to different weather conditions and temperature changes.

For interior wall surfaces, the balance of the required reinforcement in each direction should have a minimum concrete cover of $\frac{3}{4}$ in. but not more than $\frac{1}{3}$ of the wall thickness.

The minimum steel area in the wall footing (heel or toe), according to the ACI Code, Section 10.5.3, is that required for shrinkage and temperature reinforcement, which is $0.0018bh$ when $f_y = 60$ ksi and $0.0020bh$ when $f_y = 40$ ksi or 50 ksi. Because this minimum steel area is relatively small, it is a common practice to increase it to that minimum A_s required for flexure:

$$A_{s\,\min} = \left(\frac{3\sqrt{f'_c}}{f_y}\right) bd \geq \left(\frac{200}{f_y}\right) bd \tag{14.16}$$

14.10 DRAINAGE

The earth pressure discussed in the previous sections does not include any hydrostatic pressure. If water accumulates behind the retaining wall, the water pressure must be included in the design. Surface or underground water may seep into the backfill and develop the case of submerged soil. To avoid hydrostatic pressure, drainage should be provided behind the wall. If well-drained cohesionless soil is used as a backfill, the wall can be designed for earth pressure only. The drainage system may consist of one or a combination of the following:

1. Weep holes in the retaining wall of 4 in. or more in diameter and spaced about 5 ft on centers horizontally and vertically (Fig. 14.9*a*).
2. Perforated pipe 8 in. in diameter laid along the base of the wall and surrounded by gravel (Fig. 14.9*b*).
3. Blanketing or paving the surface of the backfill with asphalt to prevent seepage of water from the surface.
4. Any other method to drain surface water.

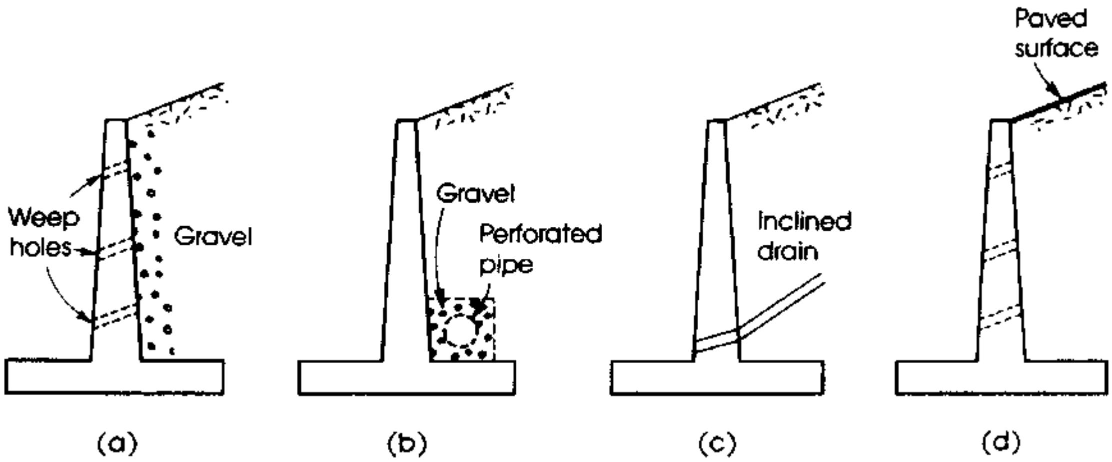

Figure 14.9 Drainage systems.

Example 14.1

The trial section of a semigravity plain concrete retaining wall is shown in Fig. 14.10. It is required to check the safety of the wall against overturning, sliding, and bearing pressure under the footing. Given: Weight of backfill is 110 pcf, angle of internal friction is $\phi = 35°$, coefficient of friction between concrete and soil is $\mu = 0.5$, allowable soil pressure is 2.5 ksf, and $f'_c = 3$ ksi.

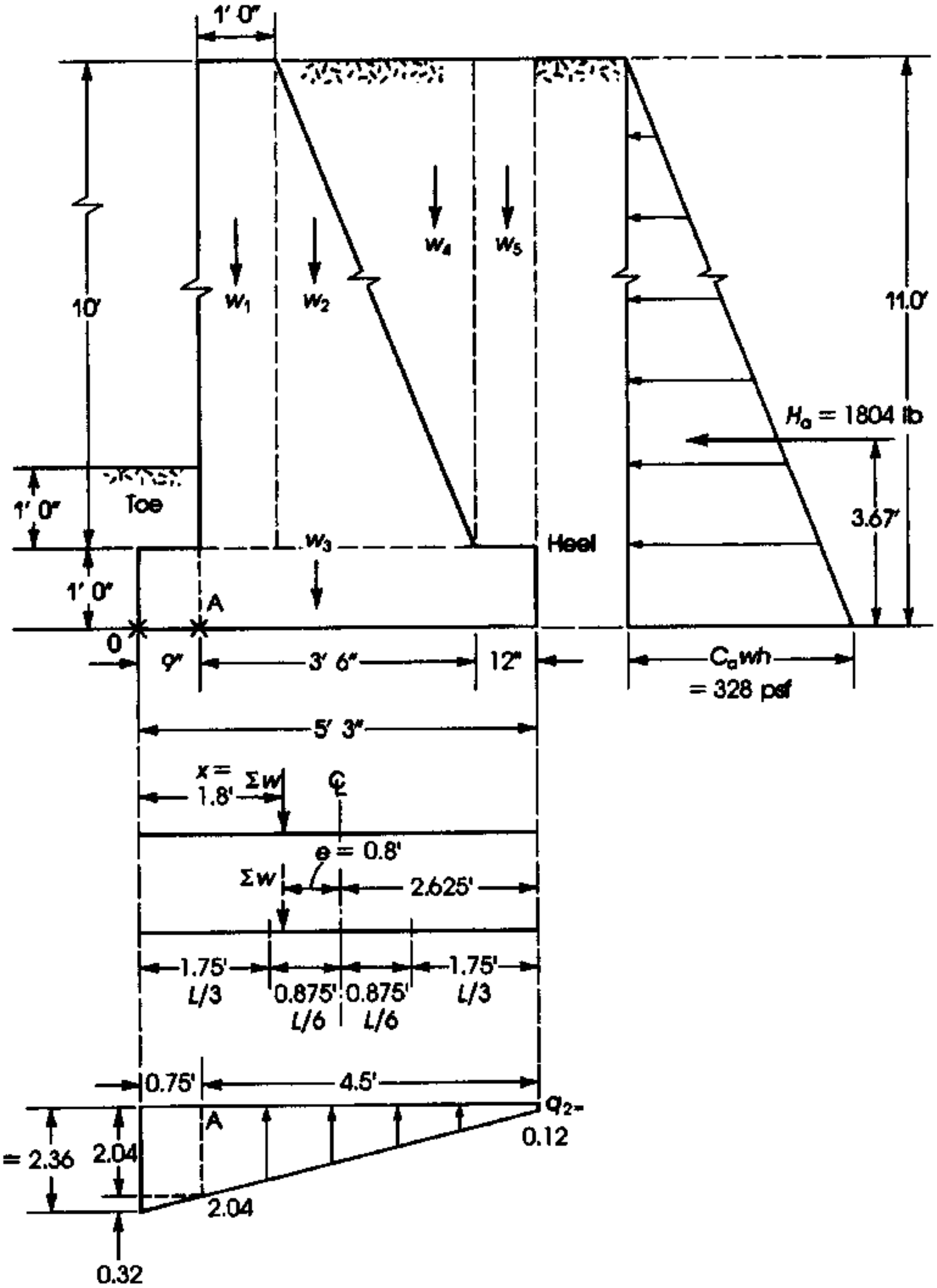

Figure 14.10 Example 14.1.

Solution

1. Using the Rankine equation,

$$C_a = \frac{1 - \sin\phi}{1 + \sin\phi} = \frac{1 - 0.574}{1 + 0.574} = 0.271$$

The passive pressure on the toe is that for a height of 1 ft, which is small and can be neglected.

$$H_a = \frac{C_a w h^2}{2} = \frac{0.271}{2}(110)(11)^2 = 1804 \text{ lb}$$

H_a acts at a distance $h/3 = \frac{11}{3} = 3.67$ ft from the bottom of the base.

2. The overturning moment is $M_O = 1.804 \times 3.67 = 6.62$ K·ft.
3. Calculate the balancing moment, M_b, taken about the toe end 0 (Fig. 14.10):

Weight (lb)	Arm (ft)	Moment (K·ft)
$w_1 = 1 \times 10 \times 145 = 1450$	1.25	1.81
$w_2 = \frac{1}{2} \times 2.5 \times 10 \times 145 = 1812$	2.60	4.71
$w_3 = 5.25 \times 1 \times 145 = 725$	2.625	2.00
$w_4 = \frac{1}{2} \times 2.5 \times 10 \times 110 = 1375$	3.42	4.70
$w_5 = \frac{12}{12} \times 10 \times 110 = 1100$	4.75	5.22

$$\sum w = R = 6.50 \text{ K} \qquad M_b = \sum M = 18.44 \text{ K·ft}$$

4. The factor of safety against overturning is $18.44/6.50 = 2.82 > 2.0$.
5. The force resisting sliding, $F = \mu R$, is $F = 0.5(6.50) = 3.25$ K. The factor of safety against sliding is $F/H_a = 3.25/1.804 = 1.8 > 1.5$.
6. Calculate the soil pressure under the base:
   a. The distance of the resultant from toe end 0 is

$$x = \frac{M_b - M_O}{R} = \frac{18.44 - 6.62}{6.50} = 1.82 \text{ ft}$$

The eccentricity is $e = 2.62 - 1.82 = 0.80$ ft. The resultant R acts just inside the middle third of the base and has an eccentricity of $e = 0.8$ ft from the center of the base (Fig. 14.10). For a 1-ft length of the footing, the effective length of footing is 5.25 ft.

   b. The moment of inertia is $I = 1.0(5.25)^3/12 = 12.1$ ft^4. Area $= 5.25$ ft^2.
   c. The soil pressures at the two extreme ends of the footing are $q_1, q_2 = R/A \pm Mc/I$. The moment M is $Re = 6.50(0.8) = 5.2$ K·ft; $c = 2.62$ ft.

$$q_1 = \frac{6.50}{5.25} + \frac{5.2(2.62)}{12.1} = 1.24 + 1.12 = 2.36 \text{ ksf}$$

$$q_2 = 1.24 - 1.12 = 0.12 \text{ ksf}$$

7. Check the bending stress in concrete at point A of the toe.
   a. Soil pressure at A (from geometry) is

$$q_A = 0.12 + \left(\frac{4.5}{5.25}\right)(2.36 - 0.12) = 2.04 \text{ ksf}$$

   b. M_A is calculated at A due to a rectangular stress and a triangular stress.

$$M_A = \frac{2.04(0.75)^2}{2} + (0.32 \times 0.75 \times 0.5)\left(0.75 \times \frac{2}{3}\right)$$

$$= 0.63 \text{ K·ft}$$

c. The flexural stress in concrete is

$$Mc/I = 0.63(12{,}000)(6)/1728 = 26 \text{ psi}$$

where $c = h/2 = 12/2 = 6$ in. and $I = 12(12)^3/12 = 1728$ in.^{4}

d. The modulus of rupture of concrete is $7.5\lambda\sqrt{f_c'} = 410$ psi > 26 psi. The factor of safety against cracking is 410/26 = 16. Therefore, the section is adequate. No other sections need to be checked.

Example 14.2

Design a cantilever retaining wall to support a bank of earth 16.5 ft high. The top of the earth is to be level with a surcharge of 330 psf. Given: The weight of the backfill is 110 pcf, the angle of internal friction is $\phi = 35°$, the coefficient of friction between concrete and soil is $\mu = 0.5$, the coefficient of friction between soil layers is $\mu = 0.7$, allowable soil bearing capacity is 4 ksf, $f_c' = 3$ ksi, and $f_y = 60$ ksi.

Solution

1. Determine the dimensions of the retaining wall using the approximate relationships shown in Fig. 14.8.
   a. Height of wall: Allowing 3 ft for frost penetration, the height of the wall becomes $h = 16.5 + 3 = 19.5$ ft.
   b. Base thickness: Assume base thickness is $0.08h = 0.08 \times 19.5 = 1.56$ ft, or 1.5 ft. The height of the stem is $19.5 - 1.5 = 18$ ft.
   c. Base length: The base length varies between $0.4h$ and $0.67h$. Assuming an average value of $0.53h$, then the base length equals $0.53 \times 19.5 = 10.3$ ft, say, 10.5 ft. The projection of the base in front of the stem varies between $0.17h$ and $0.125h$. Assume a projection of $0.17h = 0.17 \times 19.5 = 3.3$ ft, say, 3.5 ft.
   d. Stem thickness: The maximum stem thickness is at the bottom of the wall and varies between $0.08h$ and $0.1h$. Choose a maximum stem thickness equal to that of the base, or 1.5 ft. Select a practical minimum thickness of the stem at the top of the wall of 1.0 ft. The minimum batter of the face of the wall is $\frac{1}{4}$ in./ft. For an 18-ft-high wall, the minimum batter is $\frac{3}{4} \times 18 = 4.5$in., which is less than the $1.5 - 1.0 = 0.5$ ft (6 in.) provided. The trial dimensions of the wall are shown in Fig. 14.11.
2. Using the Rankine equation:

$$C_a = \frac{1 - \sin\phi}{1 + \sin\phi} = \frac{1 - 0.574}{1 + 0.574} = 0.271$$

3. The factor of safety against overturning can be determined as follows:
   a. Calculate the actual unfactored forces acting on the retaining wall. First, find those acting to overturn the wall:

$$h_s(\text{due to surcharge}) = \frac{w_s}{w} = \frac{330}{110} = 3 \text{ ft}$$

$$p_1 = C_a w h_s = 0.271 \times (110 \times 3) = 90 \text{ psf}$$

$$p_2 = C_a w h = 0.271 \times (110 \times 19.5) = 581 \text{ psf}$$

$$H_{a_1} = 90 \times 19.5 = 1755 \text{ lb} \qquad \text{arm} = \frac{19.5}{2} = 9.75 \text{ ft}$$

$$H_{a_2} = \frac{1}{2} \times 581 \times 19.5 = 5665 \text{ lb} \qquad \text{arm} = \frac{19.5}{3} = 6.5 \text{ ft}$$

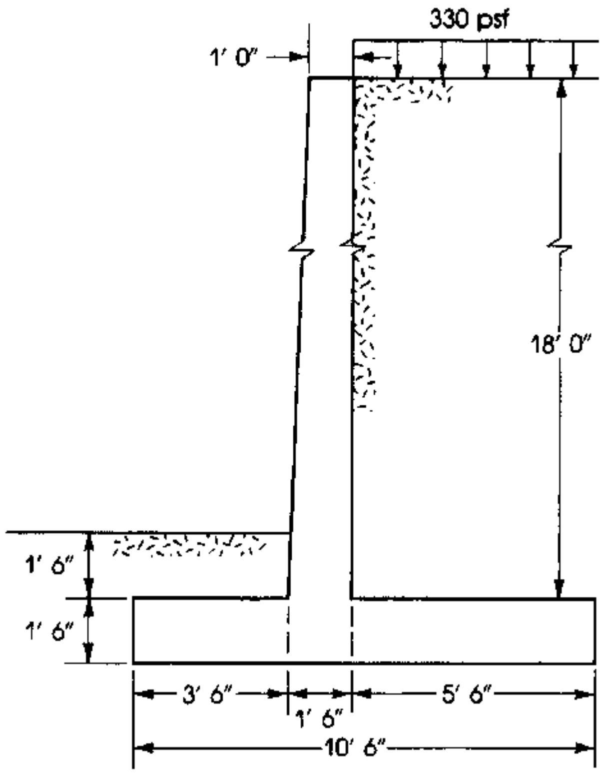

Figure 14.11 Example 14.2: trial configuration of retaining wall.

b. The overturning moment is $1.755 \times 9.75 + 5.665 \times 6.5 = 53.93$ K·ft.

c. Calculate the balancing moment against overturning (see Fig. 14.12):

Force (lb)	Arm (ft)	Moment (K·ft)
$w_1 = 1 \times 18 \times 150 = 2{,}700$	4.50	12.15
$w_2 = \frac{1}{2} \times 18 \times \frac{1}{2} \times 150 = 675$	3.83	2.59
$w_3 = 10.5 \times 1.5 \times 150 = 2{,}363$	5.25	12.41
$w_4 = 5.5 \times 21 \times 110 = 12{,}705$	7.75	98.46

$$\sum w = R = 18.44 \text{ K} \qquad \sum M = 125.61 \text{ K·ft}$$

$$\text{Factor of safety against overturning} = \frac{125.61}{53.93} = 2.33 > 2.0$$

4. Calculate the base soil pressure. Take moments about the toe end 0 (Fig. 14.12) to determine the location of the resultant R of the vertical forces.

$$x = \frac{\sum M - \sum Hy}{R} = \frac{\text{balancing } M - \text{overturning } M}{R}$$

$$= \frac{125.61 - 53.93}{18.44} = 3.89 \text{ ft} > \frac{10.5}{3} \quad \text{or} \quad 3.5 \text{ ft}$$

The eccentricity is $e = 10.5/2 - 3.89 = 1.36$ ft. The resultant R acts within the middle third of the base and has an eccentricity of $e = 1.36$ ft from the center of the base. For a 1-ft length

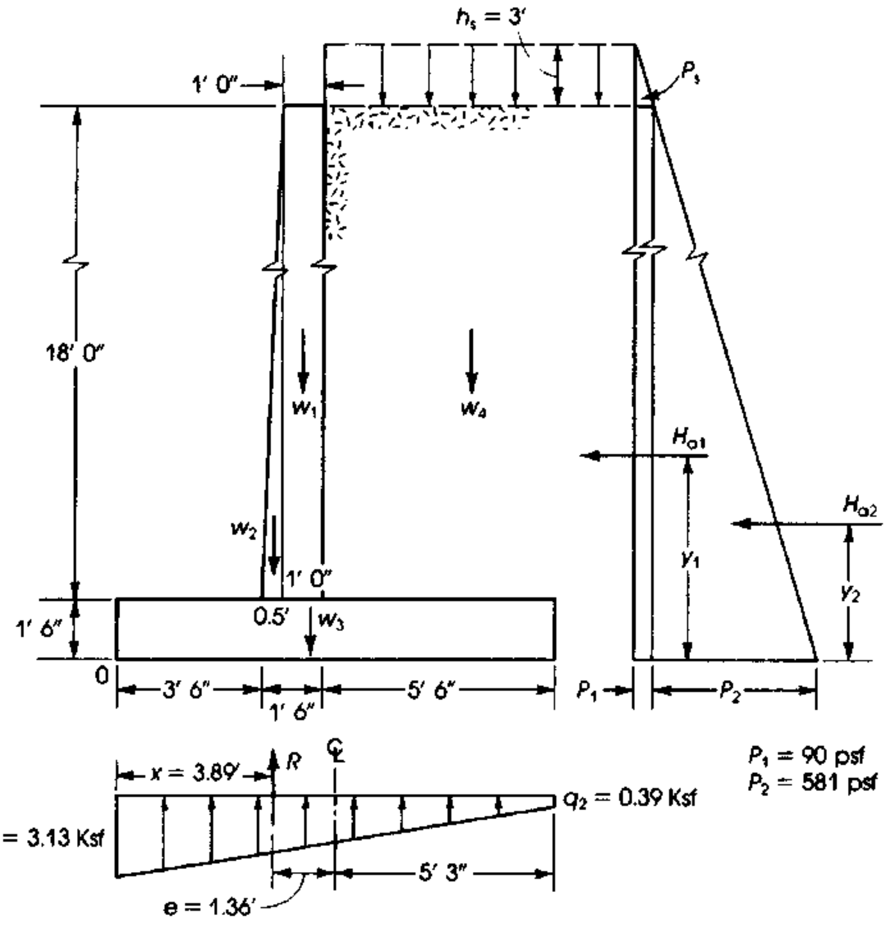

Figure 14.12 Example 14.2: forces acting on retaining wall.

of the footing, area = $10.5 \times 1 = 10.5\,\text{ft}^2$.

$$I = 1 \times \frac{(10.5)^3}{12} = 96.47\ \text{ft}^4$$

$$q_1 = \frac{R}{A} + \frac{(Re)C}{I} = \frac{18.44}{10.5} + \frac{(18.44 \times 1.36) \times 5.25}{96.47}$$

$$= 1.76 + 1.37 = 3.13\ \text{ksf} < 4\ \text{ksf}$$

$$q_2 = 1.76 - 1.37 = 0.39\ \text{ksf}$$

Soil pressure is adequate.

5. Calculate the factor of safety against sliding. A minimum factor of safety of 1.5 must be maintained.

$$\text{Force causing sliding} = H_{a1} + H_{a2} = 1.76 + 5.67 = 7.43\ \text{K}$$

$$\text{Resisting force} = \mu R = 0.5 \times 18.44 = 9.22\ \text{K}$$

$$\text{Factor of safety against sliding} = \frac{9.22}{7.43} = 1.24 < 1.5$$

The resistance provided does not give an adequate safety against sliding. In this case, a key should be provided to develop a passive pressure large enough to resist the excess force that causes sliding. Another function of the key is to provide sufficient development length for the dowels of the stem. The key is therefore placed such that its face is about 6 in. from the back face of the stem (Fig. 14.13). In the calculation of the passive pressure, the top foot of the earth at the toe side is usually neglected, leaving a height of 2 ft in this example. Assume a key depth of $t = 1.5$ ft and a width of $b = 1.5$ ft.

$$C_p = \frac{1 + \sin\phi}{1 - \sin\phi} = \frac{1}{C_a} = \frac{1}{0.271} = 3.69$$

$$H_p = \frac{1}{2}C_p w(h' + t)^2 = \frac{1}{2} \times 3.69 \times 110(2 + 1.5)^2 = 2486 \text{ lb}$$

The sliding may occur now on the surfaces AC, CD, and EF (Fig. 14.13). The sliding surface AC lies within the soil layers with a coefficient of internal friction = tan ϕ = tan 35° = 0.7, whereas the surfaces CD and EF are those between concrete and soil with a coefficient of internal friction of 0.5, as given in this example. The frictional resistance is $F = \mu_1 R_1 + \mu_2 R_2$.

$$R_1 = \text{reaction of } AC = \left(\frac{3.13 + 1.96}{2}\right) \times 4.5 = 11.44 \text{ K}$$

$$R_2 = R - R_1 = 18.44 - 11.44 = 7.0 \text{ K}$$

$$R_2 = \text{reaction of } CDF = \left(\frac{1.96 + 0.39}{2}\right) \times 6 = 7.05 \text{ K}$$

$$F = 0.7(11.44) + 0.5(7.00) = 11.50 \text{ K}$$

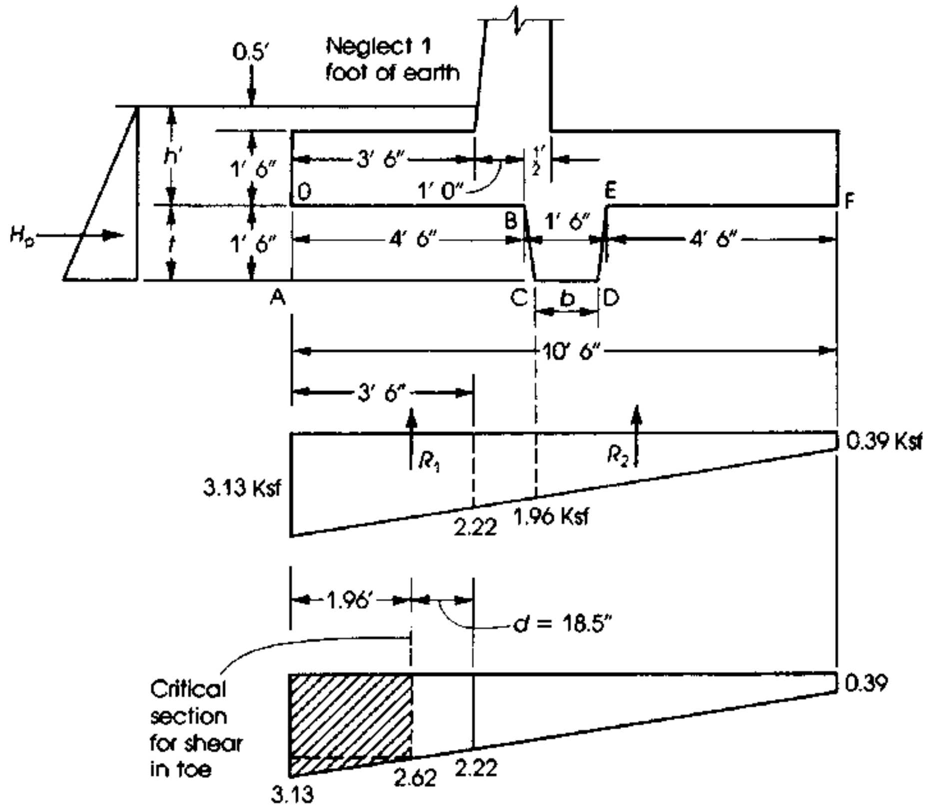

Figure 14.13 Example 14.2: footing details.

The total resisting force is

$$F + H_p = 11.50 + 2.49 = 13.99 \text{ K}$$

The factor of safety against sliding is

$$\frac{13.99}{7.43} = 1.9 \quad \text{or} \quad \frac{11.5}{7.43} = 1.55 > 1.5$$

The factor is greater than 1.5, which is recommended when passive resistance against sliding is not included.

6. Design the wall (stem). The design of the different reinforced concrete structural elements can be performed now using the ACI Code alternative design method, Appendix B and C.

a. Main reinforcement: The lateral forces applied to the wall are calculated using a load factor of 1.6. The critical section for bending moment is at the bottom of the wall, height = 18 ft. Calculate the applied ultimate forces:

$$P_1 = 1.6(C_a w h_s) = 1.6(0.271 \times 110 \times 3) = 143 \text{ lb}$$

$$P_2 = 1.6(C_a w h) = 1.6(0.271 \times 110 \times 18) = 858.3 \text{ lb}$$

$$H_{a1} = 0.143 \times 18 = 2.57 \text{ K} \quad \text{arm} = \tfrac{18}{2} = 9 \text{ ft}$$

$$H_{a2} = \tfrac{1}{2} \times 0.858 \times 18 = 7.72 \text{ K} \quad \text{arm} = \tfrac{18}{3} = 6 \text{ ft}$$

$$M_u(\text{at bottom of wall}) = 2.57 \times 9 + 7.72 \times 6 = 69.45 \text{ K·ft}$$

The total depth used is 18 in., $b = 12$ in., and $d = 18 - 2$ (concrete cover) $- 0.5$ (half the bar diameter) $= 15.5$ in.

$$R_u = \frac{M_u}{bd^2} = \frac{69.45 \times 12{,}000}{12(15.5)^2} = 289 \text{ psi}$$

The steel ratio, ρ, can be obtained from Table A.1 in Appendix A or from

$$\rho = \frac{0.85 f_c'}{f_y}\left[1 - \sqrt{\frac{2R_u}{\phi 0.85 f_c'}}\right] = 0.007$$

$$A_s = 0.007(12)(15.5) = 1.3 \text{ in.}^2$$

Use no. 8 bars spaced at 7 in. (1.35 in.^{2}). The minimum vertical A_s according to the ACI Code, Section 14.3, is

$$A_{s\,\min} = 0.0015(12)(18) = 0.32 \text{ in.}^2 < 1.35 \text{ in.}^2$$

Because the moment decreases along the height of the wall, A_s may be reduced according to the moment requirements. It is practical to use one A_s or spacing, for the lower half and a second A_s, or spacing, for the upper half of the wall. To calculate the moment at midheight of the wall, 9 ft from the top,

$$P_1 = 1.6(0.271 \times 110 \times 3) = 143 \text{ lb}$$

$$P_2 = 1.6(0.271 \times 110 \times 9) = 429 \text{ lb}$$

$$H_{a_1} = 0.143 \times 9 = 1.29 \text{ K} \qquad \text{arm} = \frac{9}{2} = 4.5 \text{ ft}$$

$$H_{a_2} = \frac{1}{2} \times 0.429 \times 9 = 1.9 \text{ K} \qquad \text{arm} = \frac{9}{3} = 3 \text{ ft}$$

$$M_u = 1.29 \times 4.5 + 1.9 \times 3 = 11.5 \text{ K·ft}$$

The total depth at midheight of wall is

$$\frac{12+18}{2} = 15 \text{ in.}$$

$$d = 15 - 2 - 0.5 = 12.5 \text{ in.}$$

$$R_u = \frac{M_u}{bd^2} = \frac{11.5 \times 12{,}000}{12 \times (12.5)^2} = 73.6 \text{ psi}$$

$$\rho = 0.0017 \quad \text{and} \quad A_s = 0.0017(12)(12.5) = 0.25 \text{ in.}^2$$

$$A_{s\ \min} = 0.0015 \times 12 \times 15 = 0.27 \text{ in.}^2 > 0.25 \text{ in.}^2$$

Use no. 4 vertical bars spaced at 8 in. (0.29 in.^{2}) with similar spacing to the lower vertical steel bars in the wall.

b. Temperature and shrinkage reinforcement: The minimum horizontal reinforcement at the base of the wall according to ACI Code, Section 14.3, is

$$A_{s\ \min} = 0.0020 \times 12 \times 18 = 0.432 \text{ in.}^2$$

(for the bottom third), assuming no. 5 bars or smaller.

$$A_{s\ \min} = 0.0020 \times 12 \times 15 = 0.36 \text{ in.}^2$$

(for the upper two-thirds). Because the front face of the wall is mostly exposed to temperature changes, use one-half to two-thirds of the horizontal bars at the external face of the wall and place the balance at the internal face.

$$0.5A_s = 0.5 \times 0.432 = 0.22 \text{ in.}^2$$

Use no. 4 horizontal bars spaced at 8 in. (A_s = 0.29 in.^{2}) at both the internal and external surfaces of the wall. Use no. 4 vertical bars spaced at 12 in. at the front face of the wall to support the horizontal temperature and shrinkage reinforcement.

c. Dowels for the wall vertical bars: The anchorage length of no. 8 bars into the footing must be at least 22 in. Use an embedment length of 2 ft into the footing and the key below the stem.

d. Design for shear: The critical section for shear is at a distance d = 15.5 in. from the bottom of the stem. At this section, the distance from the top equals 18 − 15.5/12 = 16.7 ft.

$$P_1 = 143 \text{ lb} P_1 = 1.6(0.271 \times 110 \times 3) = 143 \text{ lb}$$

$$P_2 = 1.6(0.271 \times 110 \times 16.7) = 796 \text{ lb}$$

$$H_{a_1} = 0.143 \times 16.7 = 2.39 \text{ K}$$

$$H_{a_2} = \frac{1}{2} \times 0.796 \times 16.7 = 6.6 \text{ K}$$

$$\text{Total } H = 2.39 + 6.6 = 9.0 \text{ K}$$

$$\phi V_c = \phi(2\lambda\sqrt{f'_c})bd = \frac{0.75 \times 2 \times 1}{1000} \times \sqrt{3000} \times 12 \times 15.5$$

$$= 15.28 \text{ K} > 9.0\text{K}$$

7. Design of the heel: A load factor of 1.2 is used to calculate the factored bending moment and shearing force due to the backfill and concrete, whereas a load factor of 1.6 is used for the surcharge. The upward soil pressure is neglected, because it will reduce the effect of the backfill and concrete on the heel. Referring to Fig. 14.12, the total load on the heel is

$$V_u = [1.2[(18 \times 5.5 \times 110) + (1.5 \times 5.5 \times 150)] + 1.6(3 \times 5.5 \times 100)]/1000 = 17.5 \text{ K}$$

$$M_u(\text{at face of wall}) = V_u \times \frac{5.5}{2} = 48.1 \text{ K-ft}$$

The critical section for shear is usually at a distance d from the face of the wall when the reaction introduces compression into the end region of the member. In this case, the critical section will be considered at the face of the wall, because tension and not compression develops in the concrete.

$$V_u = 17.2 \text{ K}$$

$$\phi V_c = \phi(2\lambda\sqrt{f_c'})bd = \frac{0.75 \times 2 \times 1}{1000} \times \sqrt{3000} \times 12 \times 14.5 = 14.3 \text{ K}$$

ϕV_c is less than V_u of 17.2 K, and the section must be increased by the ratio 17.5/14.3 or shear reinforcement must be provided.

$$\text{Required } d = \frac{17.2}{14.3} \times 14.5 = 17.4 \text{ in.}$$

$$\text{Total thickness required} = 17.4 + 3.5 = 20.9 \text{ in.}$$

Use a base thickness of 22 in. and $d = 18.5$ in.

$$R_u = \frac{M_u}{bd^2} = \frac{48.1 \times 12{,}000}{12 \times (18.5)^2} = 140.5 \text{ psi} \quad \rho = 0.0027$$

$$A_s = \rho bd = 0.60 \text{ in}^2$$

$$\text{Min. shrinkage } A_s = 0.0018(12)(22) = 0.475 \text{ in.}^2$$

$$\text{Min. flexural } A_s = 0.0033(12)(18.5) = 0.733 \text{ in.}^2$$

Use no. 6 bars spaced at 7 in. ($A_s = 0.76 \text{ in.}^2$). The development length for the no. 6 top bars equals $1.4l_d = 35$ in. Therefore, the bars must be extended 3 ft into the toe of the base.

Temperature and shrinkage reinforcement in the longitudinal direction is not needed in the heel or toe, but it may be preferable to use minimal amounts of reinforcement in that direction, say, no. 4 bars spaced at 12 in.

8. Design of the toe: The toe of the base acts as a cantilever beam subjected to upward pressures, as calculated in step 4. The factored soil pressure is obtained by multiplying the service load soil pressure by a load factor of 1.6, because it is primarily caused by the lateral forces. The critical section for the bending moment is at the front face of the stem. The critical section for shear is at a distance d from the front face of the stem, because the reaction in the direction of shear introduces compression into the toe.

Referring to Fig. 14.13, the toe is subjected to an upward pressure from the soil and downward pressure due to self-weight of the toe slab.

$$V_u = 1.6\left(\frac{3.13 + 2.62}{2}\right) \times 1.96 - 1.2\left(\frac{22}{12} \times 0.150\right) \times 1.96 = 837 \text{ K}$$

This is less than $\phi\ V_c$ of 14.3 K calculated for the heel in step 7.

$$M_u = 1.6\left[\frac{2.22}{2} \times (3.5)^2 + (3.13 - 2.22) \times 3.5 \times 0.5\left(\frac{2}{3} \times 3.5\right)\right]$$

$$- 1.2\left[\left(\frac{22}{12} \times 0.150\right) \times \frac{(3.5)^2}{2}\right] = 25.7 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{25.7 \times 12{,}000}{12 \times (18.5)^2} = 75 \text{ psi} \qquad \rho = 0.0017$$

$$A_s = 0.0017(12)(18.5) = 0.377 \text{ in.}^2$$

$$\text{Min. shrinkage } A_s = 0.0018(12)(22) = 0.475 \text{ in.}^2$$

$$\text{Min. flexural } A_s = 0.0033(12)(18.5) = 0.733 \text{ in.}^2$$

Use no. 6 bars spaced at 7 in., similar to the heel reinforcement. Development length of no. 6 bars equals 25 in. Extend the bars into the heel 25 in. The final reinforcement details are shown in Fig. 14.14.

9. Shear keyway between wall and footing: In the construction of retaining walls, the footing is cast first and then the wall is cast on top of the footing at a later date. A construction joint

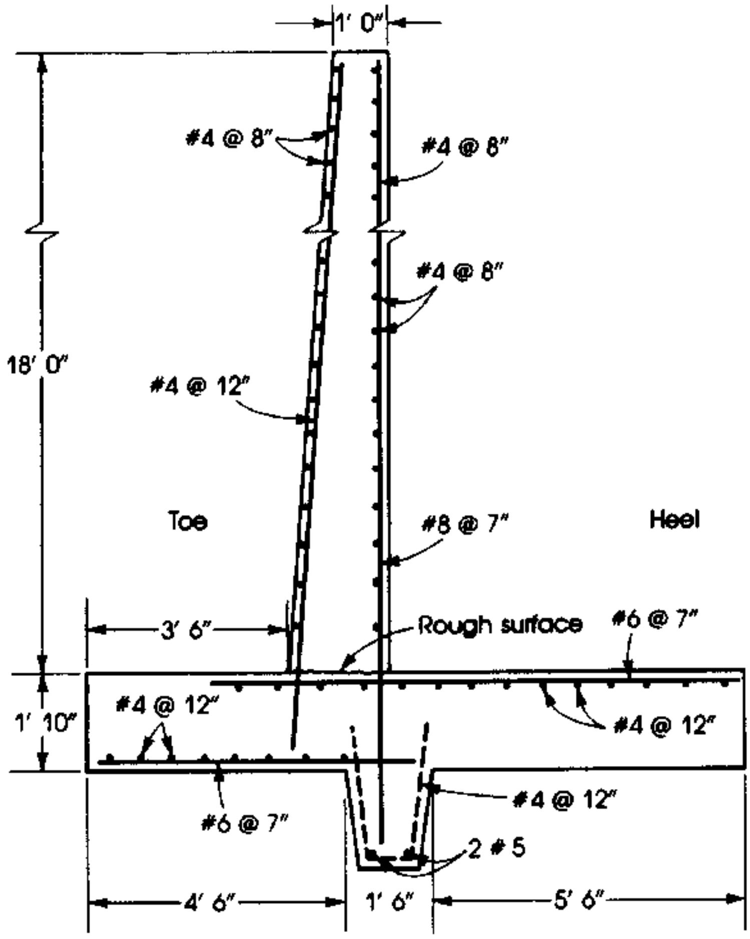

Figure 14.14 Example 14.2: reinforcement details.

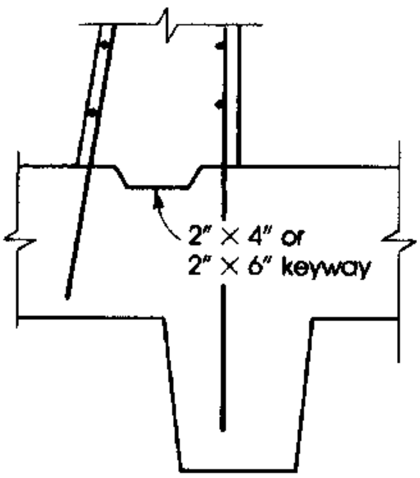

Figure 14.15 Example 14.2: keyway details.

is used at the base of the wall. The joint surface takes the form of a keyway, as shown in Fig. 14.15, or is left in a very rough condition (Fig. 14.14). The joint must be capable of transmitting the stem shear into the footing.

10. Proper drainage of the backfill is essential in this design, because the earth pressure used is for drained backfill. Weep holes should be provided in the wall, 4 in. in diameter and spaced at 5 ft in the horizontal and vertical directions.

14.11 BASEMENT WALLS

It is a common practice to assume that basement walls span vertically between the basement-floor slab and the first-floor slab. Two possible cases of design should be investigated for a basement wall.

First, when the wall only has been built on top of the basement floor slab, the wall will be subjected to lateral earth pressure with no vertical loads except its own weight. The wall in this case acts as a cantilever, and adequate reinforcement should be provided for a cantilever wall design. This case can be avoided by installing the basement and the first-floor slabs before backfilling against the wall.

Second, when the first-floor and the other floor slabs have been constructed and the building is fully loaded, the wall in this case will be designed as a propped cantilever wall subjected to earth pressure and to vertical load.

For an angle of internal friction of 35°, the coefficient of active pressure is $C_a = 0.271$. The horizontal earth pressure at the base is $p_a = C_a wh$. For w = 110 pcf and an average height of a basement of $h = 10$ ft, then

$$P_a = 0.271 \times 0.110 \times 10 = 0.3 \text{ ksf}$$

$$H_a = 0.271 \times 0.110 \times \frac{100}{2} = 1.49 \text{ K/ft of wall}$$

H_a acts at $h/3 = 10/3 = 3.33$ ft from the base. An additional pressure must be added to allow for a surcharge of about 200 psf on the ground behind the wall. The equivalent height to the surcharge is

$$h_s = \frac{200}{110} = 1.82 \text{ ft}$$
$$\rho_s = C_a w h_s = 0.271 \times 0.110 \times 1.82 = 0.054 \text{ ksf}$$
$$H_s = (C_a w h_s) \times h = 0.054 \times 10 = 0.54 \text{ K/ft of wall}$$

H_s of the surcharge acts at $h/2 = 5 ft$ from the base.

In the preceding calculations, it is assumed that the backfill is dry, but it is necessary to investigate the presence of water pressure behind the wall. The maximum water pressure occurs when the whole height of the basement wall is subjected to water pressure, and $\rho_w = wh = 62.5 \times 10 = 625$ psf.

$$H_w = \frac{wh^2}{2} = 0.625 \times 5 = 3.125 \text{ K/ft of wall}$$

The maximum pressure may not be present continuously behind the wall. Therefore, if the ground is intermittently wet, a percentage of the preceding pressure may be adopted, say, 50%:

$$\frac{P_w}{2} = \frac{0.625}{2} = 0.31 \text{ ksf}$$
$$H'_w = \frac{H_w}{2} = (0.5wh)\frac{h}{2} = \frac{3.125}{2} = 1.56 \text{ K/ft of wall}$$

H'_w acts at $h/3 = \frac{10}{3} = 3.33$ ft from the base. Water may be prevented from collecting against the wall by providing drains at the lower end of the wall.

In addition to drainage, a waterproofing or damp-proofing membrane must be laid or applied to the external face of the wall. The ACI Code, Section 14.5.3, specifies that the minimum thickness of an exterior basement wall and its foundation is 7.5 in. In general, the minimum thickness of bearing walls is $\frac{1}{25}$ of the supported height or length, whichever is shorter, or 4 in.

Example 14.3

Determine the thickness and necessary reinforcement for the basement retaining wall shown in Fig. 14.16. Given: Weight of backfill = 110 pcf, angle of internal friction = 35°, $f'_c = 3$ ksi, and $f_y = 60$ ksi.

Solution

1. The wall spans vertically and will be considered as fixed at the bottom end and propped at the top. Consider a span of $L = 9.75$ ft, as shown in Fig. 14.16. For these data, the different lateral pressures on a 1-ft length of the wall are as follows: Due to active soil pressure, $p_a = 0.3$ ksf and $H_a = 1.49$ K. Due to water pressure, $p_w = 0.31$ ksf and $H_w = 1.56$ K. Due to surcharge, $p_s = 0.054$ ksf and $H_s = 0.54$ K. H_a and H_w are due to triangular loadings, whereas H_S is due to uniform loading. Referring to Fig. 14.16 and using moment coefficients of a propped beam

Basement Wall

subjected to triangular and uniform loads, and a load factor = 1.6 (ACI Code, Appendix C)

$$M_u = 1.6(H_a + H_w)\frac{L}{7.5} + 1.6H_s\frac{L}{8}$$

$$= 1.6\left(\frac{3.05}{7.5} \times 9.75 + 0.54 \times \frac{9.75}{8}\right) = 7.41 \text{ K·ft}$$

$$R_B = 1.6\left(\frac{3.05}{3} + \frac{0.54}{2}\right) - \frac{7.41}{9.75} = 1.3 \text{ K}$$

$$R_A = 4.45 \text{ K}$$

Maximum positive bending moment within the span occurs at the section of 0 shear.

$$V_u = 1.3 - 1.6(0.054x) - 1.6\left(0.063\frac{x^2}{2}\right) = 0$$

$$x = 4.3 \text{ ft}$$

$$M_c = 1.3 \times 4.3 - 1.6\left[\frac{0.054}{2}(4.3)^2 + \frac{0.27}{2}\frac{(4.3)^2}{3}\right]$$

$$= +3.45 \text{ K·ft}$$

2. Assuming 0.01 steel ratio and $R_u = 332$ psi,

$$d = \sqrt{\frac{M_u}{R_u b}} = \sqrt{\frac{7.41 \times 12}{0.332 \times 12}} = 4.72 \text{ in.}$$

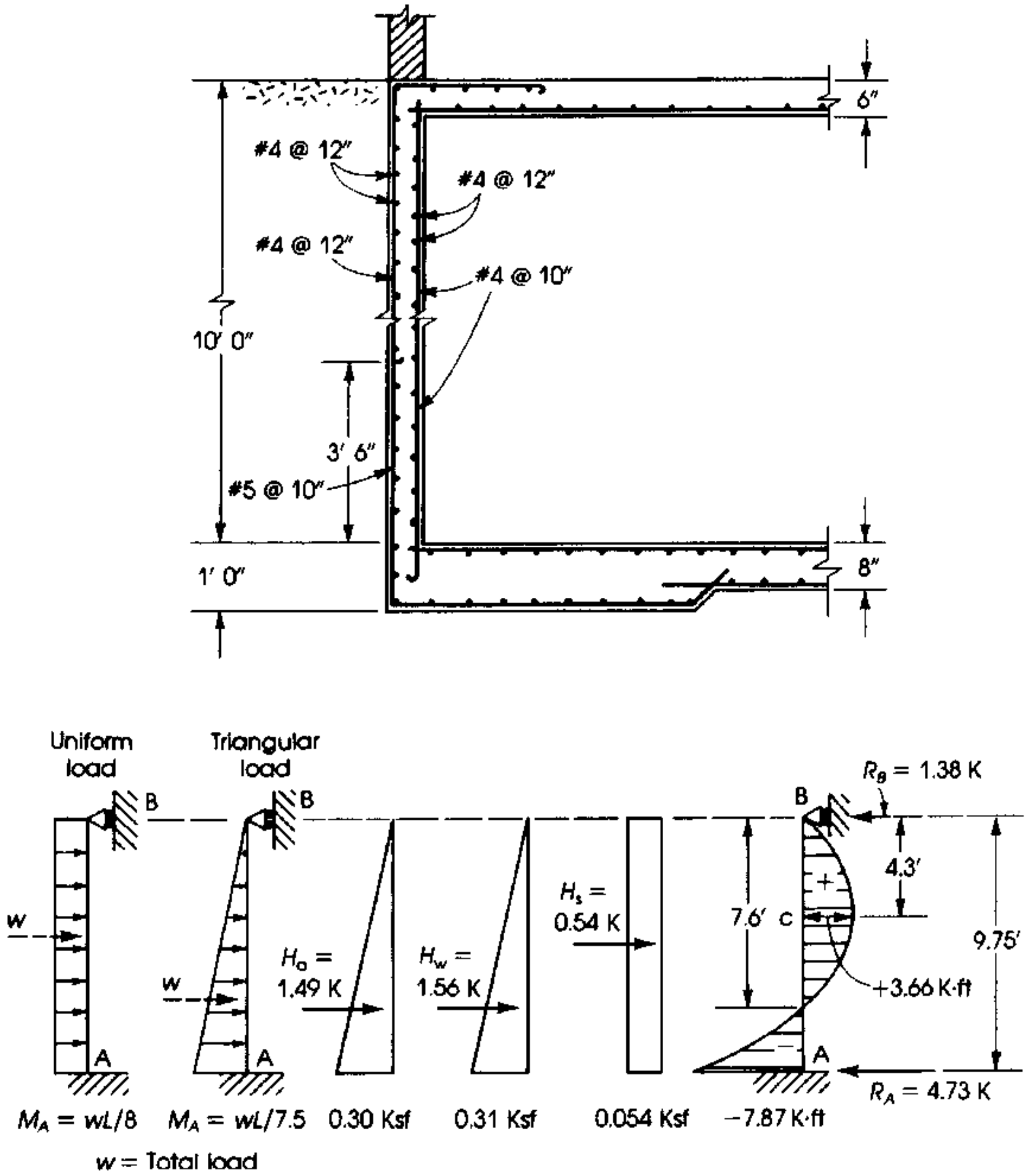

Figure 14.16 Example 14.3: basement wall.

Total depth = 4.72 + 1.5 (concrete cover) + 0.25 = 6.47 in. Use a $7\frac{1}{2}$ in. slab. $d = 5.75$ in.

$$R_u = \frac{M_u}{bd^2} = \frac{7.41 \times 12{,}000}{12 \times (5.75)^2} = 226 \text{ psi}$$

The steel ratio is $\rho = 0.0054$ and $A_s = 0.0054 \times 12 \times 5.75 = 0.369\,\text{in.}^2$

$$\text{Minimum } A_s = 0.0015\,bh = 0.0015(12)(7.5) = 0.135 \text{ in.}^2 \quad \text{(vertical bars)}$$

$$\text{Minimum } A_s(\text{flexure}) = 0.0033(12)(5.75) = 0.23 \text{ in.}^2$$

Use no. 5 bars spaced at 10 in. ($A_s = 0.37$ in^2).

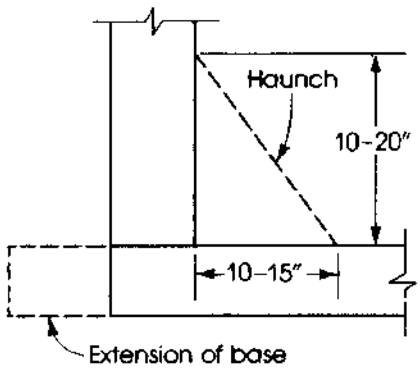

Figure 14.17 Example 14.3: adjustment of wall base.

3. For the positive moment, $M_c = 3.45$ K-ft:

$$R_u = \frac{3.45 \times 12{,}000}{12 \times (5.75)^2} = 105 \text{ psi} \quad \rho = 0.0020$$

$$A_s = 0.002 \times 12 \times 5.75 = 0.14 \text{ in.}^2 < 0.33 \text{ in.}^2$$

Use no. 4 bars spaced at 10 in. ($A_s = 0.24$ in^2).

4. Zero moment occurs at a distance of 7.6 ft from the top and 2.15 ft from the base. The development length of no. 5 bars is 14 in. Therefore, extend the main no. 5 bars to a distance of $2.15 + 1.2 = 3.35$ ft, or 3.5 ft, above the base; then use no. 4 bars spaced at 12 in. at the exterior face. For the interior face, use no. 4 bars spaced at 10 in. throughout.
5. Longitudinal reinforcement: Use a minimum steel ratio of 0.0020 (ACI Code, Section 14.3), or $A_s = 0.0020 \times 7 \times 12 = 0.17$ in^2. Use no. 4 bars spaced at 12 in. on each side of the wall.
6. If the bending moment at the base of the wall is quite high, it may require a thick wall slab, for example, 12 in. or more. In this case a haunch may be adopted, as shown in Fig. 14.17. This solution will reduce the thickness of the wall, because it will be designed for the moment at the section exactly above the haunch.
7. The basement slab may have a thickness greater than the wall thickness and may be extended outside the wall by about 10 in. or more, as required.

SUMMARY

Sections 14.1–14.3

1. A retaining wall maintains unequal levels of earth on its two faces. The most common types of retaining walls are gravity, semigravity, cantilever, counterfort, buttressed, and basement walls.
2. For a linear pressure, the active and passive pressure intensities are

$$P_a = C_a wh \quad \text{and} \quad P_p = C_p wh$$

According to Rankine's theory,

$$C_a = \left(\frac{1 - \sin\phi}{1 + \sin\phi}\right) \quad \text{and} \quad C_p = \left(\frac{1 + \sin\phi}{1 - \sin\phi}\right)$$

Values of C_a and C_p for different values of ϕ and δ are given in Tables 14.2 and 14.3.

Sections 14.4–14.5

1. When soil is saturated, the submerged unit weight must be used to calculate earth pressure. The hydrostatic water pressure must also be considered.
2. A uniform surcharge on a retaining wall causes an additional pressure height, $h_s = w_s/w$.

Sections 14.6–14.8

1. A total frictional force, F, to resist sliding effect is

$$F = \mu R + H_p \tag{14.13}$$

$$\text{Factor of safety against sliding} = \frac{F}{H_{\text{ah}}} \geq 1.5 \tag{14.14}$$

2. Factor of safety against overturing is

$$\frac{M_b}{M_0} = \frac{\sum wx}{H_a h/3} \geq 2.0 \tag{14.15}$$

3. Approximate dimensions of a cantilever retaining wall are shown in Fig. 14.8.

Sections 14.9–14.10

1. Minimum reinforcement is needed in retaining walls.
2. To avoid hydrostatic pressure on a retaining wall, a drainage system should be used that consists of weep holes, perforated pipe, or any other adequate device.
3. Basement walls in buildings may be designed as propped cantilever walls subjected to earth pressure and vertical loads. This case occurs only if the first-floor slab has been constructed. A surcharge of 200 psf may be adopted.

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PROBLEMS

14.1 Check the adequacy of the retaining wall shown in Fig. 14.18 with regard to overturning, sliding, and the allowable soil pressure. Given: Weight of backfill. = 110 pcf, the angle of internal friction is $\phi = 30°$, the coefficient of friction between concrete and soil is $\mu = 0.5$, allowable soil pressure = 3.5 ksf, and $f'_c = 3$ ksi.

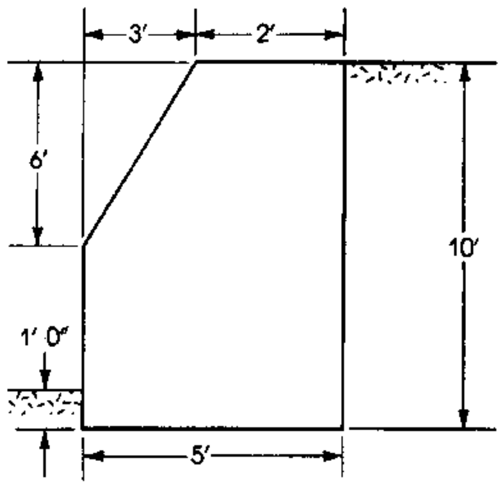


Figure 14.18 Problem 14.1: gravity wall.

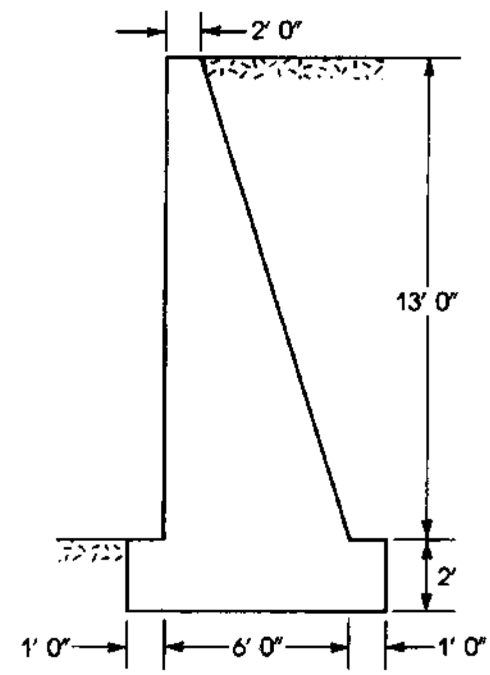


Figure 14.19 Problem 14.2: semigravity wall.

14.2 Repeat Problem 14.1 with Fig. 14.19.

14.3 For each problem in Table 14.4, determine the factor of safety against overturning and sliding. Also, determine the soil pressure under the wall footing and check if all calculated values are adequate (equal or below the allowable values). Given: Weight of soil = 110 pcf, weight of concrete = 150 pcf, and coefficient of friction between concrete and soil is 0.5 and between soil layers is 0.6. Consider that the allowable soil pressure of 4 ksf and the top of the backfill is level without surcharge. Neglect the passive soil resistance. See Fig. 14.20. ($\phi = 35°$.)

14.4 Repeat Problems 14.3*e*–*h*, assuming a surcharge of 300 psf.

Table 14.4 Problem

Problem No.	H	h_f	A	B	C	L
(a)	12	1.00	2.0	1.0	4.0	7
(b)	14	1.50	2.0	1.5	4.5	8
(c)	15	1.50	2.0	1.5	4.5	8
(d)	16	1.50	3.0	1.5	4.5	9
(e)	17	1.50	3.0	1.5	4.5	9
(f)	18	1.75	3.0	1.75	5.25	10
(g)	19	1.75	3.0	1.75	5.25	10
(h)	20	2.00	3.0	2.0	6.0	11
(i)	21	2.00	3.5	2.0	6.5	12
(j)	22	2.00	3.5	2.0	6.5	12

Refer to Fig. 14.20. All dimensions are in feet.

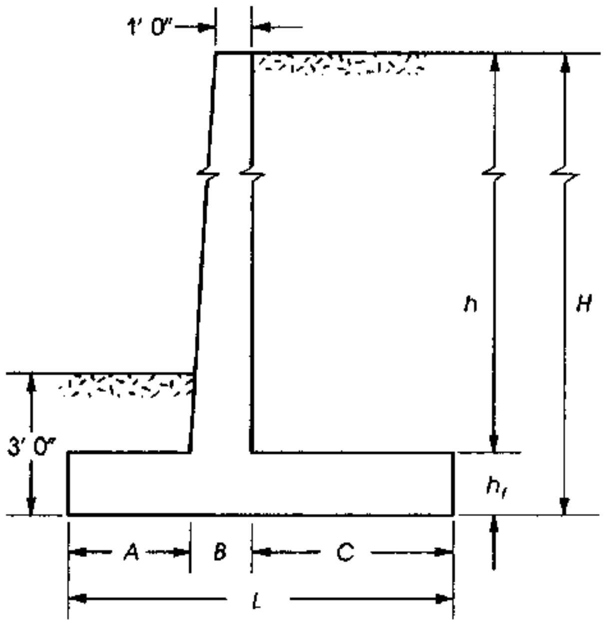


Figure 14.20 Problem 14.3.

14.5 Repeat Problems 14.3*e*–*h*, assuming that the backfill slopes at 10° to the horizontal. (Add key if needed.)

14.6 For Problems 14.3*e*–*h*, determine the reinforcement required for the stem, heel, and toe, and choose adequate bars and distribution. Use $f'_c = 3$ ksi and $f_y = 60$ ksi.

14.7 Determine the dimensions of a cantilever retaining wall to support a bank of earth 16 ft high. Assume that frost penetration depth is 4 ft. Check the safety of the retaining wall against overturning and sliding only. Given: Weight of backfill = 120 pcf, angle of internal friction = 33°, coefficient of friction between concrete and soil = 0.45, coefficient of friction between soil layers = 0.65, and allowable soil pressure = 4 ksf. Use a 1.5 × 1.5-ft key if needed.

14.8 A complete design is required for the retaining wall shown in Fig. 14.21. The top of the backfill is to be level without surcharge. Given: Weight of backfill soil = 110 pcf, angle of internal

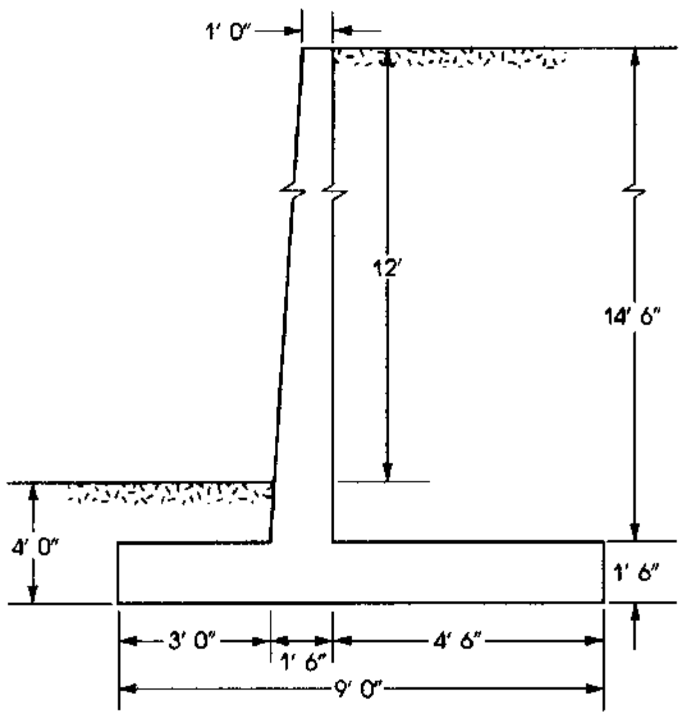


Figure 14.21 Problem 14.8: cantilever retaining wall.

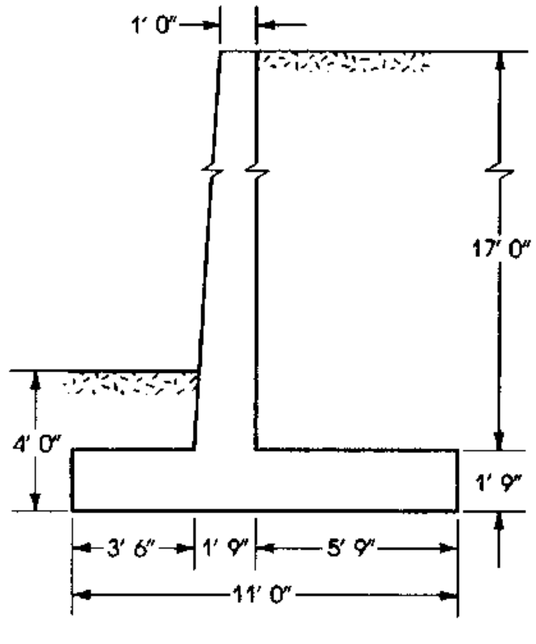


Figure 14.22 Problem 14.9: cantilever retaining wall.

friction = 35°, the coefficient of friction between concrete and soil is 0.55, and that between soil layers is 0.6. Use $f_c' = 3$ ksi, and $f_y = 60$ ksi, and an allowable soil pressure of 4 ksf.

14.9 Check the adequacy of the cantilever retaining wall shown in Fig. 14.22 for both sliding and over-turning conditions. Use a key of 1.5 × 1.5 ft if needed. Then determine reinforcement needed for the stem, heel, and toe, and choose adequate bars and distribution. Given: Weight of soil = 120 pcf,

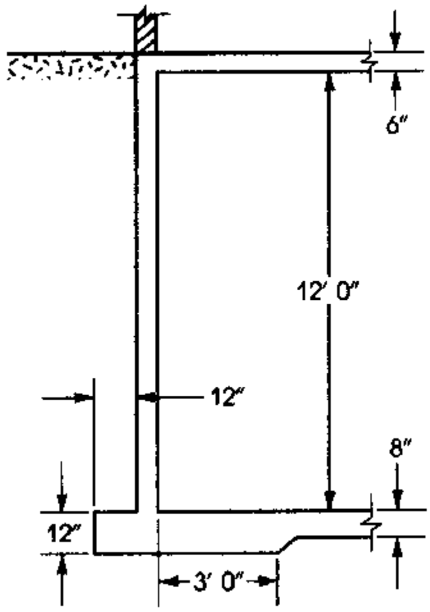


Figure 14.23 Problem 14.11: basement wall.

the angle of internal friction is $\phi = 35^\circ$, the coefficient of friction between concrete and soil is 0.52 and that between soil layers is 0.70. Use $f_c' = 3$ ksi, $f_y = 60$ ksi, an allowable soil pressure of 4 ksf, and a surcharge of 300 psf.

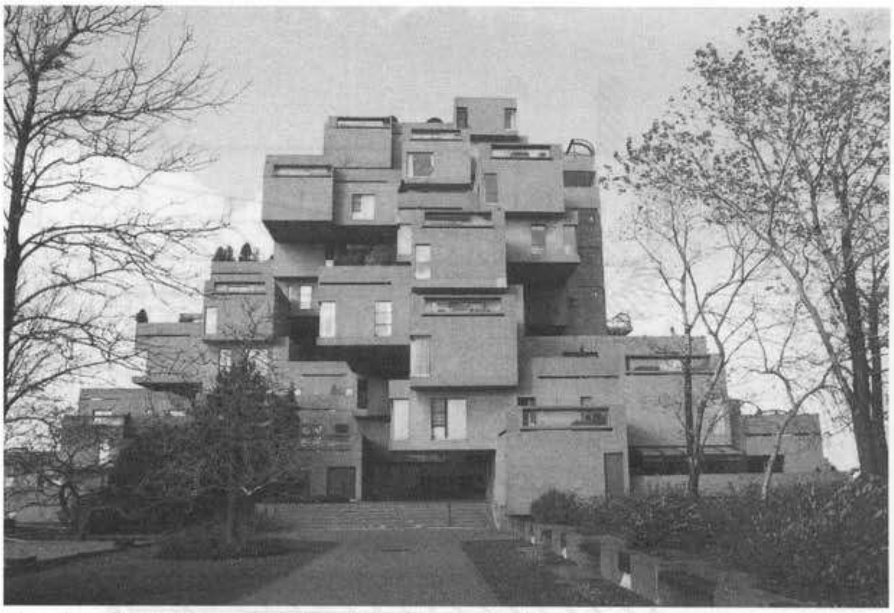
14.10 Repeat Problem 14.9, assuming the backfill slopes at 30° to the horizontal.

14.11 Determine the thickness and necessary reinforcement for the basement wall shown in Fig. 14.23. The weight of backfill is 120 pcf and the angle of internal friction is $\phi = 30^\circ$. Assume a surcharge of 400 psf and use $f_c' = 3$ ksi and $f_y = 60$ ksi.

14.12 Repeat Problem 14.11 using a basement clear height of 14 ft.

CHAPTER 15

DESIGN FOR TORSION



Apartment Building, Habitat 67, Montreal, Canada.

15.1 INTRODUCTION

Torsional stresses develop in a beam section when a moment acts on that section parallel to its surface. Such moments, called torsional moments, cause a rotation in the structural member and cracking on its surface, usually in the shape of a spiral. To illustrate torsional stresses, let a torque T be applied on a circular cantilever beam made of elastic homogeneous material, as shown in Fig. 15.1. The torque will cause a rotation of the beam. Point B moves to point B' at one end of the beam, whereas the other end is fixed. The angle θ is called the angle of twist. The plane $AO'\ OB$ will be distorted to the shape $AO'\ OB'$. Assuming that all longitudinal elements have the same length, the shear strain is

$$\gamma = \frac{(BB')}{L} = \frac{r\theta}{L}$$

where L is the length of the beam and r is the radius of the circular section.

In reinforced concrete structures, members may be subjected to torsional moments when they are curved in plan, support cantilever slabs, act as spandrel beams (end beams), or are part of a spiral stairway.

Structural members may be subjected to pure torsion only or, as in most cases, subjected simultaneously to shearing forces and bending moments. Example 15.1 illustrates the different forces that may act at different sections of a cantilever beam.

Example 15.1

Calculate the forces acting at sections 1, 2, and 3 of the cantilever beam shown in Fig. 15.2. The beam is subjected to a vertical force $P_1 = 15\,\text{K}$, a horizontal force $P_2 = 12\,\text{K}$ acting at C, and a horizontal force $P_3 = 20\,\text{K}$ acting at B and perpendicular to the direction of the force P_2.

Solution

Let N = normal force, V = shearing force, M = bending moment, and T = torsional moment. The forces are as follows.

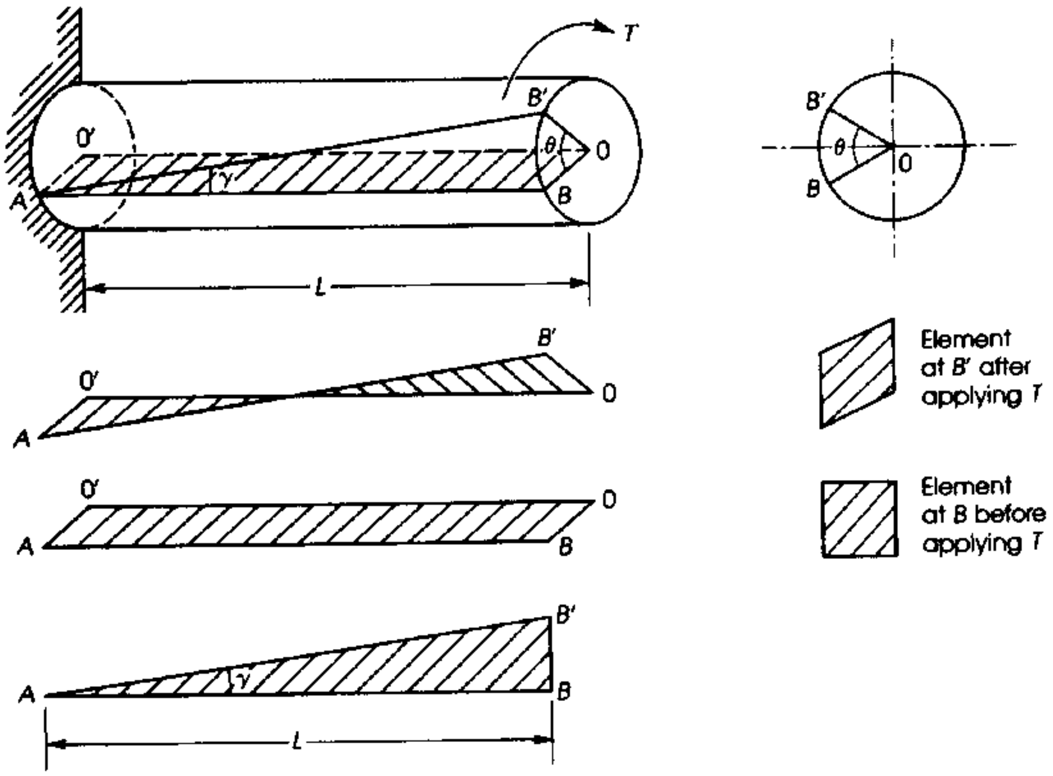


Figure 15.1 Torque applied to a cantilever beam.

Section	N (K)	M_x (K·ft)	M_y (K·ft)	V_x (K)	V_y (K)	T (K·ft)
1	0	−135 (12 × 9)	+108	+12	+15	0
	(15 × 9)					
2	−12 Compression	0	+108	+20	+15	135 (15 × 9)
3	−12 Compression	−180	+348	+20	+15	135 (15 × 9)

If P_1, P_2, and P_3 are factored loads ($P_u = 1.2P_D + 1.6P_L$), then the values in the table will be the factored design forces.

15.2 TORSIONAL MOMENTS IN BEAMS

It was shown in Example 15.1 that forces can act on building frames, causing torsional moments. If a concentrated load P is acting at point C in the frame ABC shown in Fig. 15.3a, it develops a torsional moment in beam AB of $T = PZ$ acting at D. When D is at midspan of AB, then the torsional design moment in AD equals that in DB, or $\frac{1}{2}T$. If a cantilever slab is supported by the beam AB in Fig. 15.3b, the slab causes a uniform torsional moment m_t along AB. This uniform torsional moment is due to the load on a unit width strip of the slab. If S is the width of the cantilever slab and w is load on the slab (psf), then $m_t = wS^2/2$ K·ft/ft of beam AB.

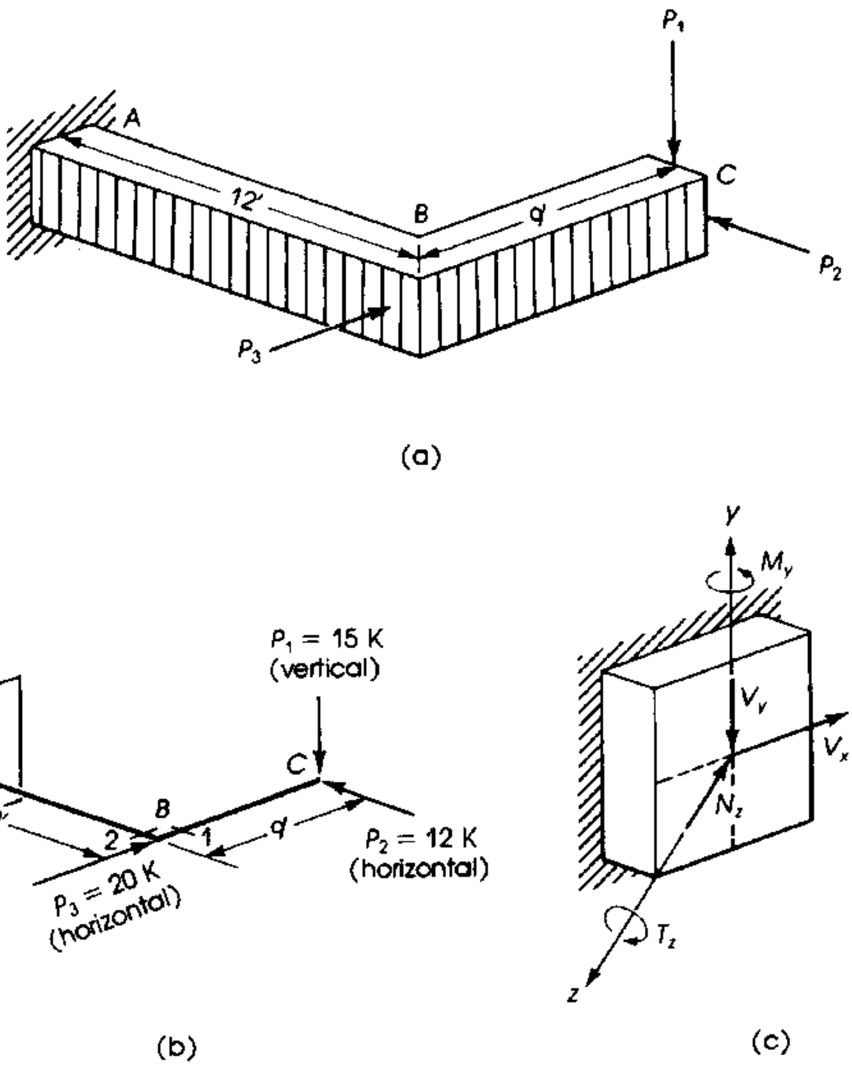


Figure 15.2 Example 15.1.

The maximum torsion design moment in beam AB is $T = \frac{1}{2}m_t L$ acting at A and B. Other cases of loading are explained in Table 15.1. In general, the distribution of torsional moments in beams has the same shape and numerically has the same values as the shear diagrams for beams subjected to a load m_t or T.

15.3 TORSIONAL STRESSES

Considering the cantilever beam with circular section of Fig. 15.1, the torsional moment T will cause a shearing force dV perpendicular to the radius of the section. From the conditions of equilibrium, the external torsional moment is resisted by an internal torque equal to and opposite to T. If dV is the shearing force acting on the area dA (Fig. 15.4), then the magnitude of the torque is $T = \int r\, dV$. Let the shearing stress be v; then

$$dV = v\, dA \quad \text{and} \quad T = \int rv\, dA$$

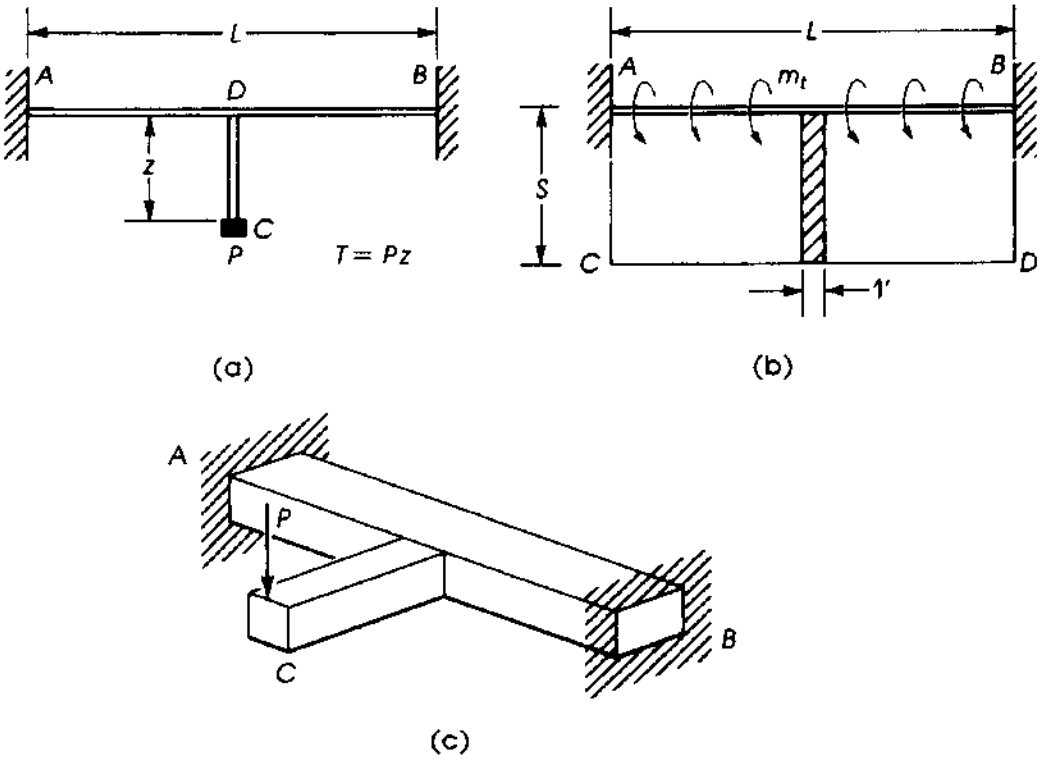


Figure 15.3 Torsional moments on *AB*.

The maximum elastic shear occurs at the external surface of the circular section at radius r with thickness dr; then the torque T can be evaluated by taking moments about the center 0 for the ring area:

$$dT = (2\pi r\,dr)vr$$

where $(2\pi r\,dr)$ is the area of the ring and v is the shear stress in the ring. Thus,

$$T = \int_0^R (2\pi r dr)vr = \int_0^R 2\pi r^2\,v\,dr \tag{15.1}$$

For a hollow section with internal radius R_1,

$$T = \int_{R1}^R 2\pi r^2\,v\,dr \tag{15.2}$$

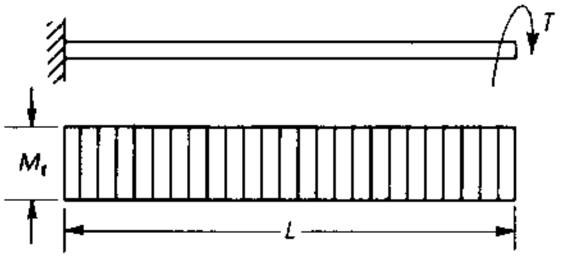
For a solid section, using Eq. 15.1 and using $v = v_{\max}\,r/R$,

$$T = \int_0^R 2\pi r^2 \left(\frac{v_{\max}r}{R}\right) dr = \left(\frac{2\pi}{R}\right) v_{\max} \int_0^R r^3 dr$$

$$= \left(\frac{2\pi}{R}\right) v_{\max} \times \frac{R^4}{4} = \left(\frac{\pi}{2}\right) v_{\max} R^3$$

$$v_{\max} = \frac{2T}{\pi r^3} \tag{15.3}$$

Table 15.1 Torsion Diagrams

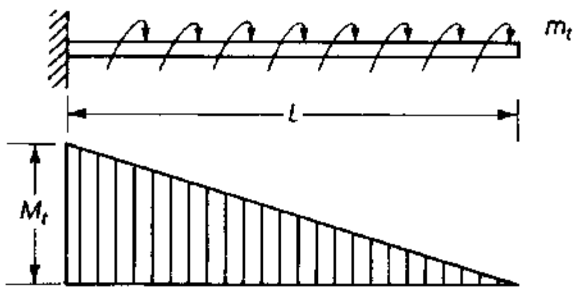


At support

$M_t = T$

For a circular section

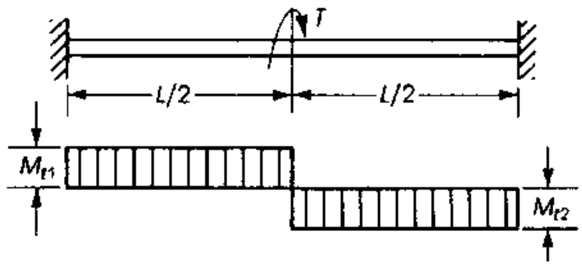
$$J = \frac{\pi R^4}{2}$$



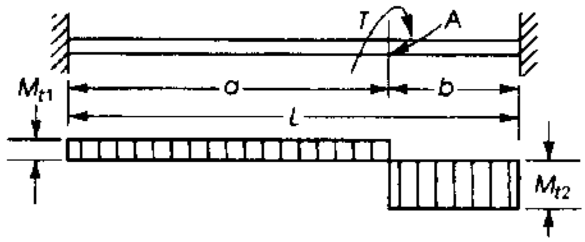
At support

$M_t = m_t L$

m_t = uniform torque

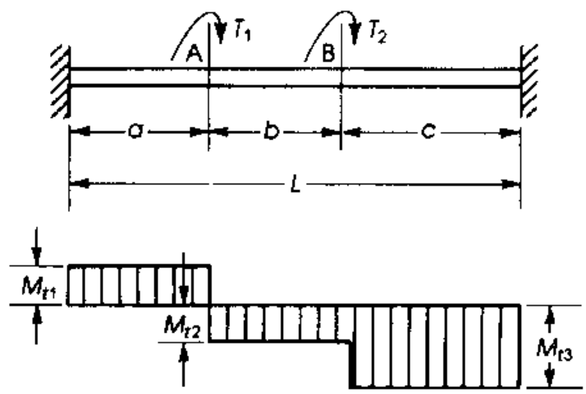


$$M_{t1} = M_{t2} = \frac{T}{2}$$



$$M_{t1} = \frac{Tb}{L}$$

$$M_{t2} = \frac{Ta}{L}$$



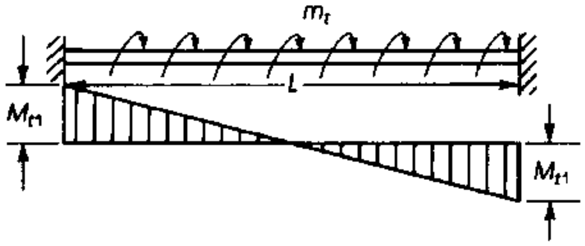
$$M_{t1} = \frac{T_1(b-c) + T_2 c}{L}$$

$$M_{t2} = \frac{T_2 c - T_1 a}{L}$$

$$M_{t3} = \frac{T_1 a - T_2(a+b)}{L}$$

Note: When $a = b = c = L/\beta$ and $T_1 = T_2$
$= M_{t1} = -M_{t3} = T \cdot M_{t2} = 0$

Table 15.1 (*continued*)



$$M_{t1} = \frac{m_t L}{2}$$

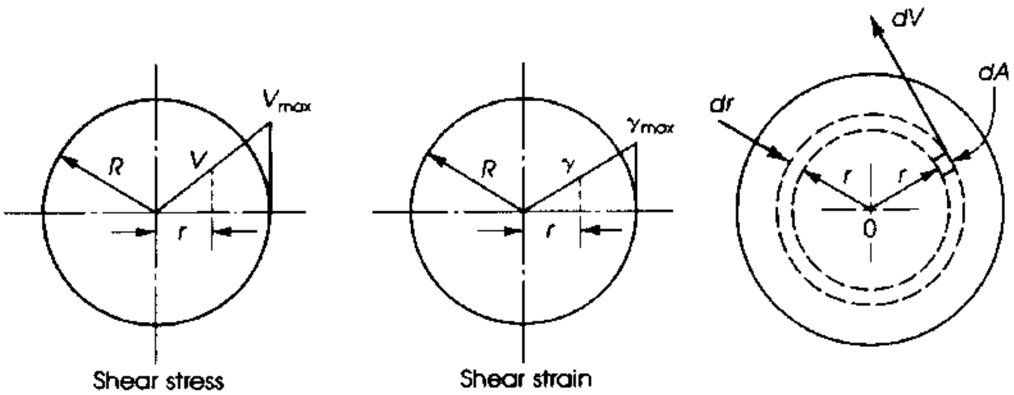


Figure 15.4 Torque in circular sections.

The polar moment of inertia of a circular section is $J = \pi\ R^4/2$. Therefore, the shear stress can be written as a function of the polar moment of inertia J as follows:

$$v_{max} = \frac{TR}{J} \tag{15.4}$$

15.4 TORSIONAL MOMENT IN RECTANGULAR SECTIONS

The determination of the stress in noncircular members subjected to torsional loading is not as simple as that for circular sections. However, results obtained from the theory of elasticity in dicate that the maximum shearing stress for rectangular sections can be calculated as follows:

$$v_{max} = \frac{T}{\alpha x^2 y} \tag{15.5}$$

where

T = the applied torque
x = short side of the rectangular section
y = the long side of the rectangular section
α = coefficient that depends on the ratio of y/x; its value is given in the following table.

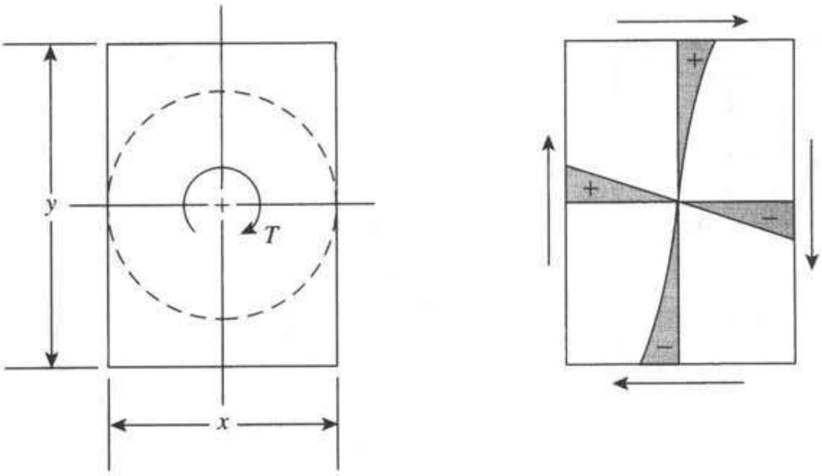


Figure 15.5 Stress distribution in rectangular sections due to pure torsion.

y/x	1.0	1.2	1.5	2.0	4	10
α	0.208	0.219	0.231	0.246	0.282	0.312

The maximum shearing stress occurs along the centerline of the longer side y (Fig. 15.5).

For members composed of rectangles, such as T-, L-, or I-sections, the value of α can be assumed equal to be $\frac{1}{3}$, and the section may be divided into several rectangular components having a long side y_i and a short side x_i. The maximum shearing stress can be calculated from

$$v_{max} = \frac{3T}{\Sigma x_i^2 y_i} \tag{15.6}$$

where $\Sigma x_i^2 y_i$ is the value obtained from the rectangular components of the section. When $y/x \leq 10$, a better expression may be used:

$$v_{max} = \frac{3T}{\sum x^2 y \left(1 - 0.63\dfrac{x}{y}\right)} \tag{15.7}$$

15.5 COMBINED SHEAR AND TORSION

In most practical cases, a structural member may be subjected simultaneously to both shear and torsional forces. Shear stresses will be developed in the section, as was explained in Chapter 8, with an average shear $= v_1$ in the direction of the shear force V (Fig. 15.6*a*). The torque T produces torsional stresses along all sides of the rectangular section $ABCD$ (Fig. 15.6*a*), with $v_3 > v_2$. The final stress distribution is obtained by adding the effect of both shear and torsion stresses to produce maximum value of $(v_1 + v_3)$ on side CD, whereas side AB will have a final stress of $(v_1 - v_3)$. Both sides AD and BC will be subjected to torsional stress v_2 only. The section must be designed for the maximum $v = (v_1 + v_3)$.

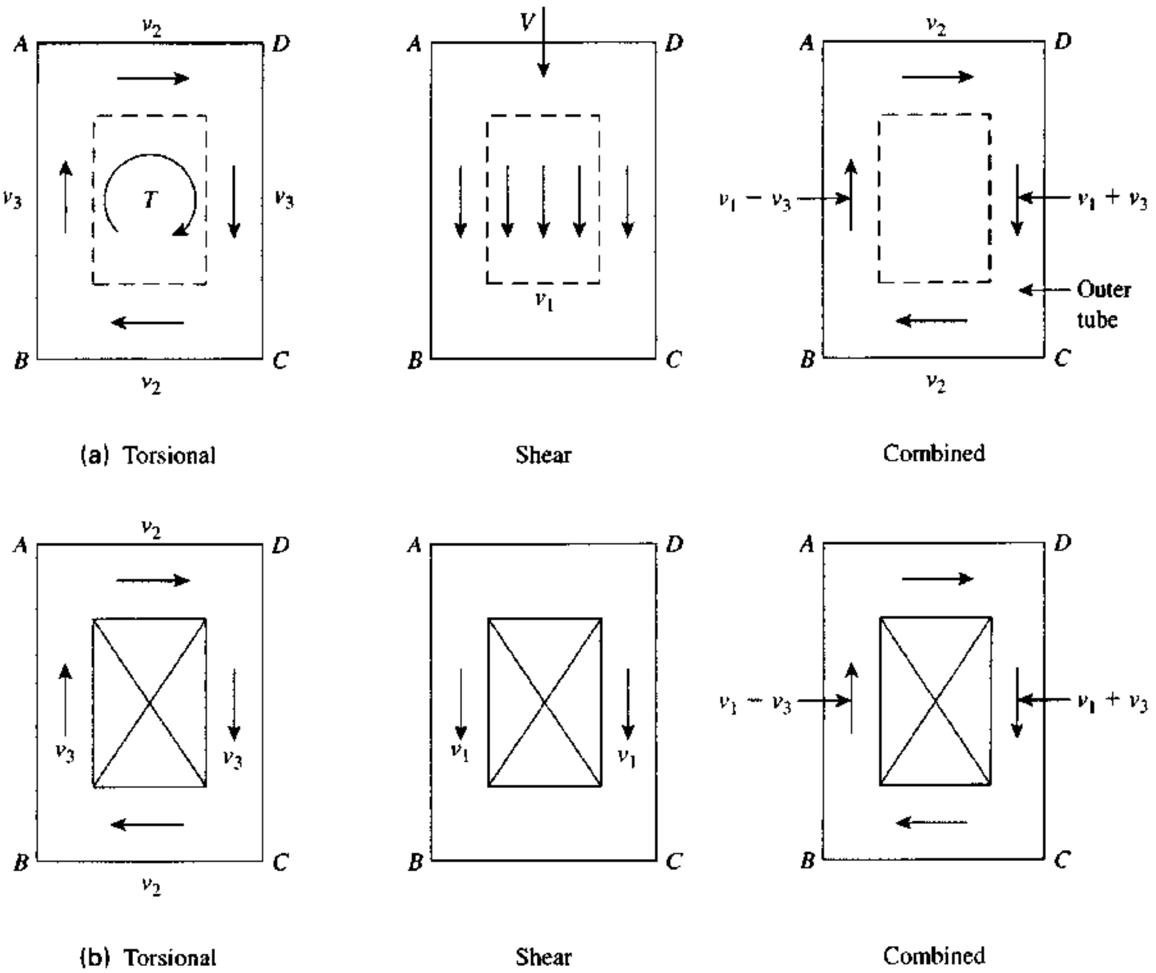


Figure 15.6 Combined shear and torsional stresses: (*a*) solid sections and (*b*) hollow sections.

15.6 TORSION THEORIES FOR CONCRETE MEMBERS

Various methods are available for the analysis of reinforced concrete members subjected to torsion or simultaneous torsion, bending, and shear. The design methods rely generally on two basic theories: the skew bending theory and the space truss analogy.

15.6.1 Skew Bending Theory

The skew bending concept was first presented by Lessig in 1959 [2] and was further developed by Goode and Helmy [3], Collins et al. in 1968 [4], and Below et al. in 1975 [5]. The concept was applied to reinforced concrete beams subjected to torsion and bending. Expressions for evaluating the torsional capacity of rectangular sections were presented by Hsu in 1968 [6,7] and were adopted by the ACI Code of 1971. Torsion theories for concrete members were discussed by Zia [8]. Empirical design formulas were also presented by Victor et al. in 1976 [9].

The basic approach of the skew bending theory, as presented by Hsu, is that failure of a rectangular section in torsion occurs by bending about an axis parallel to the wider face of the section y and inclined at about 45° to the longitudinal axis of the beam (Fig. 15.7). Based on

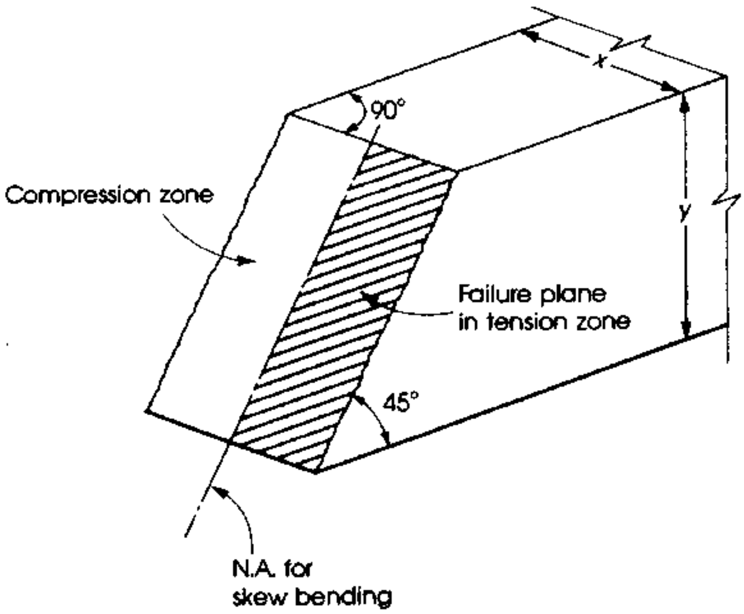


Figure 15.7 Failure surface due to skew bending.

this approach, the minimum torsional moment, T_n, can be evaluated as follows:

$$T_n = \left(\frac{x^2y}{3}\right) f_r \tag{15.8}$$

where f_r is the modulus of rupture of concrete; f_r is assumed to be $5\sqrt{f'_c}$ in this case, as compared to $7.5\lambda\sqrt{f'_c}$ adopted by the ACI Code for the computation of deflection in beams.

The torque resisted by concrete is expressed as follows:

$$T_c = \left(\frac{2.4}{\sqrt{x}}\right) x^2y\sqrt{f'_c} \tag{15.9}$$

and the torque resisted by torsional reinforcement is

$$T_s = \frac{\alpha_1(x_1y_1A_tf_y)}{s} \tag{15.10}$$

Thus, $T_n = T_c + T_s$, where T_n is the nominal torsional moment capacity of the section.

15.6.2 Space Truss Analogy

The space truss analogy was first presented by Rausch in 1929 and was further developed by Lampert [10,11], who supported his theoretical approach with extensive experimental work. The Canadian Code provisions for the design of reinforced concrete beams in torsion and bending are based on the space truss analogy. Mitchell and Collins [12] presented a theoretical model for structural concrete in pure torsion. McMullen and Rangan [13] discussed the design concepts of rectangular sections subjected to pure torsion. In 1983, Solanki [14] presented a simplified design approach based on the theory presented by Mitchell and Collins.

The concept of the space truss analogy is based on the assumption that the torsional capacity of a reinforced concrete rectangular section is derived from the reinforcement and the concrete surrounding the steel only. In this case, a thin-walled section is assumed to act as a space truss

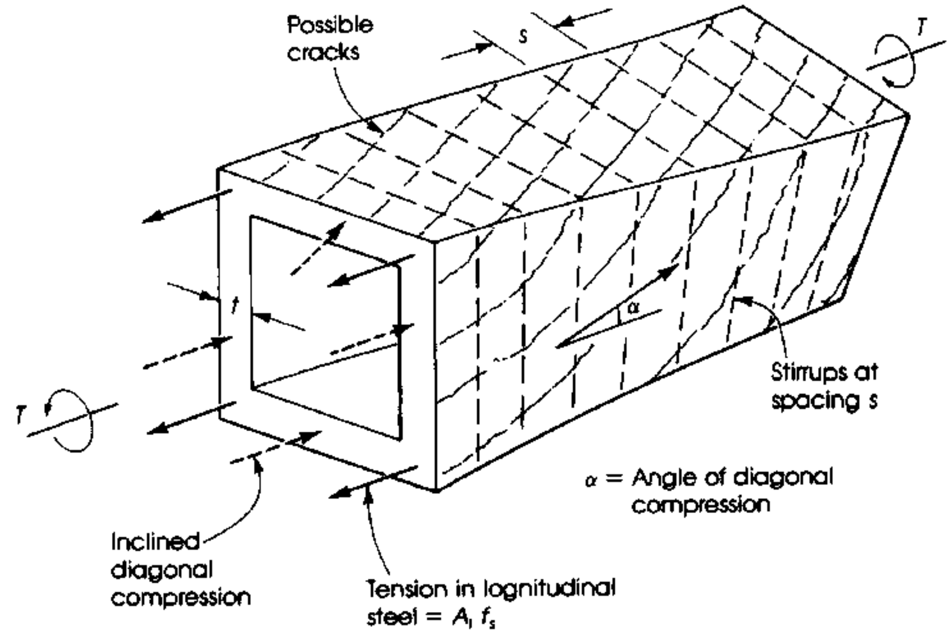


Figure 15.8 Forces on section in torsion (space truss analogy).

(Fig. 15.8). The inclined spiral concrete strips between cracks resist the compressive forces, whereas the longitudinal bars at the corners and stirrups resist the tensile forces produced by the torsional moment.

The behavior of a reinforced concrete beam subjected to pure torsion can be represented by an idealized graph relating the torque to the angle of twist, as shown in Fig. 15.9. It can be seen that prior to cracking, the concrete resists the torsional stresses and the steel is virtually unstressed. After cracking, the elastic behavior of the beam is not applicable, and hence a sudden

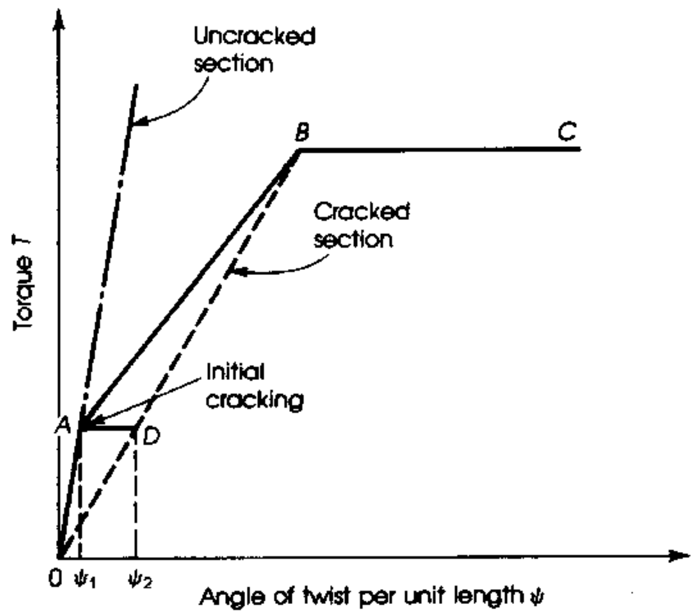


Figure 15.9 Idealized torque versus twist relationship.

change in the angle of twist occurs, which continues to increase until the maximum torsional capacity is reached. An approximate evaluation of the torsional capacity of a cracked section may be expressed as follows:

$$T_n = 2\left(\frac{A_t f_y}{s}\right) x_1 y_1 \tag{15.11}$$

where

A_t = of one leg of stirrups
s = spacing of stirrups

x_1 and y_1 = short and long distances, center to center of closed rectangular stirrups or corner bars

The preceding expression neglects the torsional capacity due to concrete. Mitchell and Collins [12] presented the following expression to evaluate the angle of twist per unit length ψ:

$$\psi = \left(\frac{P_0}{2A_0}\right)\left[\left(\frac{\varepsilon_1}{\tan\alpha}\right) + \left(\frac{P_h(\varepsilon_h \tan\alpha)}{P_0}\right) + \frac{2\varepsilon_d}{\sin\alpha}\right] \tag{15.12}$$

where

ε_1 = strain in the longitudinal reinforcing steel
ε_h = strain in the hoop steel (stirrups)
ε_d = concrete diagonal strain at the position of the resultant shear flow
P_h = hoop centerline perimeter
α = angle of diagonal compression = $(\varepsilon_d + \varepsilon_1) \Big/ \left[\varepsilon_d + \varepsilon_h\left(\frac{P_h}{P_0}\right)\right]$
A_0 = area enclosed by shear, or
= torque/$2q$ where q = shear flow)
P_0 = perimeter of the shear flow path (perimeter of A_0).

The preceding twist expression is analogous to the curvature expression in flexure (Fig. 15.10):

$$\phi = \text{curvature} = \frac{\varepsilon_c + \varepsilon_s}{d_t} \tag{15.13}$$

where ε_c and ε_s are the strains in concrete and steel, respectively. A simple equation is presented by Solanki [14] to determine the torsional capacity of a reinforced concrete beam in pure torsion, based on the space truss analogy, as follows:

$$T_u = (2A_0)\left[\left(\frac{\sum A_s f_{sy}}{P_0}\right) \times \left(\frac{A_h f_{hy}}{s}\right)\right]^{1/2} \tag{15.14}$$

where A_0, P_0, and s are as explained before and

$\Sigma A_s f_{sy}$ = yield force of all the longitudinal steel bars
$A_h f_{hy}$ = yield force of the stirrups

The ACI Code adapted this theory to design concrete structural members subjected to torsion or shear and torsion in a simplified approach.

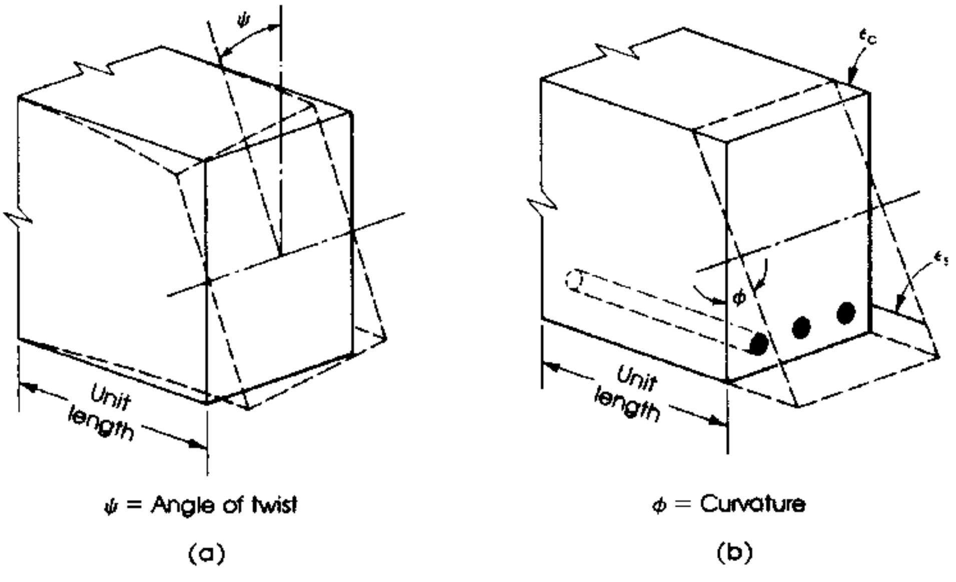


Figure 15.10 (*a*) Torsion and (*b*) flexure.

15.7 TORSIONAL STRENGTH OF PLAIN CONCRETE MEMBERS

Concrete structural members subjected to torsion will normally be reinforced with special torsional reinforcement. In case that the torsional stresses are relatively low and need to be calculated for plain concrete members, the shear stress, v_{tc}, can be estimated using Eq. 15.6:

$$v_{tc} = \frac{3T}{\phi \Sigma x^2 y} \leq 6\sqrt{f_c'}$$

and the angle of twist is $\theta = 3TL/x^3yG$, where T is the torque applied on the section (less than the cracking torsional moment) and G is the shear modulus and can be assumed to be equal to 0.45 times the modulus of elastic of concrete, E_c; that is, $G = 25{,}700\sqrt{f_c'}$. The torsional cracking shear, v_c, in plain concrete may be assumed equal to $6\sqrt{f_c'}$. Therefore, for plain concrete rectangular sections,

$$T_c = 2\phi x^2 y \sqrt{f_c'} \tag{15.15}$$

and for compound rectangular sections,

$$T_c = 2\phi \sqrt{f_c'} \Sigma x^2 y \tag{15.16}$$

15.8 TORSION IN REINFORCED CONCRETE MEMBERS (ACI CODE PROCEDURE)

15.8.1 General

The design procedure for torsion is similar to that for flexural shear. When the factored torsional moment applied on a section exceeds that which the concrete can resist, torsional cracks develop, and consequently torsional reinforcement in the form of closed stirrups or hoop reinforcement must be provided. In addition to the closed stirrups, longitudinal steel bars are provided in the corners of the stirrups and are well distributed around the section. Both types of reinforcement, closed stirrups and longitudinal bars, are essential to resist the diagonal tension forces caused by torsion; one type will not be effective without the other. The stirrups must be closed, because torsional stresses occur on all faces of the section.

The reinforcement required for torsion must be added to that required for shear, bending moment, and axial forces. The reinforcement required for torsion must be provided such that the torsional moment strength of the section ϕT_n is equal to or exceeds the applied factored torsional moment T_u computed from factored loads.

$$\phi T_n \geq T_u \tag{15.17}$$

When torsional reinforcement is required, the torsional moment strength ϕT_n must be calculated assuming that all the applied torque, T_u, is to be resisted by stirrups and longitudinal bars with concrete torsional strength, $T_c = 0$. At the same time, the shear resisted by concrete, v_c, is assumed to remain unchanged by the presence of torsion.

15.8.2 Torsional Geometric Parameters

In the ACI Code, Section 11.5 the design for torsion is based on the space truss analogy, as shown in Fig. 15.8. After torsional cracking occurs, the torque is resisted by closed stirrups, longitudinal bars, and concrete compression diagonals. The concrete shell outside the stirrups becomes relatively ineffective and is normally neglected in design. The area enclosed by the centerline of the outermost closed stirrups is denoted by A_{oh}, the shaded area in Fig. 15.11. Because other terms are used in the design equations, they are introduced here first to make the equation easier to comprehend. Referring to Fig. 15.11, the given terms are defined as follows:

A_{cp} = enclosed by outside perimeter of concrete section, in.^{2}

P_{cp} = perimeter of concrete gross area, A_{cp}, in.

A_{oh} = area enclosed by centerline of the outermost closed transverse torsional reinforcement, in.^{2} (shaded area in Fig. 15.11)

A_0 = gross area enclosed by shear flow path and may be taken equal to 0.85 A_{oh} (A_0 may also be determined from analysis [18,19].)

P_h = perimeter of concrete of outermost closed transverse torsional reinforcement

θ = angle of compression diagonals between 30° and 60° (may be taken equal to 45° for reinforced concrete members)

In T- and L-sections, the effective overhang width of the flange on one side is limited to the projection of the beam above or below the slab, whichever is greater, but not greater than four times the slab thickness (ACI Code, Sections 11.5.1 and 13.2.4).

15.8.3 Cracking Torsional Moment, T_{cr}

The cracking moment under pure torsion, T_{cr}, may be derived by replacing the actual section, prior to cracking, with an equivalent thin-walled tube, $t = 0.75\ A_{\text{cp}}/P_{\text{cp}}$, and an area enclosed by the wall centerline, $A_0 = 2\ A_{\text{cp}}/3$. When the maximum tensile stress (principal stress) reaches $4\lambda\sqrt{f'_c}$, cracks start to occur and the torque T in general is equal to

$$T = 2A_0\tau t \tag{15.18}$$

where τ = the torsional shear stress = $4\lambda\sqrt{f'_c}$ for torsional cracking.

Replacing τ by $4\lambda\sqrt{f'_c}$,

$$T_{\text{cr}} = 4\lambda\sqrt{f'_c}\left(\frac{A^2_{\text{cp}}}{P_{\text{cp}}}\right) = T_n \quad \text{and} \quad T_u = \phi T_{\text{cr}} \tag{15.19}$$

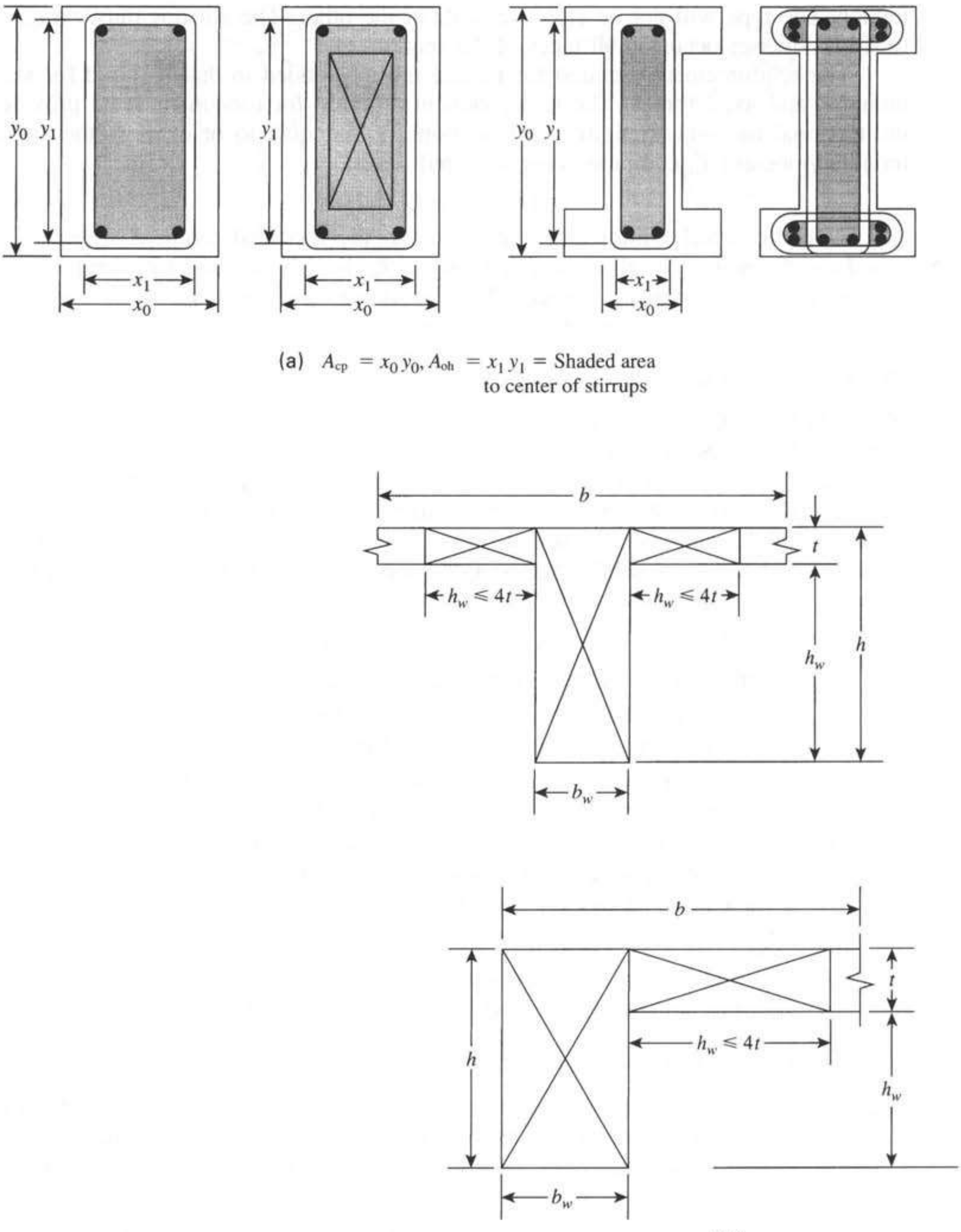


Figure 15.11 (*a*) Torsional geometric parameters; (*b*) effective flange width for T- and L-sections and component rectangles.

Assuming that a torque less than or equal to $T_{cr}/4$ will not cause a significant reduction in the flexural or shear strength in a structural member, the ACI Code, Section 11.5.1, permits neglect of torsion effects in reinforced concrete members when the factored torsional moment $T_u \leq \phi T_{cr}/4$, or

$$T_u \leq \phi\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{P_{cp}}\right) = T_a \tag{15.20}$$

When T_u exceeds the value in Eq. 15.20, all T_u must be resisted by closed-stirrup and longitudinal bars. The torque, T_u, is calculated at a section located at distance d from the face of the support and $T_u = \phi T_n$, where $\phi = 0.75$.

Example 15.2

For the three sections shown in Fig. 15.12, and based on the ACI Code limitations, it is required to compute the following:

a. The cracking moment ϕT_{cr}

b. The maximum factored torque ϕT_n that can be applied to each section without using torsional web reinforcement

Assume $f'_c = 4$ ksi, $f_y = 60$ ksi, a 1.5-in. concrete cover, and no. 4 stirrups.

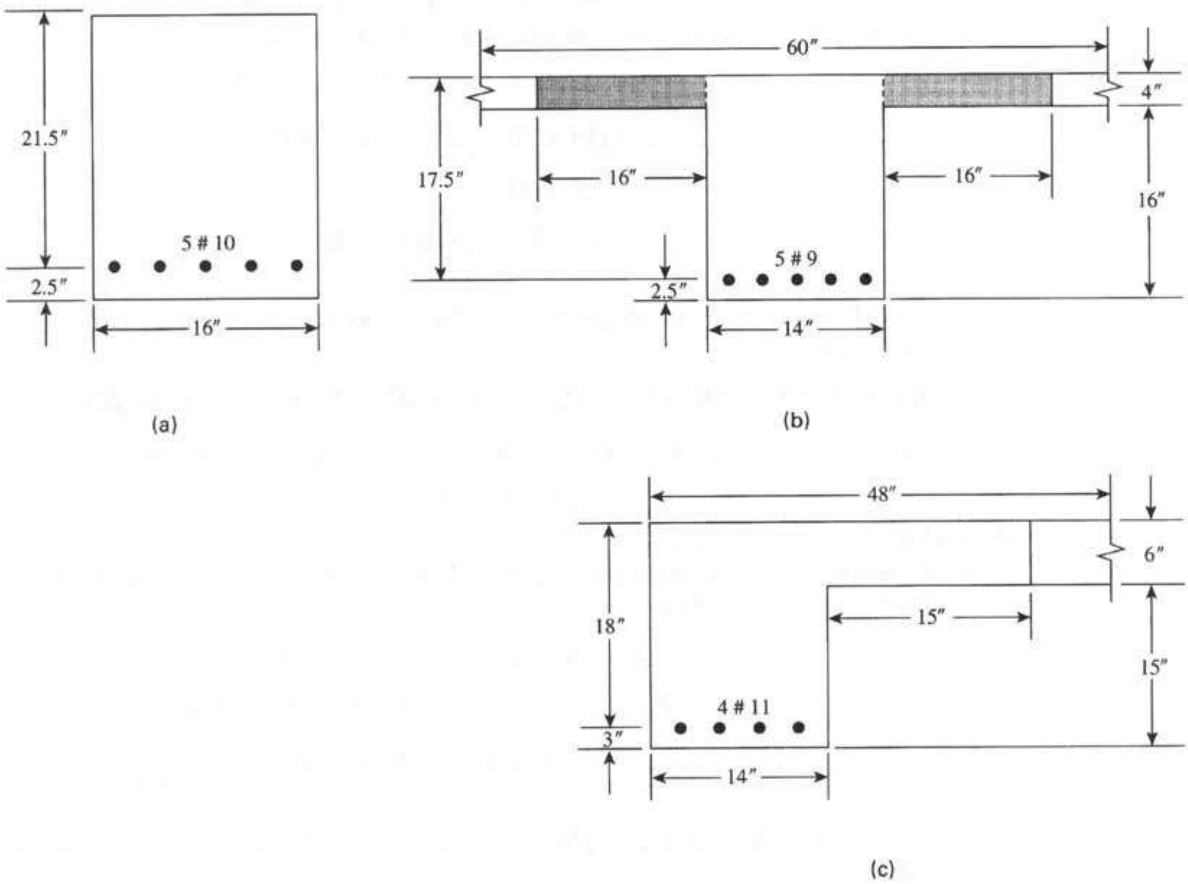


Figure 15.12 Example 15.2.

Solution

1. Section 1

 a. Cracking moment, ϕT_{cr}, can be calculated from Eq. 15.19.

$$\phi T_{cr} = \phi 4\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{P_{cp}}\right)$$

 For this section, $A_{cp} = x_0 y_0$, the gross area of the section, where $x_0 = 16$ in. and $y_0 = 24$ in.

$$A_{cp} = 16(24) = 384 \text{ in.}^2$$

$$P_{cp} = \text{perimeter of the gross section}$$

$$= 2(x_0 + y_0) = 2(16 + 24) = 80 \text{ in.}$$

$$\phi T_{cr} = \frac{0.75(4)(1)\sqrt{4000}(384)^2}{80} = 349.7 \text{ K·in.}$$

 b. The allowable ϕT_n that can be applied without using torsional reinforcement is computed from Eq. 15.20:

$$T_a = \frac{\phi T_r}{4} = \frac{349.7}{4} = 87.4 \text{ K·in.}$$

2. Section 2

 a. First calculate A_{cp} and P_{cp} for this section and apply Eq. 15.19 to calculate ϕT_{cr}. Assuming flanges are confined with closed stirrups, the effective flange part to be used on each side of the web is equal to four times the flange thickness, or $4(4) = 16$ in $= h_w = 16$ in.

$$A_{cp} = \text{web area } (b_w h) + \text{area of effective flanges}$$

$$= (14 \times 20) + 2(16 \times 4) = 408 \text{ in.}^2$$

$$P_{cp} = 2(b + h) = 2(14 + 2 \times 16 + 20) = 132 \text{ in.}^2$$

$$\phi T_{cr} = \frac{0.75(4)(1)\sqrt{(4000)}(408)^2}{132} = 239.3 \text{ K·in.}$$

 Note: If the flanges are neglected and the torsional reinforcement is confined in the web only, then

 $A_{cp} = 14(20) = 280$ in.^{2} $P_{cp} = 2(14 + 20) = 68$ in. $\phi T_{cr} = 219$ K·in.

 b. The allowable ϕT_n that can be applied without using torsional reinforcement is

$$\phi T_{cr}/4 = 239.3/4 = 59.8 \text{ K·in.}$$

3. Section 3

 a. Assuming flange is confined with closed stirrups, effective flange width is equal to $b_w = 15$ in. $< 4 \times 6 = 24$ in.

$$A_{cp} = (14 \times 21) + (15 \times 6) = 384 \text{ in.}^2$$

$$P_{cp} = 2(b + h) = 2(14 + 15 + 21) = 100 \text{ in.}$$

$$\phi T_{cr} = \frac{0.75(4)(1)\sqrt{(4000)}(384)^2}{100} = 279.8 \text{ K·in.}$$

 Note: If the flanges are neglected, then $A_{cp} = 294$ in.^{2}, $P_{cp} = 70$ in., and $\phi T_{cr} = 234.3$ K·in.

 b. The allowable $\phi T_n = \phi T_{cr}/4 = 279.8/4 = 70$ K·in.

15.8.4 Equilibrium Torsion and Compatibility Torsion

Structural analysis of concrete members gives the different forces acting on the member, such as normal forces, bending moments, shear forces, and torsional moments, as explained in the simple problem of Example 15.1. The design of a concrete member is based on failure of the member under factored loads. In statically indeterminate members, a redistribution of moments occurs before failure; consequently, design moments may be reduced, whereas in statically determinate members, such as a simple beam or a cantilever beam, no moment redistribution occurs.

In the design of structural members subjected to torsional moments two possible cases may apply after cracking.

1. The *equilibrium torsion case* occurs when the torsional moment is required for the structure to be in equilibrium and T_u cannot be reduced by redistribution of moments, as in the case of simple beams. In this case torsion reinforcement must be provided to resist all of T_u. Figure 15.13 shows an edge beam supporting a cantilever slab where no redistribution of moments will occur [18,19].
2. The *compatibility torsion case* occurs when the torsional moment, T_u, can be reduced by the redistribution of internal forces after cracking while compatibility of deformation is maintained in the structural member. Figure 15.14 shows an example of this case, where two transverse beams are acting on an edge beam producing twisting moments. At torsional

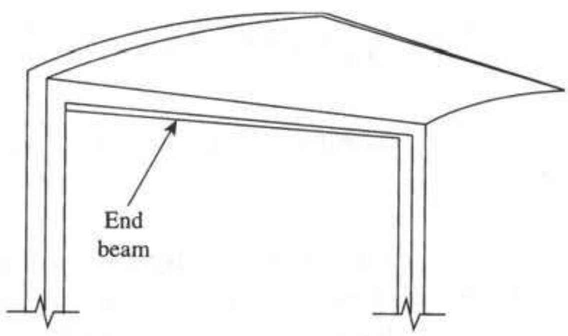


Figure 15.13 Design torque may not be reduced. Moment redistribution is not possible [19].

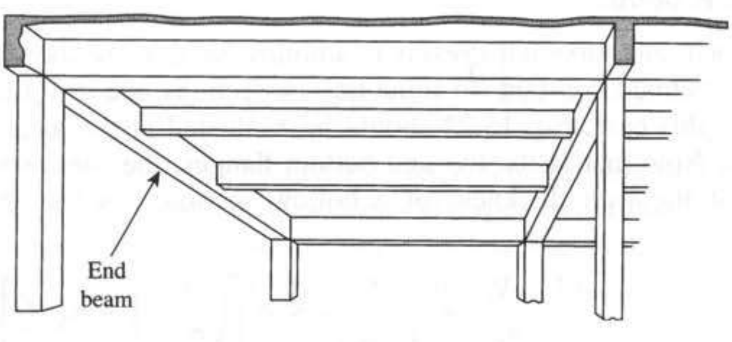


Figure 15.14 Design torque may be reduced in a spandrel beam. Moment redistribution is possible [19].

cracking, a large twist occurs, resulting in a large distribution of forces in the structure [18,19]. The cracking torque, T_{cr}, under combined flexure, shear, and torsion is reached when the principal stress in concrete is about $4\lambda\sqrt{f_c'}$. When $T_u > T_{cr}$, a torque equal to T_{cr} (Eq. 15.19), may be assumed to occur at the critical sections near the faces of the supports.

The ACI Code limits the design torque to the smaller of T_u from factored loads or ϕT_{cr} from Eq. 15.19.

15.8.5 Limitation of Torsional Moment Strength

The ACI Code, Section 11.5.3, limits the size of the cross-sectional dimension by the following two equations:

1. For solid sections,

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u P_h}{1.7A_{oh}^2}\right)^2} \le \phi\left[\left(\frac{V_c}{b_w d}\right) + 8\sqrt{f_c'}\right] \qquad (15.21)$$

2. For hollow sections,

$$\left(\frac{V_u}{b_w d}\right) + \left(\frac{T_u P_h}{1.7A_{oh}^2}\right) \le \phi\left[\left(\frac{V_c}{b_w d}\right) + 8\sqrt{f_c'}\right] \qquad (15.22)$$

where $V_c = 2\lambda\sqrt{f_c'}b_w d$ = shear strength for normal-weight concrete. All other terms were defined in Section 15.8.2.

This limitation is based on the fact that the sum of the stresses due to shear and torsion (on the left-hand side) may not exceed the cracking stress plus $8\sqrt{f_c'}$. The same condition was applied to the design of shear without torsion in Chapter 8. The limitation is needed to reduce cracking and to prevent crushing of the concrete surface due to inclined shear and torsion stresses.

15.8.6 Hollow Sections

Combined shear and torsional stresses in a hollow section are shown in Fig. 15.6, where the wall thickness is assumed constant. In some hollow sections, the wall thickness may vary around the perimeter. In this case, Eq. 15.22 should be evaluated at the location where the left-hand side is maximum. Note that at the top and bottom flanges, the shear stresses are usually negligible. In general, if the wall thickness of a hollow section t is less than A_{oh}/P_h, then Eq. 15.22 becomes

$$\frac{V_u}{b_w d} + \frac{T_u}{1.7A_{oh}t} \le \phi\left[\left(\frac{V_c}{b_w d}\right) + 8\sqrt{f_c'}\right] \qquad (15.23)$$

(ACI Code, Section 11.5.3).

15.8.7 Web Reinforcement

As was explained earlier, the ACI Code approach for the design of the members due to torsion is based on the space truss analogy in Fig. 15.8. After torsional cracking, two types of reinforcement are required to resist the applied torque, T_u: transverse reinforcement, A_t, in the form of closed stirrups, and longitudinal reinforcement, A_l, in the form of longitudinal bars. The ACI Code, Section 11.5.3, presented the following expression to compute A_t and A_l:

1. Closed stirrups A_t can be calculated as follows.

$$T_n = \frac{2A_0 A_t f_{yt} \cot\theta}{s} \tag{15.24}$$

where

$T_n = \dfrac{T_u}{\theta}$ and $\phi = 0.75$
A_t = area of one leg of the transverse closed stirrups
f_{yt} = yield strength of $A_t \leq 60\,\text{ksi}$
s = spacing of stirrups

A_0 and θ were defined in Section 15.8.2. Equation 15.24 can be written as follows:

$$\frac{A_t}{s} = \frac{T_n}{2A_0 f_{yt} \cot\theta} \tag{15.25}$$

If $\theta = 45^\circ$, then cot θ = 1.0, and if f_{yv} = 60 ksi, then Eq. 15.25 becomes

$$\frac{A_t}{s} = \frac{T_n}{120A_0} \tag{15.26}$$

where T_n is in kip in. Spacing of stirrups, s, should not exceed the smaller of $P_h/8$ or 12 in. For hollow sections in torsion, the distance measured from the centerline of stirrups to the inside face of the wall shall not be less than 0.5 A_{oh}/P_h.

2. The additional longitudinal reinforcement, A_l, required for torsion should not be less than the following:

$$A_l = \left(\frac{A_t}{s}\right) P_h \left(\frac{f_{yt}}{f_y}\right) \cot^2\theta \tag{15.27}$$

If $\theta = 45^\circ$ and $f_{yt} = f_y = 60\,\text{ksi}$ for both stirrups and longitudinal bars, then Eq. 15.27 becomes

$$A_l = \left(\frac{A_t}{s}\right) P_h = 2\left(\frac{A_t}{s}\right)(x_1 + y_1) \tag{15.28}$$

P_h was defined in Section 15.8.2. Note that reinforcement required for torsion should be added to that required for the shear, moment, and axial force that act in combination with torsion. Other limitations for the longitudinal reinforcement, A_l, are as follows:

a. The smallest bar diameter of a longitudinal bar is that of no. 3 or stirrup spacing $s/24$, whichever is greater.

b. The longitudinal bars should be distributed around the perimeter of the closed stirrups with a maximum spacing of 12 in.

c. The longitudinal bars must be inside the stirrups with at least one bar in each corner of the stirrups. Corner bars are found to be effective in developing torsional strength and in controlling cracking.

d. Torsional reinforcement should be provided for a distance $(b_t + d)$ beyond the point theoretically required, where b_t is the width of that part of the cross-section containing the stirrups resisting torsion.

15.8.8 Minimum Torsional Reinforcement

Where torsional reinforcement is required, the minimum torsional reinforcement may be computed as follows (ACI Code, Section 11.5.5):

1. Minimum transverse closed stirrups for combined shear and torsion (see Section 8.6):

$$A_v + 2A_t \geq \frac{50 b_w s}{f_{yt}} \qquad \text{(for } f'_c < 4.5\,\text{ksi)}$$

$$\geq 0.75\sqrt{f'_c}\left(\frac{b_w s}{f_{yt}}\right) \qquad \text{(for } f'_c \geq 4.5\,\text{ksi)} \qquad (15.29)$$

where

A_v = area of two legs of a closed stirrup determined from shear
A_t = area of one leg of closed stirrup determined from torsion
s = spacing of stirrups
f_{yt} = yield strength of closed stirrups $\leq$ 60 ksi

Spacing of stirrups, s, should not exceed $P_h/8$ or 12 in., whichever is smaller. This spacing is needed to control cracking width.

2. Minimum total area of longitudinal torsional reinforcement:

$$A_{l\,\min} = \left(\frac{5\sqrt{f'_c}A_{cp}}{f_y}\right) - \left(\frac{A_t}{s}\right) P_h \left(\frac{f_{yt}}{f_y}\right) \qquad (15.30)$$

where A_t/s shall not be taken less than $25\, b_w/f_{yt}$.

The minimum A_l in Eq. 15.30 was determined to provide a minimum ratio of the volume of torsional reinforcement to the volume of concrete of about 1% for reinforced concrete subjected to pure torsion.

15.9 SUMMARY OF ACI CODE PROCEDURES

The design procedure for combined shear and torsion can be summarized as follows:

1. Calculate the factored shearing force, V_u, and the factored torsional moment, T_u, from the applied forces on the structural member. Critical values for shear and torsion are at a section distance d from the face of the support.
2. a. Shear reinforcement is needed when $V_u > \phi V_c/2$, where $V_c = 2\lambda\sqrt{f'_c}b_w d$.

b. Torsional reinforcement is needed when

$$T_u > \phi\lambda\sqrt{f'_c}\left(\frac{A^2_{cp}}{P_{cp}}\right) \tag{15.20}$$

If web reinforcement is needed, proceed as follows.

3. Design for shear:

a. Calculate the nominal shearing strength provided by the concrete, V_c. Determine the shear to be carried by web reinforcement:

$$V_u = \phi V_c + V_s \text{ or } V_s = \frac{V_u - \phi V_c}{\phi}$$

b. Compare the calculated V_s with maximum permited value of $(8\sqrt{f'_c}b_w d)$ according to the ACI Code Section 11.4.7.9. If calculated V_s is less, proceed with the design; if not, increase the dimensions of the concrete section.

c. The shear web reinforcement is calculated as follows:

$$A_v = \frac{V_s s}{f_{yt} d}$$

where

A_v = area of two legs of the stirrup
s = spacing of stirrups.

The shear reinforcement per unit length of beam is

$$\frac{A_v}{s} = \frac{V_s}{f_{yt} d}$$

d. Check A_v/s calculated with the minimum A_v/s:

$$(\text{min})\frac{A_v}{s} = 0.75\sqrt{f'_c}\left(\frac{b_w}{f_{yt}}\right) \geq 50\left(\frac{b_w}{f_{yt}}\right)$$

The minimum A_v, specified by the code under the combined action of shear and torsion, is given in step 5.

4. Design for torsion:

a. Check if the factored torsional moment, T_u, causes equilibrium or compatibility torsion. For equilibrium torsion, use T_u. For compatibility torsion, the design torsional moment is the smaller of T_u from factored load and

$$T_{u2} = \phi 4\lambda\sqrt{f'_c}\left(\frac{A^2_{cp}}{P_{cp}}\right) \tag{15.19}$$

b. Check that the size of the section is adequate. This is achieved by checking either Eq. 15.21 for solid sections or Eq. 15.22 for hollow sections. If the left-hand-side value is greater than $\phi(V_c/b_w d + 8\sqrt{f'_c})$, then increase the cross-section. If it is less than that value, proceed. For hollow sections, check if the wall thickness t is less than A_{oh}/P_h. If it is less, use Eq. 15.23 instead of Eq. 15.22; otherwise, use Eq. 15.22.

c. Determine the closed stirrups required from Eq. 15.25:

$$\frac{A_t}{s} = \frac{T_n}{2A_0 f_{yt} \cot\theta} \tag{15.25}$$

A_t/s should not be less than 25 b_w/f_{yt}. Also, the angle θ may be assumed to be 45°, $T_n = T_u/\phi$, and $\phi = 0.75$.

Assume $A_0 = 0.85\ A_{oh} = 0.85\ (x_1\ y_1)$, where x_1 and y_1 are the width and depth of the section to the centerline of stirrups; see Fig. 15.11. Values of A_0 and θ may be obtained from analysis [18]. For $\theta = 45°$ and $f_y = 60$ ksi,

$$\frac{A_t}{s} = \frac{T_n}{120A_0} \tag{15.26}$$

The maximum allowable spacing, s, is the smaller of 12 in. or $P_h/8$.

d. Determine the additional longitudinal reinforcement:

$$A_l = \left(\frac{A_t}{s}\right) P_h \left(\frac{f_{yt}}{f_y}\right) \cot^2\theta \tag{15.27}$$

but not less than

$$A_{l\,\min} = \left(\frac{5\sqrt{f'_c}A_{cp}}{f_y}\right) - \left(\frac{A_t}{s}\right) P_h \left(\frac{f_{yt}}{f_y}\right) \tag{15.30}$$

For $\theta = 45°$ and $f_{yt} = 60$ ksi, then $A_l = (A_t/s)P_h$. (15.28)

Bars should have a diameter of at least stirrup spacing, $s/24$, but not less than no. 3 bars. The longitudinal bars should be placed inside the closed stirrups with maximum spacing of 12 in. At least one bar should be placed at each corner of stirrups. Normally, one-third of A_l is added to the tension reinforcement, one-third at midheight of the section, and one-third at the compression side.

5. Determine the total area of closed stirrups due to V_u and T_u.

$$A_{vt} = (A_v + 2A_t) \geq \frac{50\ b_w s}{f_{yt}} \tag{15.29}$$

Choose proper closed stirrups with a spacing s as the smaller of 12 in. or $P_h/8$.

The stirrups should be extended a distance $(b_t + d)$ beyond the point theoretically no longer required, where b_t = width of cross-section resisting torsion.

Example 15.3: (Equilibrium Torsion)

Determine the necessary web reinforcement for the rectangular section shown in Fig. 15.15. The section is subjected to an factored shear $V_u = 48$ K and an equilibrium torsion $T_u = 360$ K·in at a section located at a distance d from the face of the support. Use normal-weight concrete with $f'_c = 4$ ksi, and $f_y = 60$ ksi.

Solution

The following steps explain the design procedure:

1. Design forces are $V_u = 48$ K and an equilibrium torsion $T_u = 400$ K·in.
2. a. Shear reinforcement is needed when $V_u > \phi V_c/2$.

$$\phi V_c = \phi 2\lambda\sqrt{f'_c}bd = 0.75(2)(1)\sqrt{4000}(16)(20.5) = 31.1\text{ K}$$

$$V_u = 48\text{ K} > \frac{\phi V_c}{2} = 15.55\text{ K}$$

Shear reinforcement is required.

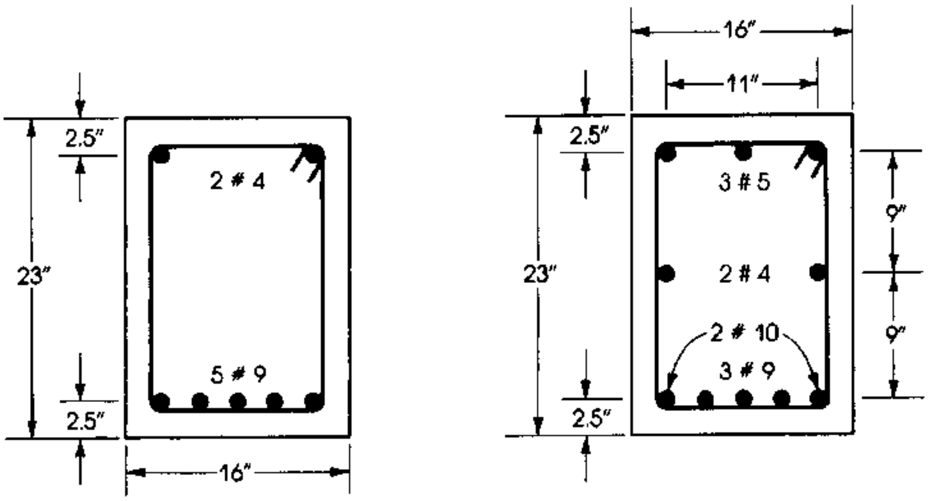


Figure 15.15 Example 15.3.

b. Torsional reinforcement is needed when

$$T_u > \phi\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{P_{cp}}\right) = T_a$$

$$A_{cp} = x_0 y_0 = 16(23) = 368 \text{ in.}^2$$

$$P_{cp} = 2(x_0 + y_0) = 2(16 + 23) = 78 \text{ in.} \qquad (15.20)$$

$$T_a = \frac{0.75(1)\sqrt{4000}(368)^2}{78} = 82.36 \text{ K·in.}$$

$$T_u = 360 \text{ K·in.} > 82.36 \text{ K·in.}$$

Torsional reinforcement is needed. Note that if T_u is less than 82.36 K·in., torsional reinforcement is not required, but shear reinforcement may be required.

3. Design for shear:

a. $V_u = \phi V_c + \phi V_s$, $\phi V_c = 35.26$ K. $48 = 31.1 + 0.75V_s$, $V_s = 22.5$ K

b. Maximum $V_s = 8\sqrt{f'_c}bd = 8\sqrt{4000}(16)(20.5) = 166\text{ K} > V_s$.

c. $A_v/s = V_s/f_y d = 22.5/(60 \times 20.5) = 0.018 \text{ in.}^2/\text{in.}$ (two legs)

$A_v/2s = 0.018/2 = 0.009 \text{ in.}^2/\text{in.}$ (one leg)

4. Design for torsion:

a. Design $T_u = 360$ K·in. Determine sectional properties, assuming 1.5-in. concrete cover and no. 4 stirrups:

$$x_1 = \text{width to center of stirrups} = 16 - 2(1.5 + 0.25) = 12.5 \text{ in.}$$

$$y_1 = \text{depth to center of stirrups} = 23 - 2(1.5 + 0.25) = 19.5 \text{ in.}$$

Practically, x_1 can be assumed to be $b - 3.5$ in. and $y_1 = h - 3.5$ in.

$$A_{oh} = x_1 y_1 = (12.5 \times 19.5) = 244 \text{ in.}^2$$

$$A_0 = 0.85 A_{oh} = 207.2 \text{ in.}^2$$

$$P_h = 2(x_1 + y_1) = 2(12.5 + 19.5) = 64 \text{ in.}$$

For $\theta = 45°$ and $\cot\theta = 1.0$.

b. Check the adequacy of the size of the section using Eq. 15.21:

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u P_h}{1.7A_{oh}^2}\right)^2} \leq \phi\left[\left(\frac{V_c}{b_w d}\right) + 8\sqrt{f_c'}\right]$$

$$\phi V_c = 31.1 \text{ K} \quad \text{and} \quad V_c = 41.5 \text{ K}$$

$$\text{Left-hand side} = \sqrt{\left(\frac{48{,}000}{16 \times 20.5}\right)^2 + \left(\frac{360{,}000 \times 64}{1.7(244)^2}\right)^2} = 271 \text{ psi}$$

$$\text{Right-hand side} = 0.75\left(\frac{41{,}500}{16 \times 20.5} + 8\sqrt{4000}\right) = 475 \text{ psi} > 271 \text{ psi}$$

The section is adequate.

c. Determine the required closed stirrups due to torsion from Eq. 15.25:

$$\frac{A_t}{s} = \frac{T_n}{2A_0 f_{yt} \cot\phi}$$

$$T_n = \frac{T_u}{\phi} = \frac{360}{0.75} = 480 \text{ K·in.} \quad \cot\theta = 1.0 \quad \text{and} \quad A_0 = 207.2 \text{ in.}^2$$

$$\frac{A_t}{s} = \frac{480}{2 \times 207.2 \times 60} = 0.019 \text{ in.}^2/\text{in.} \quad \text{(per one leg)}$$

d. Determine the additional longitudinal reinforcement from Eq. 15.27:

$$A_l = \left(\frac{A_t}{s}\right) P_h \left(\frac{f_{yt}}{f_y}\right) \cot^2\theta$$

$$\frac{A_t}{s} = 0.019, \quad P_h = 64 \text{ in.} \quad f_{yt} = f_y = 60 \text{ ksi} \quad \cot\theta = 1.0$$

$$A_l = 0.019(64) = 1.21 \text{ in.}^2$$

$$\text{Min. } A_l = 5\sqrt{f_c'}A_{cp}/f_y - \left(\frac{A_t}{s}\right) P_h \left(\frac{f_{yt}}{f_y}\right)$$

$$A_{cp} = 368 \text{ in.}^2 \quad \frac{A_t}{s} = 0.019$$

$$f_{yt} = f_{yl} = 60 \text{ Ksi}$$

$$\text{Min. } A_l = \left[\frac{5\sqrt{4000}(368)}{60{,}000}\right] - (0.019 \times 64 \times 1.0) = 0.72 \text{ in.}^2$$

$$A_l = 1.21 \text{ in.}^2 \text{ controls}$$

5. Determine total area of closed stirrups:

a. For one leg of stirrups, $A_{vt}/s = A_t/s + A_v/2s$.

$$\text{Required } A_{vt} = \frac{0.018}{2} + 0.019 = 0.028 \text{ in.}^2/\text{in.} \quad \text{(per one leg)}$$

Using no. 4 stirrups, area of one leg is 0.2 in.^{2}

$$\text{Spacing of stirrups} = \frac{0.2}{0.028} = 7.14 \text{ in.} \quad \text{or} \quad 7.0 \text{ in.}$$

b. Maximum $s = P_h/8 = \frac{64}{8} = 8$ in. or 12 in., whichever is smaller. The value of s used is 7.0 in. < 8 in.

c. Minimum $A_{vt}/s = 50b_w/f_{yt} = 50(16)/60{,}000 = 0.0133$ in.^{2}/in. This is less than 0.028 in.^{2}/in. provided.

6. To find the distribution of longitudinal bars, note that total $A_l = 1.21$ in.^{2} Use one-third at the top, or 1.21/3 = 0.4 in.^{2}, to be added to the compression steel A'_s. Use one-third, or 0.4 in.^{2}, at the bottom, to be added to the tension steel, and one-third, or 0.4 in.^{2}, at middepth.

   a. The total area of top bars is 0.4(2 no. 4) + 0.4 = 0.8 in.^{2}; use three no. 5 bars (A_s = 0.91 in.^{2}).

   b. The total area of bottom bars is 5 (five no. 9) + 0.4 = 5.4 in.^{2}; use three no. 9 and two no. 10 bars at the corners (total A_s = 5.53 in.^{2}).

   c. At middepth, use two no. 4 bars (A_s = 0.4 in.^{2}).

   Reinforcement details are shown in Fig. 15.15. Spacing of longitudinal bars is equal to 9 in., which is less than the maximum required of 12 in. The diameter of no. 4 bars used is greater than the minimum of no. 3 or stirrup spacing, or $s/24 = 0.21$ in.

Example 15.4: Compatibility Torsion

Repeat Example 15.3 if the factored torsional torque is a compatibility torsion.

Solution

Referring to the solution of Example 15.3,

1. Design forces are $V_u = 48$ K and compatibility torsion is 360 K·in.
2. Steps (a) and (b) are the same as in Example 15.3. Web reinforcement is required.
3. Step (c) is the same.
4. Design for torsion:
   Because this is a compatibility torsion of 360 K in., the design T_u is the smaller of 360 K in. or ϕT_{cr} given in Eq. 15.19.

$$\phi T_{cr} = \phi 4\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{P_{cp}}\right) = \frac{0.75(4)(1)\sqrt{4000}(368)^2}{78} = 329.4 \text{ K·in.} \qquad (15.19)$$

   Because $\phi T_{cr} < 360$ K·in., use T_u = 329.4 K·in. Repeat all the steps of Example 15.3 using T_u = 329.3 K·in. to determine that the section is adequate.

$$\frac{A_t}{s} = 0.018 \text{ in.}^2/\text{in.} \qquad \text{(one leg)}$$

$$A_l = 0.018(64) = 1.152 \text{ in.}^2$$

   Use 1.2 in.^{2} > min. A_l.

5. Required $A_{vt} = 0.018/2 + 0.018 = 0.027$ in.^{2} in. (one leg).

$$s = \frac{0.2}{0.027} = 7.4 \text{ in.}$$

   Use 7 in. Choose bars, stirrups, and spacing similar to Example 15.3.

Example 15.5: L-Section with Equilibrium Torsion

The edge beam of a building floor system is shown in Fig. 15.16. The section at a distance d from the force of the support is subjected to $V_u = 53$ K and an equilibrium torque $T_u = 240$ K·in. Design the necessary web reinforcement using $f'_c = 4$ ksi and $f_y = 60$ ksi for all steel bars and stirrups.

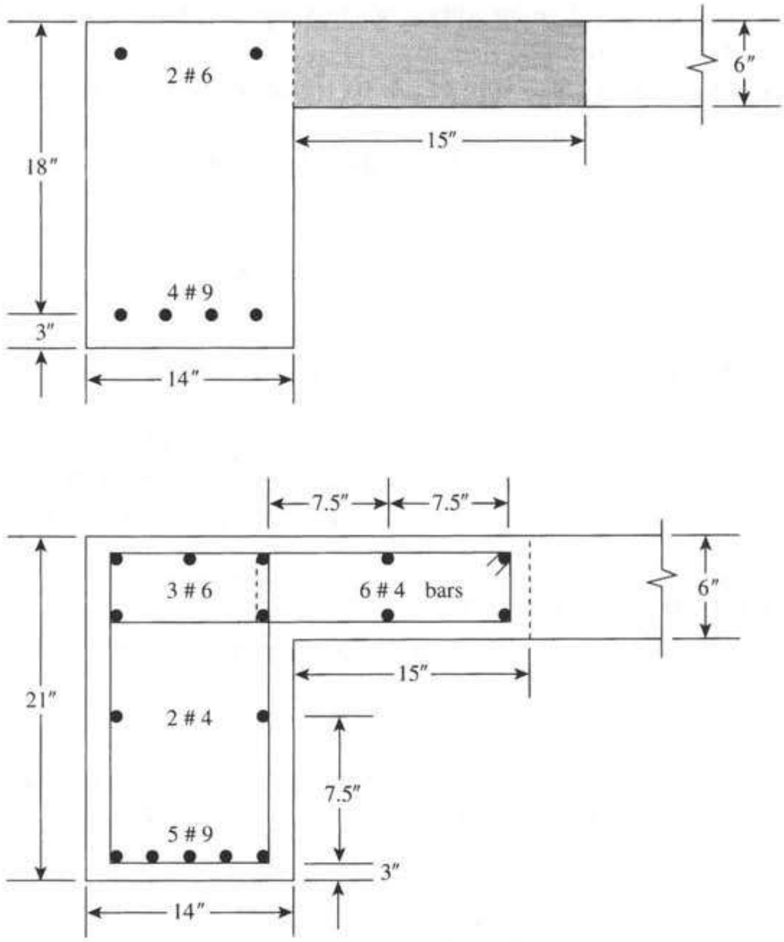


Figure 15.16 Example 15.5.

Solution

1. Design forces are $V_u = 60\,\text{K}$ and $T_u = 270\,\text{K·in.} = 22.5\,\text{K·ft.}$
2. **a.** Shear reinforcement is needed when $V_u > \phi\ V_c/2$.

$$\phi V_c = \phi 2\lambda\sqrt{f_c'}b_w d = 0.75(2)(1)\sqrt{4000}(14)(18) = 23.9\,\text{K}$$

$$V_u > \frac{\phi V_c}{2} = 11.95\,\text{K}$$

Shear reinforcement is required.

b. Check if torsional reinforcement is needed. Assuming that flange is contributing to resist torsion, the effective flange length is $h_w = 15\,\text{in.} < 4 \times 6 = 24\,\text{in.}$

$$x_0 = 14 \text{ in.} \quad \text{and} \quad y_0 = 21 \text{ in.}$$

$$A_{cp} = (14 \times 21)(\text{web}) + (15 \times 6)(\text{flange}) = 384 \text{ in.}^2$$

$$P_{cp} = 2(21 + 29) = 100 \text{ in.}$$

$$T_a \text{ (Eq. 15.20)} = \frac{0.75(1)\sqrt{4000}(384)^2}{100} = 70\,\text{K·in.}$$

$$T_u > T_a$$

Torsional reinforcement is required.

3. Design for shear:

a.
$$V_u = \phi V_c + \phi V_s$$
$$53 = 23.9 + 0.75 V_s$$
$$V_s = 38.8\,\text{K}$$

b. Maximum $V_s = 8\sqrt{f_c'} b_w d = 127.5\,\text{K} > V_s$

c.
$$A_v/s = \frac{V_s}{f_y d} = \frac{38.8}{60 \times 18} = 0.036 \text{ in.}^2/\text{in.} \qquad \text{(two legs)}$$
$$A_v/2s = \frac{0.036}{2} = 0.018 \text{ in.}^2/\text{in.}$$

4. Design for torsion: $T_u = 240\,\text{K·in}$.

a. Determine section properties assuming a concrete cover of 1.5 in. and no. 4 stirrups.

Web x_1	$= b - 3.5 \text{ in.} = 14 - 3.5 = 10.5$ in.	y_1	$= h - 3.5 = 21 - 3.5 = 17.5$ in.	
Flange x_1	$=$ 15 in. (stirrups extend to the web)	y_1	$= 6 - 3.5 = 2.5$ in.	
A_{oh}	$= (15 \times 2.5) + (10.5 \times 17.5) = 221 \text{ in.}^2$	A_0	$= 0.85 A_{oh} = 188 \text{ in.}^2$	
P_h	$= 2(15 + 2.5) + 2(10.5 + 17.5) = 91$ in.	θ	$= 45°$	$\cot\theta = 1.0$

b. Check the adequacy of the section using Eq. 15.21: $V_u = 53\,\text{K}$, $\phi\, V_c = 23.9\,\text{K}$ $V_c = 31.9\,\text{K}$ $T_u = 240\,\text{K·in}$.

$$\text{left-hand side} = \sqrt{\left(\frac{53{,}000}{14 \times 18}\right)^2 + \left[\frac{240{,}000 \times 91}{1.7(184)^2}\right]^2} = 434\,\text{psi}$$

$$\text{right-hand side} = 0.75\left[\frac{31{,}900}{14 \times 18} + 8\sqrt{4000}\right] = 475\,\text{psi}$$

The section is adequate.

c. Determine the torsional closed stirrups, A_t/s, from Eq. 15.25:

$$\frac{A_t}{s} = \frac{T_n}{2A_0 f_{yt}} = \frac{240}{0.75 \times 2 \times 188 \times 60} = 0.014 \text{ in.}^2/\text{in.} \qquad \text{(for one leg)}$$

d. Calculate the additional longitudinal reinforcement from Eq. 15.28 (for $f_y = 60\,\text{ksi}$ and $\cot\theta = 1.0$):

$$A_l = \left(\frac{A_t}{s}\right) P_h = 0.014(91) = 1.28 \text{ in.}^2$$

$A_{l\,\min}$ (from Eq. 15.30) is

$$A_l = \left[\frac{5\sqrt{4000}(384)}{60{,}000}\right] - (0.014 \times 91) = 0.75 \text{ in.}^2$$

The contribution of the flange may be neglected with slight difference in results, and less labor cost.

5. Determine the total area of the closed stirrups.

a. For one leg, $A_{vt}/s = A_t/s + A_v/2s$.

$$\text{Required } A_{vt} = 0.014 + 0.018 = 0.032 \text{ in.}^2/\text{in.} \quad \text{(per leg)}$$

Choose no. 4 closed stirrups, area $= 0.2 \text{ in.}^2$

$$\text{Spacing of stirrups} = \frac{0.2}{0.032} = 6.25 \text{ in.}$$

Use 6 in.

b. Max. $s = P_h/8 = 91/8 = 11.4$ in. Use $s = 6$ in., as calculated.

c. $A_{vt}/s = 50b_w/f_{yt} = 50(14)/60{,}000 = 0.017$ in.^{2}/in., which is less than the 0.032 in.^{2}/in. used. Use no. 4 closed stirrups spaced at 6 in.

6. Find the distribution of longitudinal bars. Total A_l is 1.28 in.^{2} Use one-third, or 0.43 in.^{2}, at the top, at the bottom, and at middepth.

a. Total top bars $= 0.88 + 0.43 = 1.31$ in.^{2}; use three no. 6 bars (1.32 in.^{2}).

b. Total bottom bars $= 4.0 + 0.43 = 4.43$ in.^{2}; use five no. 9 bars (5.0 in.^{2}).
Total A_l used $= (1.32 - 0.88) + (5 - 4) = 1.44$ in.^{2}

c. Use two no. 4 bars at middepth (0.40 in.^{2}). Reinforcement details are shown in Fig. 15.16. Spacing of longitudinal bars is at 7.5 in. < 12 in. The diameter of no. 4 bars used is 0.5 in., which is greater than no. 3 or stirrup spacing, $s/24 = \frac{6}{24} = 0.25$ in. Add no. 4 longitudinal bars on all corners of closed stirrups in beam web and flange.

SUMMARY

Sections 15.1–15.7

1. Torsional stresses develop in a beam when a moment acts on the beam section parallel to its surface.
2. In most practical cases, a structural member may be subjected to combined shear and torsional moments.
3. The design methods for torsion rely generally on two basic theories: the skew bending theory and the space truss theory. The ACI Code adopted the space truss theory.

Sections 15.8–15.9

A summary of the relative equations in U.S. customary units and SI units is given here.

Note that $(1.0\sqrt{f'_c})$ in psi is equivalent to $(0.08\sqrt{f'_c})$ in MPa N/mm^2, 1 in. ≈ 25 mm, and $f_{yt} \le 400$ MPa.

Equation	U.S. Customary Units	SI Units
15.16	$T_c = 2\phi\sqrt{f'_c}\sum x^2y$	$T_c = 0.17\phi\sqrt{f'_c}\sum x^2y$
15.17	$\phi T_n \ge T_u$	Same
15.19	$T_{cr} = 4\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{P_{cp}}\right)$	$T_{cr} = (\lambda\sqrt{f'_c}/3)(A_{cp}^2/P_{cp})$
15.20	$T_u = \phi\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{P_{cp}}\right)$	$T_u \le \phi\lambda(\sqrt{f'_c}/12)(A_{cp}^2/P_{cp})$
15.21	$\sqrt{\left(\frac{V_u}{b_wd}\right)^2 + \left(\frac{T_uP_h}{1.7A_{oh}^2}\right)^2} \le \phi\left[\left(\frac{V_c}{b_wd}\right) + 8\sqrt{f'_c}\right]$	(U.S.)
	$\sqrt{\left(\frac{V_u}{b_wd}\right)^2 + \left(\frac{T_uP_h}{1.7A_{oh}^2}\right)^2} \le \phi\left[\left(\frac{V_c}{b_wd}\right) + (2\sqrt{f'_c}/3)\right]$	(SI)
15.24	$T_n = \frac{2A_0A_tf_{yt}\cot\theta}{s}$	Same

(Note that f_{yt} is in MPa, S is in mm, A_0 and A_t are in mm^2, and T_n is in kN m.)

Equation	U.S. Customary Units	SI Units
15.25	$\frac{A_t}{s} = \frac{T_n}{2A_0 f_{yt} \cot\theta}$	Same
15.27	$A_l = \frac{A_t P_h (f_{yt}/f_y) \cot^2\theta}{s}$	Same
15.29	$A_v + 2A_t \geq \frac{50 b_w s}{f_{yt}}$	$(A_v + 2A_t) \geq 0.35 b_w s / f_{yt}$
15.30	$A_{l\,\min} = \left[\frac{5\sqrt{f'_c}A_{cp}}{f_y}\right]$	$A_{l\,\min} = [(5\sqrt{f'_c}A_{cp})/12 f_y]$
	$-(A_t/s)P_h(f_{yt}/f_y)$	$-\left(\frac{A_t}{s}\right)P_h\left(\frac{f_{yt}}{f_y}\right)$

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PROBLEMS

For each problem, compute the cracking moment ϕT_{cr} and the maximum factored torque ϕT_n that can be applied without using torsional web reinforcement. Use $f'_c = 4$ksi and $f_y = 60$ ksi.

15.1 A rectangular section with $b = 16$ in. and $h = 24$ in.

15.2 A rectangular section with $b = 12$ in. and $h = 20$ in.

15.3 A T-section with $b = 48$ in., $b_w = 12$ in., $t = 4$ in., and $h = 25$ in. Assume flanges are confined with closed stirrups.

15.4 A T-section with $b = 60$ in., $b_w = 16$ in., $t = 4$ in., and $h = 30$ in. Assume flanges are confined with closed stirrups.

15.5 An inverted L-section with $b = 32$ in., $b_w = 14$ in., $t = 6$ in., and $h = 24$ in. The flange does not have closed stirrups.

15.6 An inverted L-section with $b = 40$ in., $b_w = 12$ in., $t = 6$ in., and $h = 30$ in. The flange contains confined closed stirrups.

15.7 Determine the necessary web reinforcement for a simple beam subjected to an equilibrium factored torque $T_u = 220$ K·in. and $V_u = 36$ K. The beam section has $b = 14$ in., $h = 22$ in., and $d = 19.5$ in., and is reinforced on the tension side by four no. 9 bars. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

15.8 Repeat Problem 15.7 using $f'_c = 5$ ksi and $f_y = 60$ ksi.

15.9 The section of an edge (spandrel) beam is shown in Fig. 15.17. The critical section of the beam is subjected to an equilibrium torque $T_u = 300$ K·in and a shear $V_u = 60$ K. Determine the necessary web reinforcement using $f'_c = 4$ ksi and $f_y = 60$ ksi. Consider that the flange is not reinforced with closed stirrups.

15.10 Repeat Problem 15.9. Considering that the flange is effective and contains closed stirrups.

15.11 The T-section shown in Fig. 15.18 is subjected to a factored shear $V_u = 28$ K and a factored equilibrium torque $T_u = 300$ K·in. and $M_u = 250$ K·ft. Design the necessary flexural and web reinforcement. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

15.12 Repeat Problem 15.11 if $V_u = 36$ K, $T_u = 360$ K·in., $M_u = 400$ K·ft., and $h = 24$ in.

15.13 Repeat Problem 15.11 using $f'_c = 3$ ksi and $f_y = 60$ ksi.

15.14 Repeat Problem 15.11 if T_u is a compatibility torsion.

15.15 Repeat Problem 15.13 if T_u is a compatibility torsion.

15.16 Repeat Problem 15.7 if T_u is a compatibility torsion.

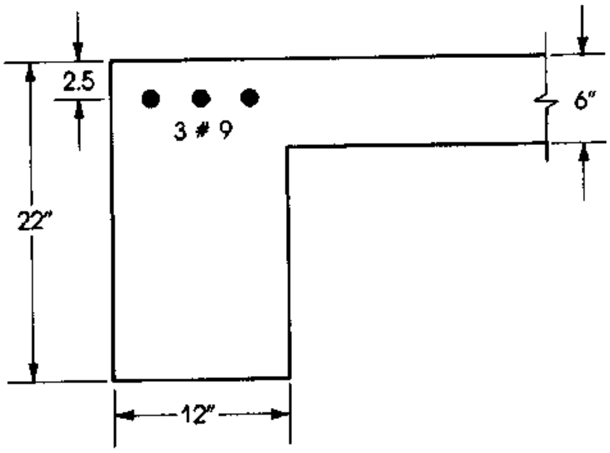


Figure 15.17 Problem 15.9.

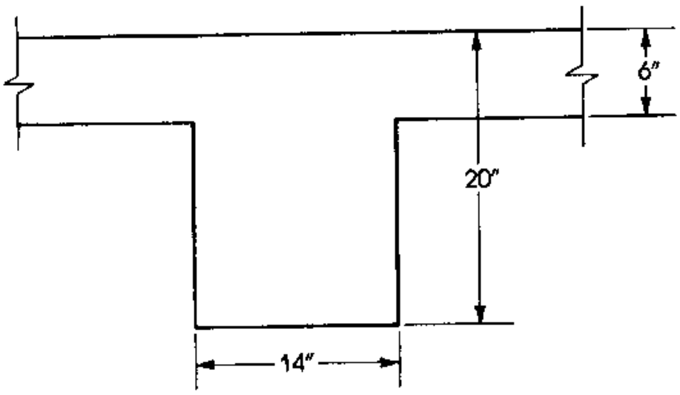


Figure 15.18 Problem 15.11.

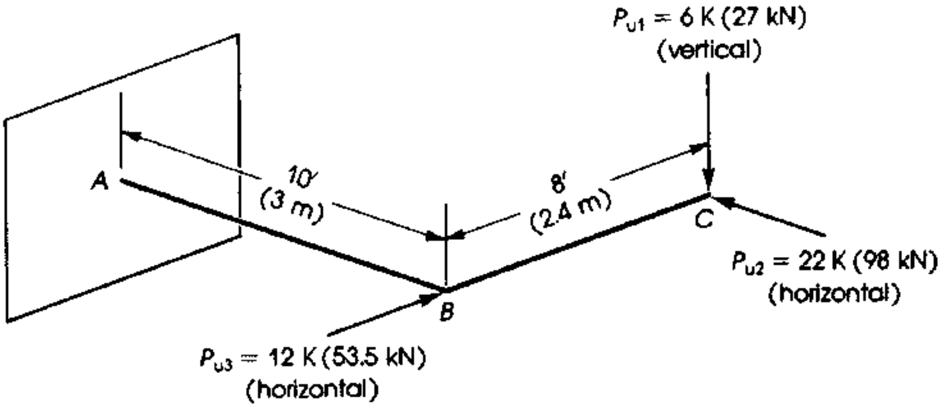


Figure 15.19 Problem 15.17.

15.17 The cantilever beam shown in Fig. 15.19 is subjected to the factored load shown.

a. Draw the axial and shearing forces and the bending and torsional moment diagrams.

b. Design the beam section at A using a steel percentage less than or equal to ρ_{max} for bending moment. Use $b = 16$ in. (300 mm), $f'_c = 4$ ksi, and $f_y = 60$ ksi.

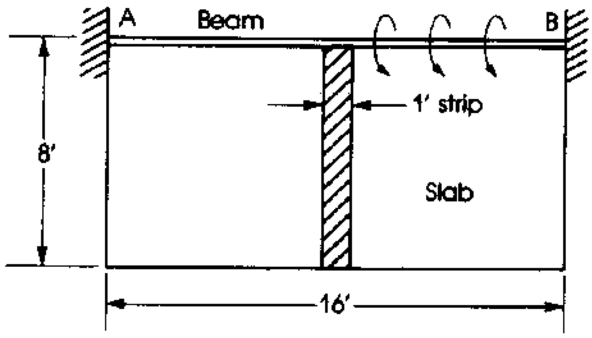


Figure 15.20 Problem 15.18.

15.18 The size of the slab shown in Fig. 15.20 is 16 by 8 ft; it is supported by the beam AB, which is fixed at both ends. The uniform dead load on the slab (including its own weight) equals 100 psf, and the uniform live load equals 80 psf. Design the section at support A of beam AB using $f'_c = 4$ ksi, $f_y = 60$ ksi, $b_w = 14$ in., $h = 20$ in., a slab thickness of 5 in., and the ACI Code requirements.

CHAPTER 16

CONTINUOUS BEAMS AND FRAMES

Reinforced concrete parking structure, Minneapolis, Minnesota.

16.1 INTRODUCTION

Reinforced concrete buildings consist of different types of structural members, such as slabs, beams, columns, and footings. These structural members may be cast in separate units as precast concrete slabs, beams, and columns or with the steel bars extending from one member to the other, forming a monolithic structure. Precast units are designed as structural members on simple supports unless some type of continuity is provided at their ends. In monolithic members, continuity in the different elements is provided, and the structural members are analyzed as statically indeterminate structures.

The analysis and design of continuous one-way slabs were discussed in Chapter 9, and the design coefficients and reinforcement details were shown in Figs. 9.8 and 9.9. In one-way floor systems, the loads from slabs are transferred to the supporting beams, as shown in Fig. 16.1*a*. If the factored load on the slab is w_u psf, the uniform load on beams *AB* and *BC* per unit length is $w_u s$ plus the self-weight of the beam. The uniform load on beams *DE* and *EF* is $w_u s/2$ plus the self-weight of the beam. The load on column *B* equals $W_u LS$, whereas the loads on columns *E*, *A*, and *D* are $W_u LS/2$, $W_u SL/2$, and $W_u LS/4$, respectively.

In two-way rectangular slabs supported by adequate beams on four sides, the floor loads are transferred to the beam from tributary areas bounded by 45° lines, as shown in Fig. 16.1*b*. Part of the floor loads are transferred to the long beams *AB, BC, DE*, and *EF* from trapezoidal areas, whereas the rest of the floor loads are transferred to the short beams *AD, BE*, and *CF* from triangular areas. In square slabs, loads are transferred to all surrounding beams from triangular floor areas. Interior beams carry loads from both sides, whereas end beams carry loads from one side only. Beams in both directions are usually cast monolithically with the slabs; therefore, they should be analyzed as statically indeterminate continuous beams. The beams transfer their loads in turn to the supporting columns. The load on column *B* equals $W_u LS$, while the loads on columns *E*, *A*, and *D* are $W_u LS/2$, $W_u SL/2$, and $W_u LS/4$, respectively. The tributary area for each column extends from the centerlines of adjacent spans in each direction.

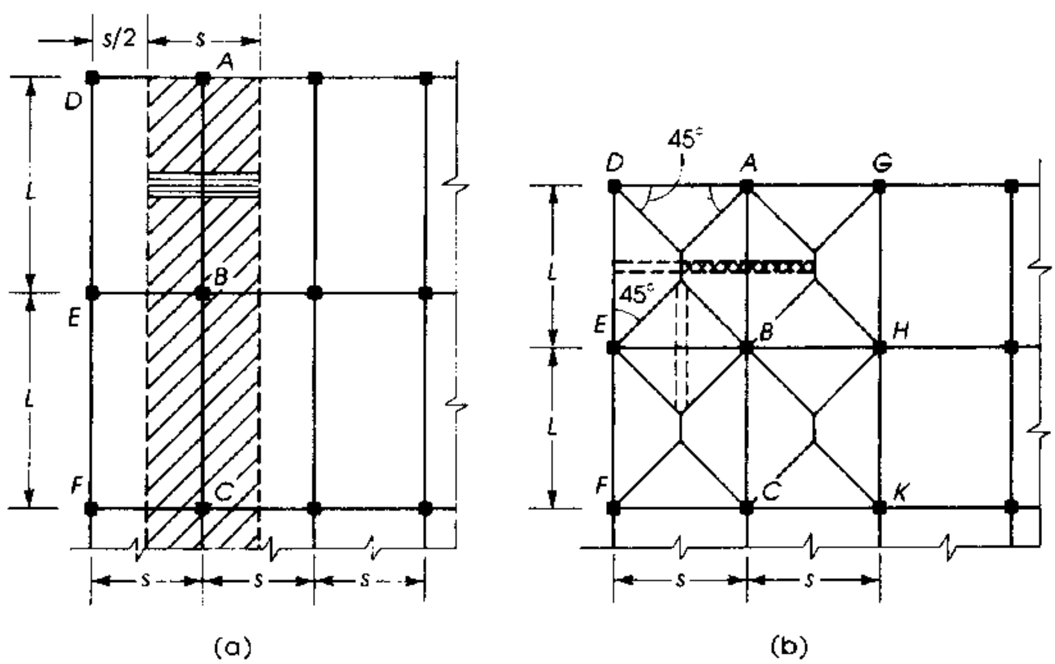


Figure 16.1 Slab loads on supporting beams: (*a*) one-way direction, $L/s > 2$; and (*b*) two-way direction, $L/s \leq 2$.

16.2 MAXIMUM MOMENTS IN CONTINUOUS BEAMS

16.2.1 Basic Analysis

The computation of bending moments and shear forces in reinforced concrete continuous beams is generally based on the elastic theory. When reinforced concrete sections are designed using the strength design method, the results are not entirely consistent with the elastic analysis. However, the ACI Code does not include provisions for a plastic design or limit design of reinforced concrete continuous structures except in allowing moment redistribution, as is explained later in this chapter.

16.2.2 Loading Application

The bending moment at any point in a continuous beam depends not only on the position of loads on the same span, but also on the loads on the other spans. In the case of dead loads, all spans must be loaded simultaneously, because the dead load is fixed in position and magnitude. In the case of moving loads or occasional live loads, the pattern of loading must be considered to determine the maximum moments at the critical sections. Influence lines may be used to determine the position of the live load to calculate the maximum and minimum moments. However, in this chapter, simple rules based on load-deflection curves are used to determine the loading pattern that produces maximum moments.

16.2.3 Maximum and Minimum Positive Moments within a Span

The maximum positive bending moment in a simply supported beam subjected to a uniform load w K/ft is at midspan, and $M = wl_2/8$. If one or both ends are continuous, the restraint at the continuous end will produce a negative moment at the support and slightly shift the

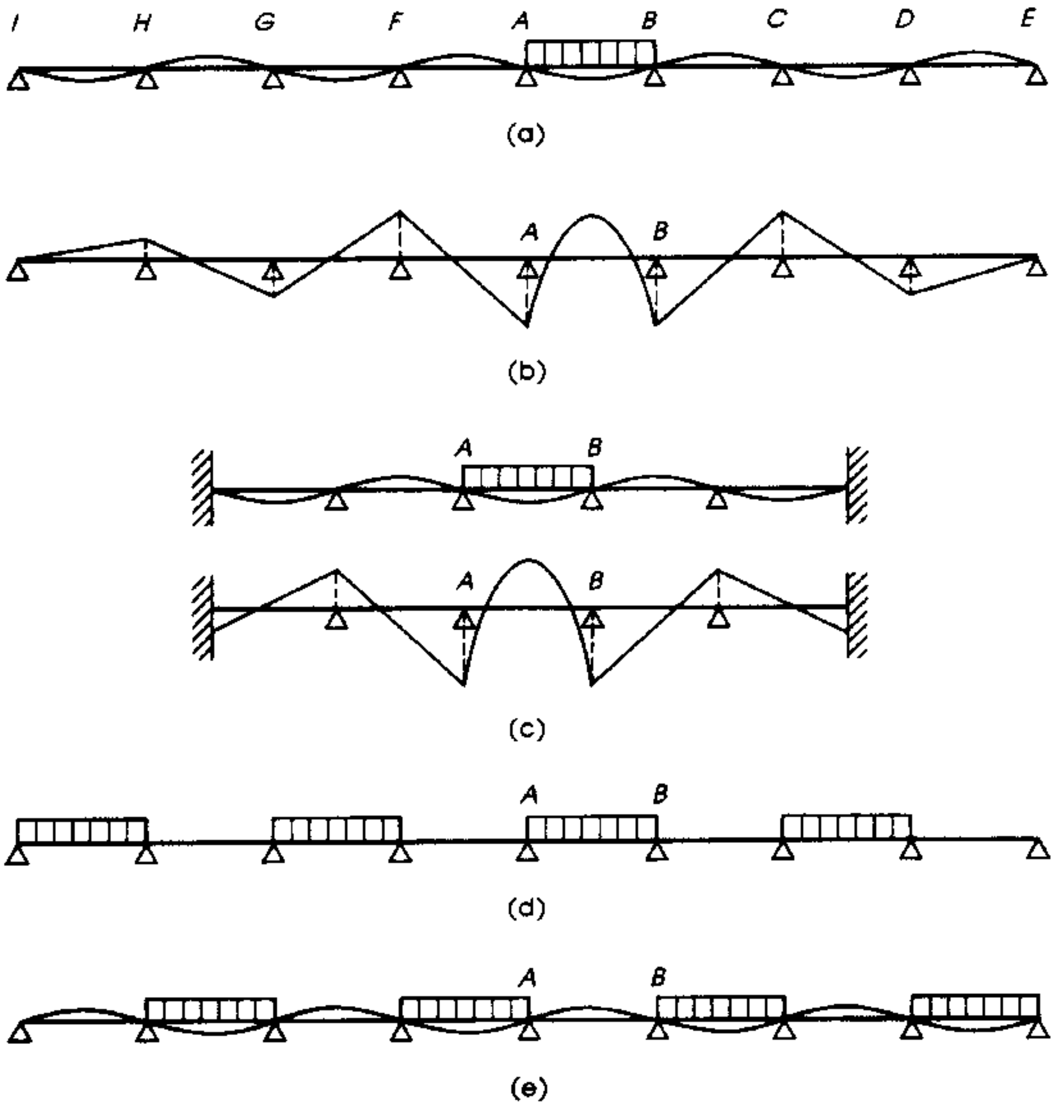


Figure 16.2 Loadings for maximum and minimum moment within span *AB*.

location of the maximum positive moment from midspan. The deflected shape of the continuous beam for a single-span loading is shown in Fig. 16.2*a*; downward deflection indicates a positive moment and upward deflection indicates a negative moment. If all spans deflected downward are loaded, each load will increase the positive moment at the considered span *AB* (Fig. 16.2*d*). Therefore, to calculate the maximum positive moment within a span, the live load is placed on that span and on every alternate span on both sides. The factored live load moment, calculated as explained before, must be added to the factored dead-load moment at the same section to obtain the maximum positive moment.

The bending moment diagram due to a uniform load on *AB* is shown in Fig. 16.2*b*. The deflections and the bending moments decrease rapidly with the distance from the loaded span *AB*. Therefore, to simplify the analysis of continuous beams, the moments in any span can be computed by considering the loaded span and two spans on either side of the considered span *AB*, assuming fixed supports at the far ends (Fig. 16.2*c*).

If the spans adjacent to span AB are loaded, the deflection curve will be as shown in Fig. 16.2e. The deflection within span AB will be upward, and a negative moment will be produced in span AB. This negative moment must be added to the positive moment due to dead load to obtain the final bending moment. Therefore, to calculate the minimum positive moment (or maximum negative moment) within a span AB, the live load is placed on the adjacent spans and on every alternate span on both sides of AB (Fig. 16.2e).

16.2.4 Maximum Negative Moments at Supports

In this case, it is required to determine the maximum negative moment at any support, say, support A (Fig. 16.3). When span AB is loaded, a negative moment is produced at support A. Similarly, the loading of span AF will produce a negative moment at A. Therefore, to calculate the maximum negative moment at any support, the live load is placed on the two adjacent spans and on every alternate span on both sides (Fig. 16.3).

In the structural analysis of continuous beams, the span length is taken from center to center of the supports, which are treated as knife-edge supports. In practice, the supports are always made wide enough to take the loads transmitted by the beam, usually the moments acting at the face of supports. To calculate the design moment at the face of the support, it is quite reasonable to deduct a moment equal to $V_u c/3$ from the factored moment at the centerline of the support, where V_u is the factored shear and c is the column width.

16.2.5 Moments in Continuous Beams

Continuous beams and frames can be analyzed using approximate methods or computer programs, which are available commercially. Other methods, such as the displacement and force methods of analysis based on the calculation of the stiffness and flexibility matrices, may also be adopted. Slope deflection and moment-distribution methods may also be used. These methods are explained in books dealing with the structural analysis of beams and frames. However, the ACI Code, Section 8.3, gives approximate coefficients for calculating the bending moments and shear forces in continuous beams and slabs. These coefficients were given in Chapter 9. The moments obtained using the ACI coefficients will be somewhat larger than those arrived at by exact analysis. The limitations stated in the use of these coefficients must be met.

Example 16.1

The slab-beam floor system shown in Fig. 16.4 carries a uniform live load of 130 psf and a dead load that consists of the slab's own weight plus 80 psf. Using the ACI moment coefficients, design a typical interior continuous beam and draw detailed sections. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, beam width $(b) = 12$ in., 12-by-12 in. columns, and a slab thickness of 5.0 in.

Solution

1. Design of slabs: The floor slabs act as one-way slabs, because the ratio of the long to the short side is greater than 2. The design of a typical continuous slab was discussed in Example 9.4.

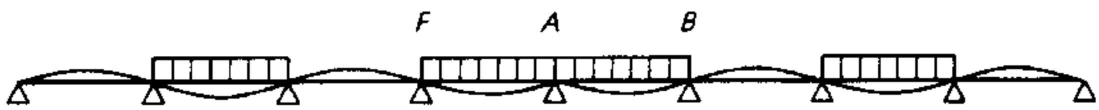


Figure 16.3 Loading for maximum negative moment at support A.

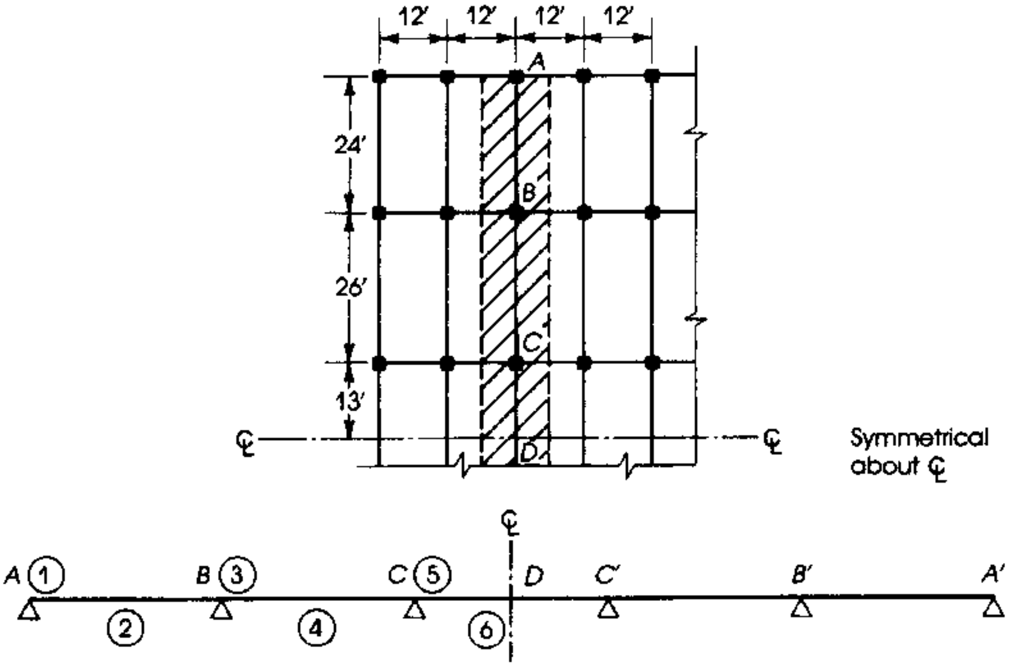


Figure 16.4 Example 16.1.

2. Loads on slabs:

$$\text{Dead load} = \frac{5}{12} \times 150 + 80 = 142.5 \text{ psf}$$

$$\text{Live load} = 130 \text{ psf}$$

$$\text{Factored load } (w_u) = 1.2(142.5) + 1.6(130) = 379 \text{ psf}$$

Loads on beams: A typical interior beam *ABC* carries slab loads from both sides of the beam, with a total slab width of 12 ft.

$$\text{Factored load on beam} = 12 \times 379 + 1.2 \times \text{ (self-wseight of beam web)}$$

The depth of the beam can be estimated using the coefficients of minimum thickness of beams shown in Table A.6. For $f_y = 60$ ksi, the minimum thickness of the first beam *AB* is $L/18.5 = (24 \times 12)/18.5 = 15.6$ in. Assume a total depth of 22 in. and a web depth of $22 - 5 = 17$ in. Therefore, the factored load on beam *ABCD* is

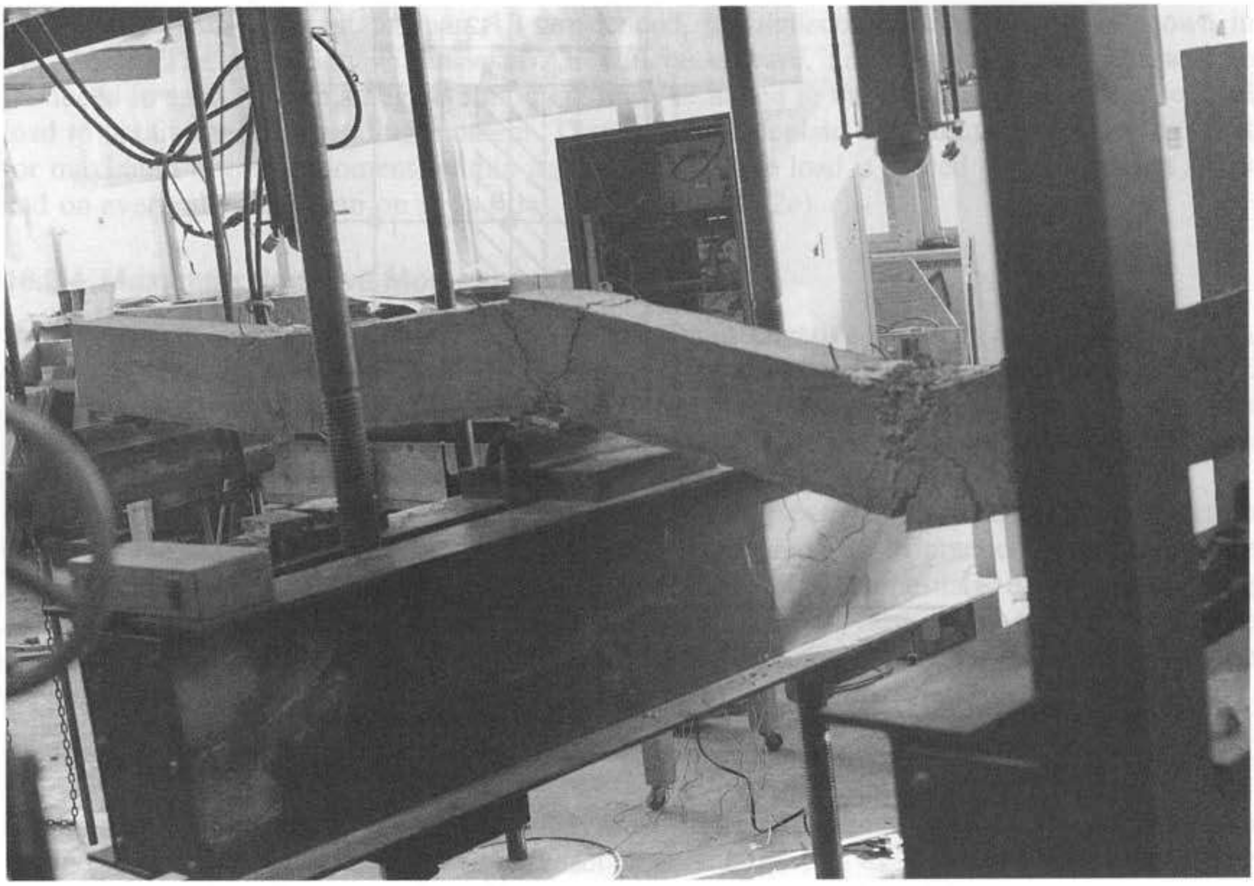
$$w_u = 12 \times 379 + 1.2\left(\frac{17 \times 12}{144} \times 150\right) = 4804 \text{ lb/ft}$$

Use 4.8 K/ft.

3. Moments in beam *ABC*: Moment coefficients are shown in Fig. 9.8. The beam is continuous on five spans and symmetrical about the centerline at *D*. Therefore, it is sufficient to design half of the beam *ABCD*, because the other half will have similar dimensions and reinforcement. Because the spans *AB* and *BC* are not equal and the ratio $\frac{26}{24}$ is less than 1.2, the ACI moment coefficients can be applied to this beam. Moreover, the average of the adjacent clear span is used to calculate the negative moments at the supports.

   Moments at critical sections are calculated as follows (Fig. 16.4):

$$M_u = \text{coefficient} \times w_u l_n^2$$



Test on a continuous reinforced concrete beam. Plastic hinges developed in the positive and negative maximum moment regions.

Location	**1**	**2**	**3**	**4**	**5**	**6**
Moment coefficient	$-\frac{1}{16}$	$+\frac{1}{14}$	$-\frac{1}{10}$	$+\frac{1}{16}$	$-\frac{1}{11}$	$+\frac{1}{16}$
M_u (K·ft)	−158.7	181.4	−276.5	187.5	−272.7	187.5

4. Determine beam dimensions and reinforcement.

a. Maximum negative moment is −276.5 K·ft. Using $\rho_{\max} = 0.016$, $R_u = 740$ psi.

$$R_{u\ \max} = 820 \text{ psi} \qquad \rho_{\max} = 0.01806 \text{ (Table 4.1)} \qquad \phi = 0.9$$

$$d = \sqrt{\frac{M_u}{R_u b}} = \sqrt{\frac{276.5 \times 12}{0.74 \times 12}} = 19.3 \text{ in.}$$

For one row of reinforcement, total depth is $19.3 + 2.5 = 21.8$ in., say, 23 in., and actual d is 20.5 in. $A_s = 0.016 \times 12 \times 19.3 = 3.7\,\text{in.}^2$; use four no. 9 bars in one row. Note that total depth used here is 23 in., which is more than the 22 in. assumed to calculate the weight of the beam. The additional load is negligible, and there is no need to revise the calculations.

b. The sections at the supports act as rectangular sections with tension reinforcement placed within the flange. The reinforcements required at the supports are as follows:

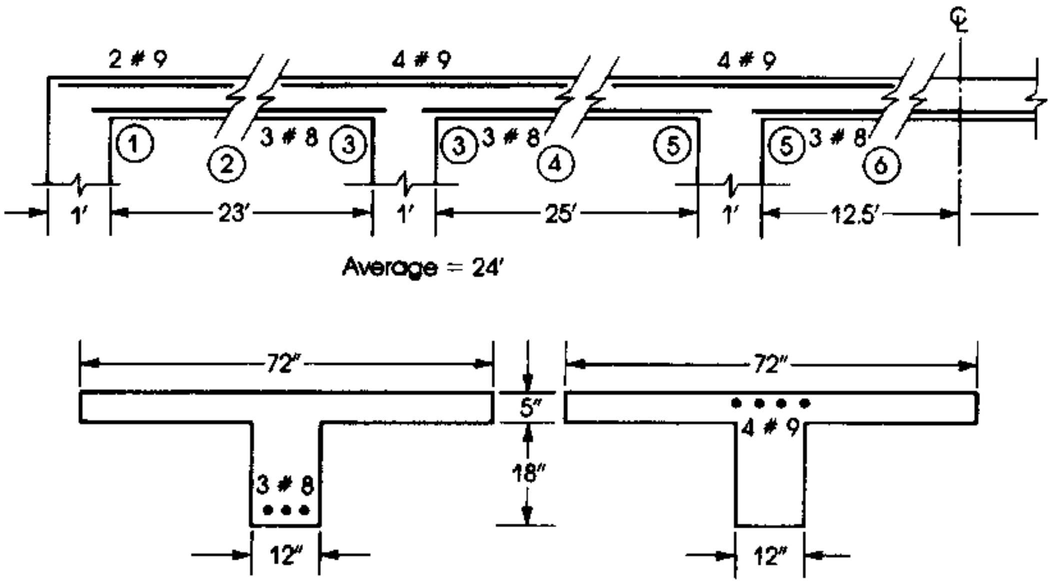


Figure 16.5 Example 16.1: reinforcement details.

Location	**1**	**3**	**5**
M_u(K·ft)	−158.7	−276.5	−272.7
$R_u = \dfrac{M_u}{bd^2}$ (psi)	378	658	649
ρ(%)	0.77	1.48	1.45
A_s(in.^{2})	1.9	3.7	3.6
No. 9 bars	2	4	4

c. For the midspan T-sections, $M_u = +187.5$ K·ft. For $a = 1.0$ in. and flange width = 72 in.,

$$A_s = \frac{M_u}{\phi f_y\left(d - \dfrac{a}{2}\right)} = \frac{187.5 \times 12}{0.9 \times 60(20.5 - 1/2)} = 2.1 \text{ in.}^2$$

$$\text{Check } a\text{: } a = \frac{A_s f_y}{0.85 f'_c b} = \frac{2.1 \times 60}{0.85 \times 3 \times 72} = 0.7 \text{ in.}$$

Revised a gives $A_s = 2.07\,\text{in.}^2$ Therefore, use three no. 8 bars ($A_s = 2.35\,\text{in.}^2$) for all midspan sections. Reinforcement details are shown in Fig. 16.5.

5. Design the beam for shear, as explained in Chapter 8.
6. Check deflection and cracking, as explained in Chapter 6.

16.3 BUILDING FRAMES

A building frame is a three-dimensional structural system consisting of straight members that are built monolithically and have rigid joints. The frame may be one bay long and one story high, such as the portal frames and gable frames shown in Fig. 16.6*a*, or it may consist of multiple

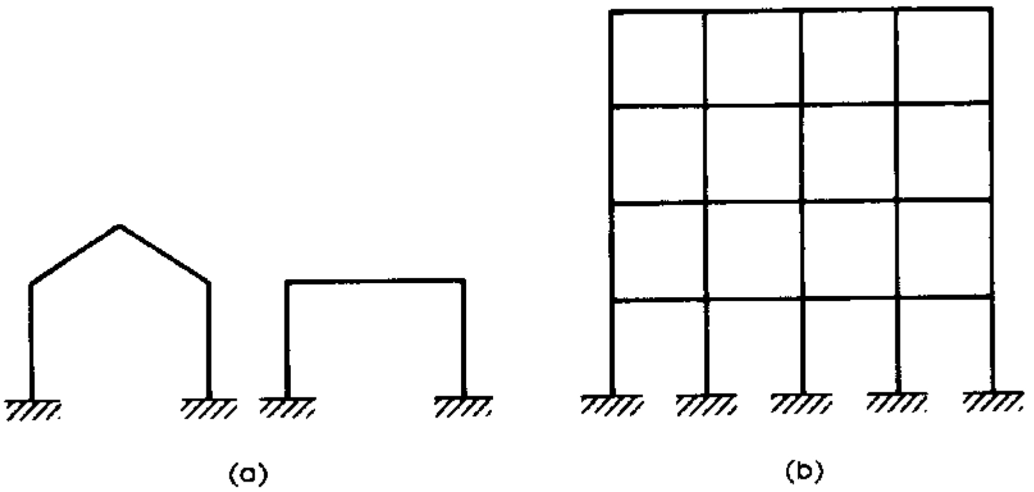


Figure 16.6 (*a*) Gable and portal frames (schematic) and (*b*) multibay, multistory frame.

bays and stories, as shown in Fig. 16.6*b*. All members of the frame are considered continuous in the three directions, and the columns participate with the beams in resisting external loads. Besides reducing moments due to continuity, a building frame tends to distribute the loads more uniformly on the frame. The effects of lateral loads, such as wind and earthquakes, are also spread over the whole frame, increasing its safety. For design purposes, approximate methods may be used by assuming a two-dimensional frame system.

A frame subjected to a system of loads may be analyzed by the equivalent frame method. In this method, the analysis of the floor under consideration is made assuming that the far ends of the columns above and below the slab level are fixed (Fig. 16.7). Usually, the analysis is performed using the moment–distribution method.

In practice, the size of panels, distance between columns, number of stories, and the height of each story are known because they are based upon architectural design and utility considerations. The sizes of beams and columns are estimated first, and their relative stiffnesses based on the gross concrete sections are used. Once the moments are calculated, the sections assumed previously are checked and adjusted as necessary. More accurate analysis can be performed using computers, which is recommended in the structural analysis of statically indeterminate structures with several redundants. Methods of analysis are described in many books on structural analysis.

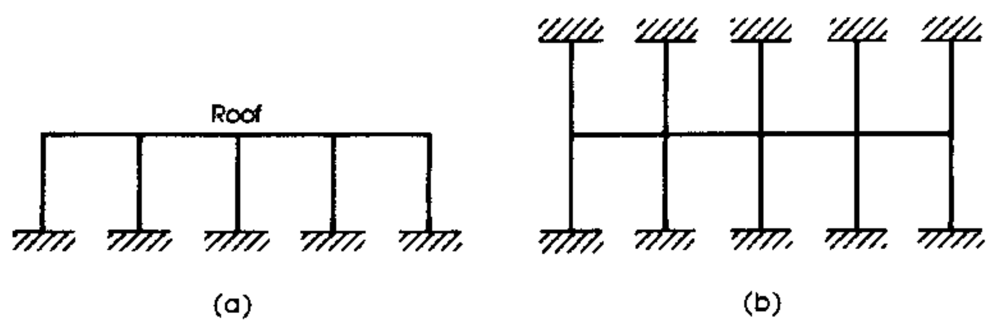


Figure 16.7 Assumption of fixed column ends for frame analysis.

16.4 PORTAL FRAMES

A portal frame consists of a reinforced concrete stiff girder poured monolithically with its supporting columns. The joints between the girder and the columns are considered rigidly fixed, with the sum of moments at the joint equal to 0. Portal frames are used in building large-span halls, sheds, bridges, and viaducts. The top member of the frame may be horizontal (portal frame) or inclined (gable frame) (Fig. 16.8). The frames may be fixed or hinged at the base.

A statically indeterminate portal frame may be analyzed by the moment-distribution method or any other method used to analyze statically indeterminate structures. The frame members are designed for moments, shear, and axial forces, whereas the footings are designed to carry the forces acting at the column base.

Girders and columns of frames may be of uniform or variable depths, as shown in Fig. 16.8. The forces in a single-bay portal frame of uniform sections may be calculated as follows.

16.4.1 Two Hinged Ends

The forces in the members of a portal frame with two hinged ends [2] can be calculated using the following expressions (Fig. 16.9).

For the case of a uniform load on top member BC, let

$$K = 3 + 2\left(\frac{I_2}{I_1} \times \frac{h}{L}\right)$$

where

I_1 and I_2 = column and beam moments of inertia
h and L = height and span of frame

The bending moments at joints B and C are

$$M_B = M_C = -\frac{wL^2}{4K}$$

$$\text{Maximum positive moment at midspan } BC = \frac{wL^2}{8} + M_B$$

The horizontal reaction at A is $H_A = M_B/h = H_D$. The vertical reaction at A is $V_A = WL/2 = V_D$. For a uniform load on half the beam BC, Fig. 16.9*b*: $M_B = M_C = -WL_2/8K$, $H_A = H_D = M_B/h$, $V_A = 3WL/8$, and $V_D = WL/8$.

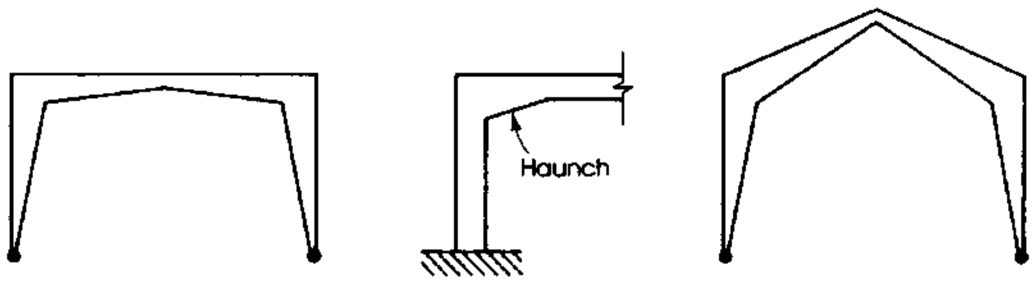


Figure 16.8 Portal and gable frames.

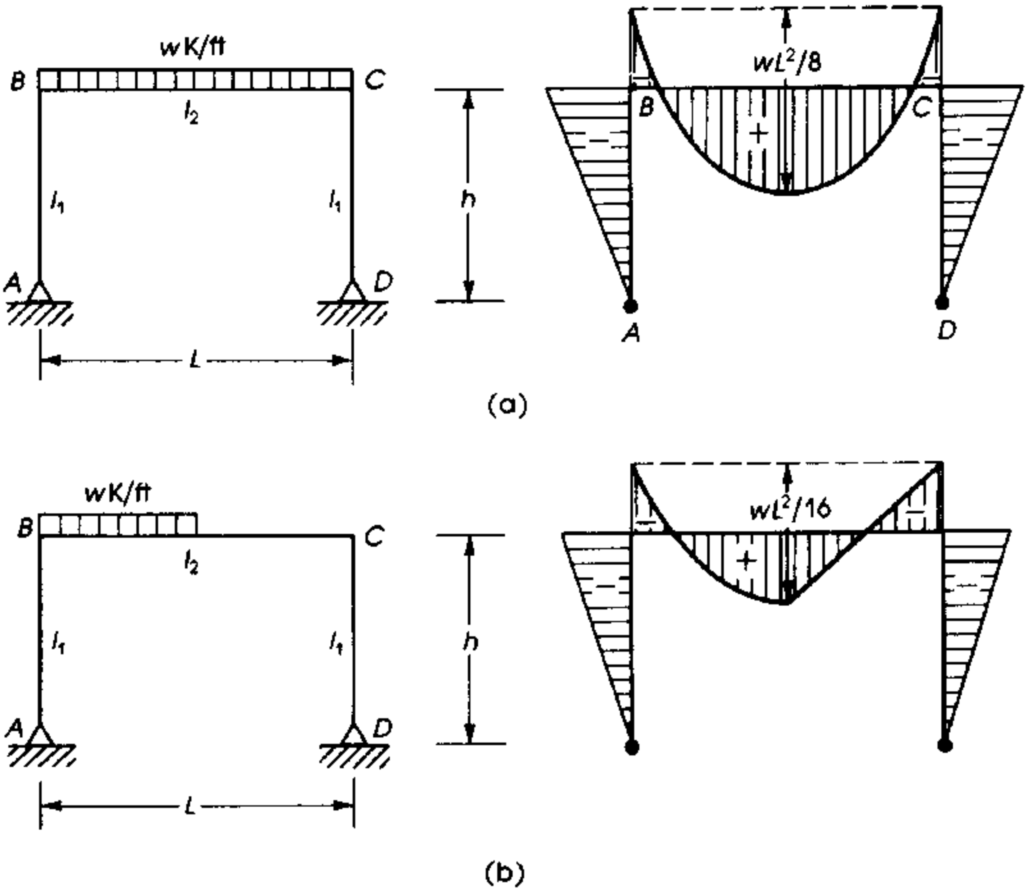


Figure 16.9 Portal frame with two hinged ends. Bending moments are drawn on the tension side.

16.4.2 Two Fixed Ends

The forces in the members of a portal frame with two fixed ends [2] can be calculated as follows (Fig. 16.10).

For a uniform load on top member BC, let

$$K_1 = 2 + \left(\frac{I_2}{I_1} \times \frac{h}{L}\right)$$

$$M_B = M_C = -\frac{wL^2}{6K_1}$$

$$M_A = M_D = \frac{M_B}{2} \qquad M \text{ (midspan)} = \frac{wL^2}{8} + M_B$$

$$H_A = H_D = \frac{3M_A}{h} \quad \text{and} \quad V_A = V_D = \frac{wL}{2}$$

For a uniform load on half the top member BC, let

$$K_2 = 1 + 6\left(\frac{I_2}{I_1} \times \frac{h}{L}\right)$$

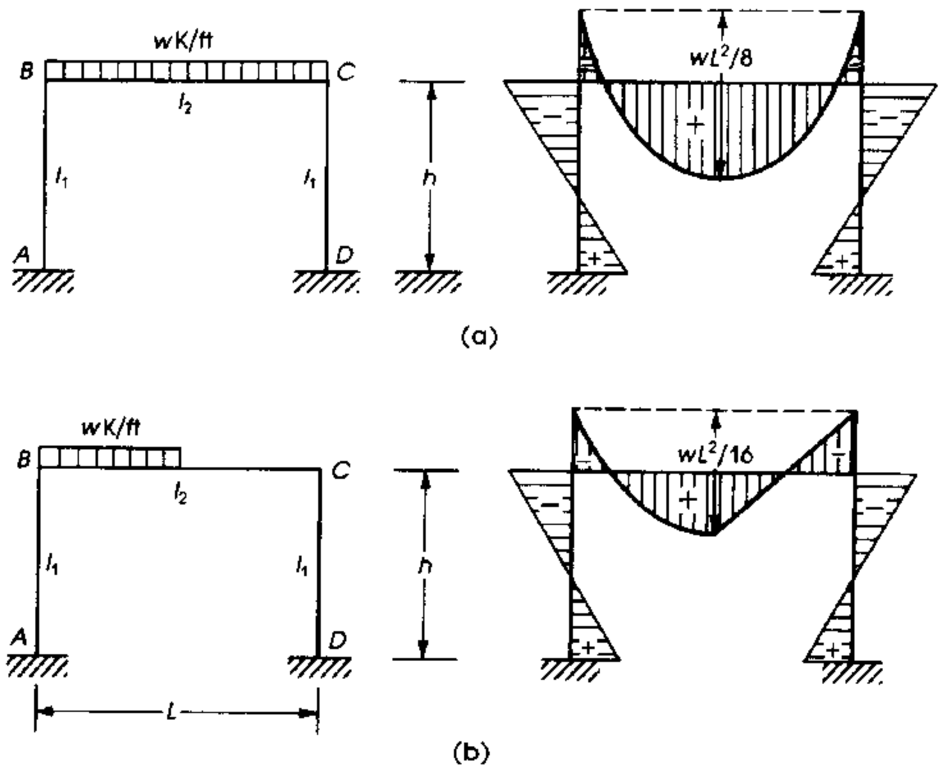


Figure 16.10 Portal frame with fixed ends. Bending moments are drawn on the tension side.

Then

$$M_A = \frac{wL^2}{8}\left(\frac{1}{3K_1} - \frac{1}{8K_2}\right) \qquad M_B = \frac{wl^2}{8}\left(\frac{2}{3K_1} + \frac{1}{8K_2}\right)$$

$$M_C = \frac{wL^2}{8}\left(\frac{2}{3K_1} - \frac{1}{8K_2}\right) \qquad M_D = \frac{wL^2}{8}\left(\frac{1}{3K_1} - \frac{1}{8K_2}\right)$$

$$H_A = H_D = \frac{wl^2}{8} \times \frac{1}{K_1 h}$$

$$V_A = \frac{wL}{2} - V_D \quad \text{and} \qquad V_D = \frac{wL}{8}\left(1 - \frac{1}{4K_2}\right)$$

16.5 GENERAL FRAMES

The main feature of a frame is its rigid joints, which connect the horizontal or inclined girders of the roof to the supporting structural members. The continuity between the members tends to distribute the bending moments inherent in any loading system to the different structural elements according to their relative stiffnesses. Frames may be classified as

1. Statically determinate frames (Fig. 16.11*a*)
2. Statically indeterminate frames (Fig. 16.12)
3. Statically indeterminate frames with ties (Fig. 16.13).

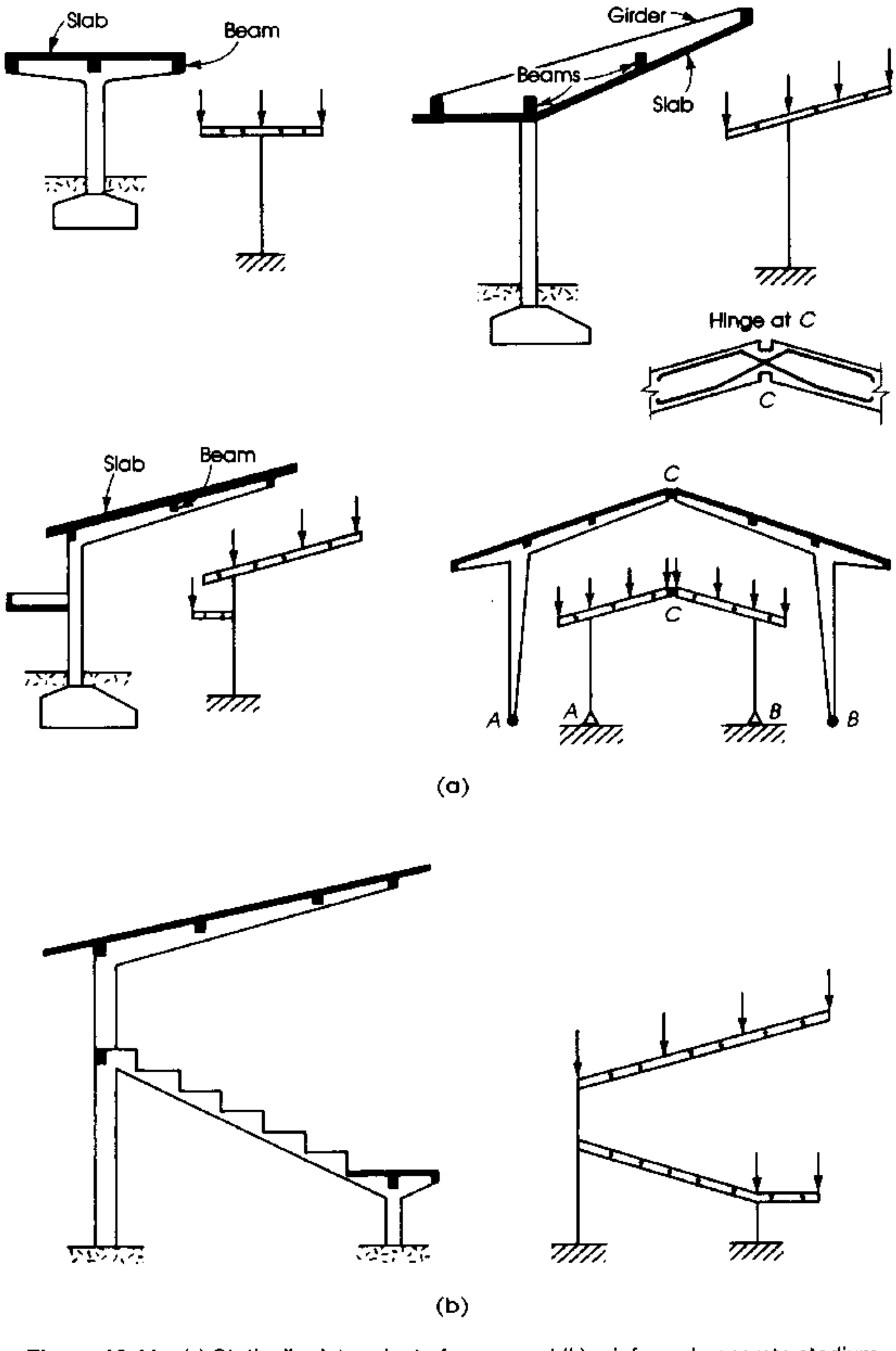


Figure 16.11 (*a*) Statically determinate frames and (*b*) reinforced concrete stadium.

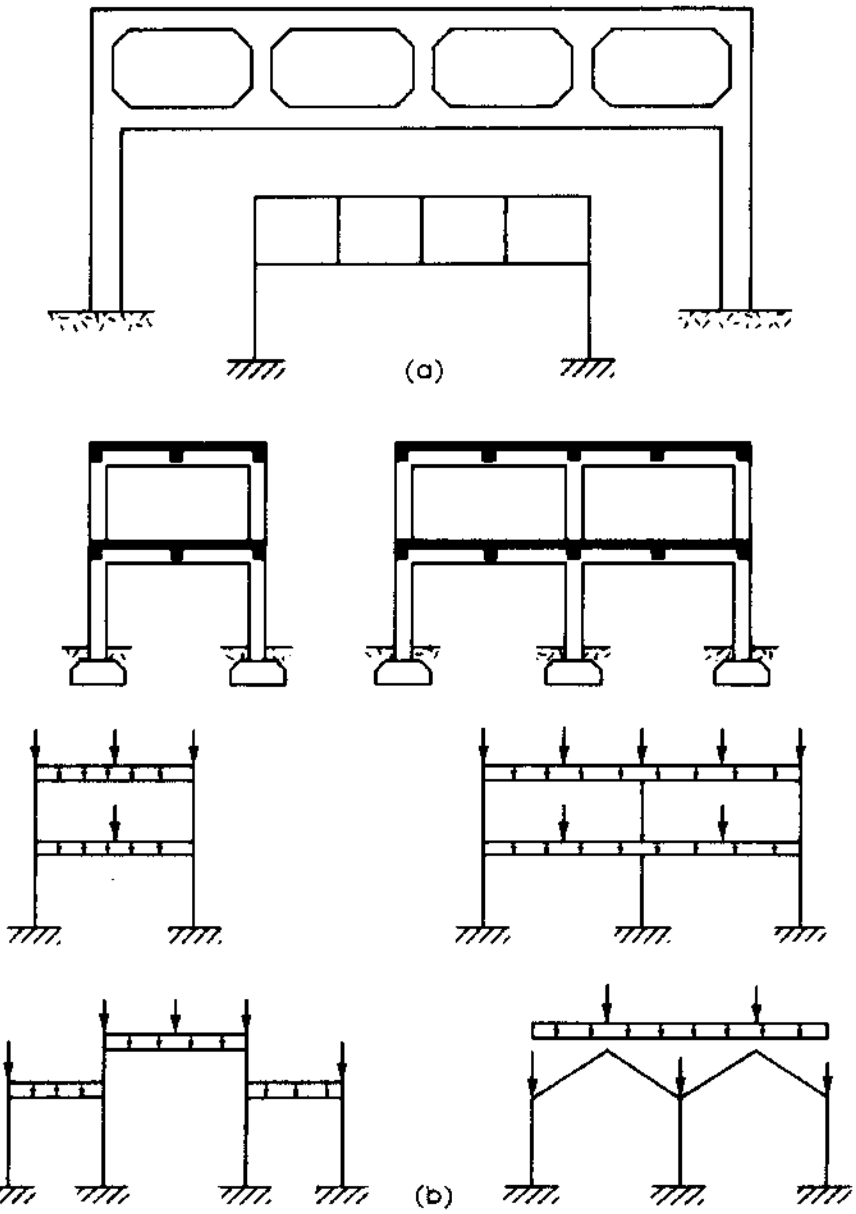


Figure 16.12 (*a*) Vierendeel girder and (*b*) statically indeterminate frames.

Different methods for the analysis of frames and other statically indeterminate structures are described in books dealing with structural analysis. Once the bending moments, shear, and axial forces are determined, the sections can be designed as the examples in this book are. Analysis may also be performed using computer programs.

16.6 DESIGN OF FRAME HINGES

The main types of hinges used in concrete structures are Mesnager hinges, Considére hinges, and lead hinges [19]. The description of each type is given next.

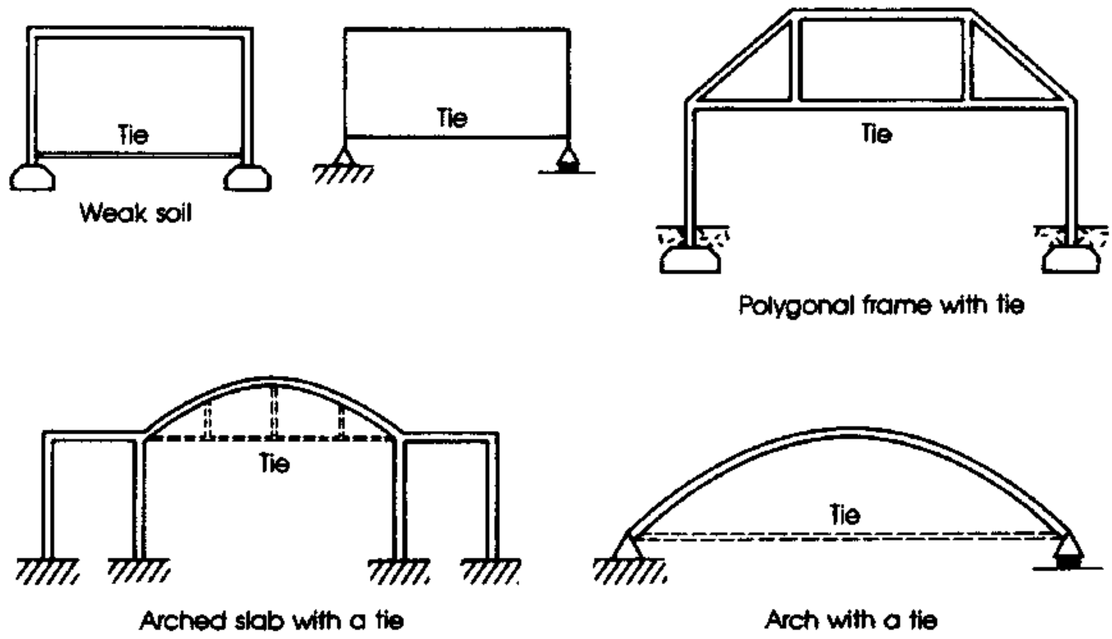


Figure 16.13 Structures with ties.

16.6.1 Mesnager Hinge

The forces that usually act on a hinge are a horizontal force, H, and a vertical force, P. The resultant of the two forces, R, is transferred to the footing through the crossing bars A and B shown in Fig. 16.14. The inclination of bars A and B to the horizontal varies between 30° and 60°, with a minimum distance a, measured from the lower end of the frame column, equal to $8D$, where D is the diameter of the inclined bars. The gap between the frame column and the top of the footing y varies between 1 in. and $1.3h'$, where h' is the width of the concrete section at the hinge level. A practical gap height ranges between 2 and 4 in. The rotation of the frame ends is taken by the hinges, and the gap is usually filled with bituminous cork or similar flexible material. The bitumen protects the cork in contact with the soil from deterioration. The crossing bars A and B are subjected to compressive stresses that must not exceed one-third the yield strength of the steel bars f_y under service loads or 0.55 f_y under factored loads. The low stress is assumed because any rotation at the hinge tends to bend the bars and induces secondary flexural stresses. It is generally satisfactory to keep the compression stresses low rather than to compute secondary stresses. The areas of bars A and B are calculated as follows:

$$\text{Area of bars } A\text{:} \quad A_{s_1} = \frac{R_1}{0.55 f_y} \tag{16.1}$$

$$\text{Area of bars } B\text{:} \quad A_{s_2} = \frac{R_2}{0.55 f_y} \tag{16.2}$$

where R_1 and R_2 are the components of the resultant R in the direction of the inclined bars A and B using factored loads. The components R_1 and R_2 are usually obtained by statics as follows:

$$H + R_2 \sin\theta = R_1 \sin\theta \quad \text{and} \quad R_2 = R_1 - \frac{H}{\sin\theta} \tag{16.3a}$$

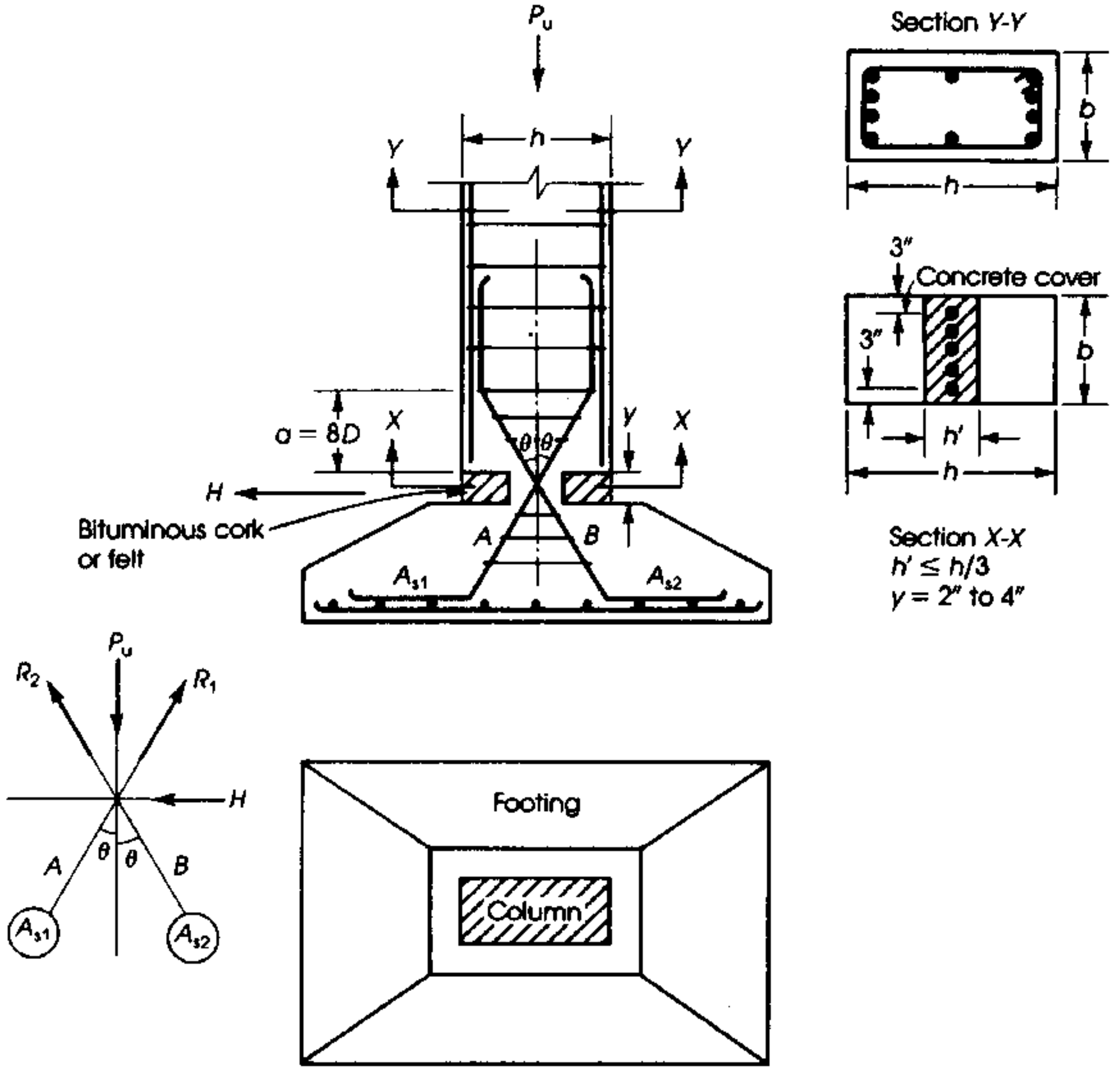


Figure 16.14 Hinge details.

Also, $(R_1 + R_2)\cos\theta = P_u$, so

$$R_1 = \frac{P_u}{\cos\theta} - R_2 = \frac{P_u}{\cos\theta} - \left[R_1 - \frac{H}{\sin\theta}\right] \tag{16.3b}$$

$$R_1 = \frac{1}{2}\left[\frac{P_u}{\cos\theta} + \frac{H}{\sin\theta}\right]$$

The inclined hinge bars transmit their force through the bond along the embedded lengths in the frame columns and footings. Consequently, the bars exert a bursting force, which must be resisted by ties. The ties should extend a distance $a = 8D$ (the larger bar diameter of bars A and B) in both columns and footings. The bursting force F can be estimated as

$$F = \frac{P_u}{2}\tan\theta + \frac{Ha}{0.85d} \tag{16.4}$$

If the contribution of concrete is neglected, then the area of tie reinforcement, A_{st}, required to resist F is

$$A_{st} = \frac{F}{\phi f_y} = \frac{F}{0.85 f_y} \tag{16.5}$$

The stress in the ties can also be computed as follows:

$$f_s(\text{ties}) = \frac{\dfrac{P_u}{2}\tan\theta + \dfrac{Ha}{0.85d}}{0.005ab + A_{st}\ (\text{ties})} \leq 0.85\, f_y \tag{16.6}$$

where

A_{st} = area of ties within a distance $a = 8D$
d = effective depth of column section
b = width of column section

This type of hinge is used for moderate forces and limited by the maximum number of inclined bars that can be placed within the column width.

16.6.2 Considère Hinge

The difference between the *Considère hinge* and the Mesnager one is that the normal force P_u is assumed to be transmitted to the footing by one or more short, spirally reinforced columns extending deep into the footing, whereas the horizontal force H is assumed to be resisted by the inclined bars A and B (Fig. 16.15). The load capacity of the spirally reinforced short column

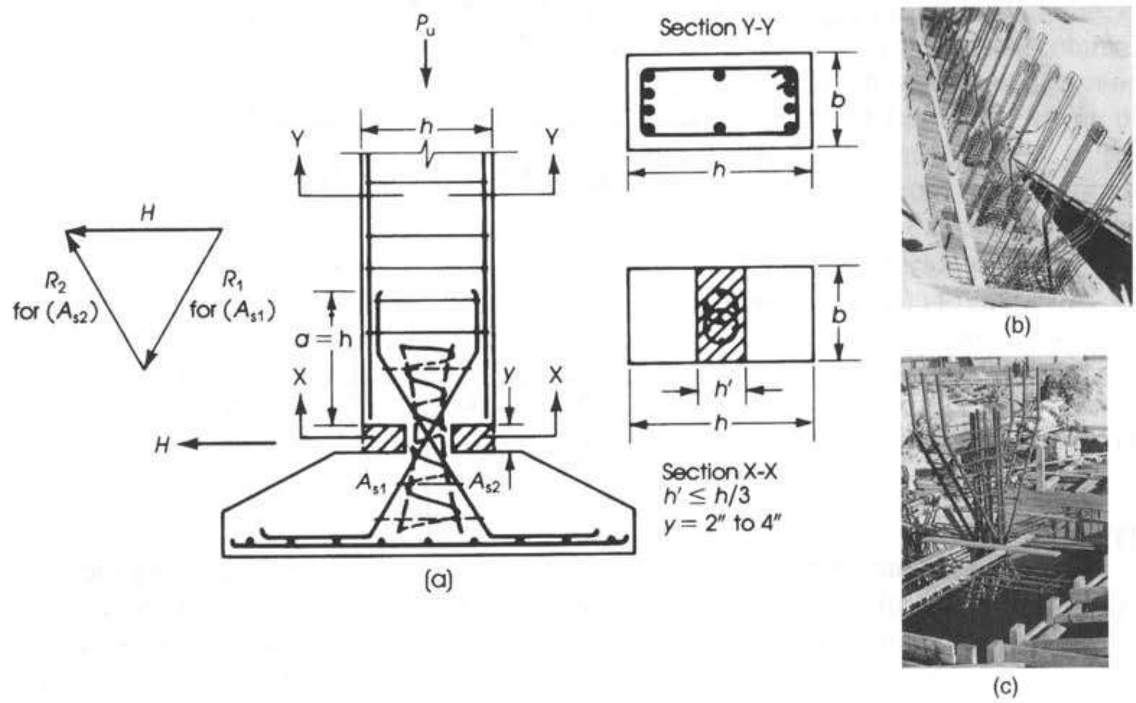


Figure 16.15 (*a*) Considère hinge, (*b*) Mesnager hinges for a series of portal frames, and (*c*) Considère hinge.

may be calculated using Eq. 10.7, neglecting the factor 0.85 for minimum eccentricity.

$$P_u = \phi P_n = 0.75[0.85\, f_c'(A_g - A_{st}) + A_{st} f_y] \tag{16.7}$$

where A_g is the area of concrete hinge section, or bh', and A_{st} is the area of longitudinal bars within the spirals. Ties should be provided in the column up to a distance equal to the long side of the column section h.

16.6.3 Lead Hinges

Lead hinges are sometimes used in reinforced concrete frames. In this type of hinge, a lead plate, usually 0.75 to 1.0 in. thick, is used to transmit the normal force, P_u, to the footing. The horizontal force H is resisted by vertical bars placed at the center of the column and extended to the footing (Fig. 16.16). At the base of the column, the axial load P_u should not exceed the bearing strength specified by the ACI Code, Section 10.14, of $\phi(0.85 f_c' A_1)$, where $\phi = 0.65$ and $A_1 = bh'$. The area of the vertical bars is $A_s. = H/0.6 f_y$, where H = factored horizontal force.

Example 16.2

An 84- by 40-ft hall is to be covered by reinforced concrete slabs supported on hinged-end portal frames spaced at 12 ft on centers (Fig. 16.17). The frame height is 15 ft, and no columns are allowed within the hall area. The dead load on the slabs is that due to self-weight plus 75 psf from roof finish. The live load on the slab is 85 psf. Design a typical interior frame using normal-weight concrete with $f_c' = 4$ ksi and $f_y = 60$ ksi for the frame and a column width of $b = 16$ in.

Solution

The main structural design of the building will consist of the following:

- Design of one-way slabs
- Analysis of the portal frame
- Design of the frame girder due to moment
- Design of the frame girder due to shear
- Design of columns
- Design of hinges
- Design of footings

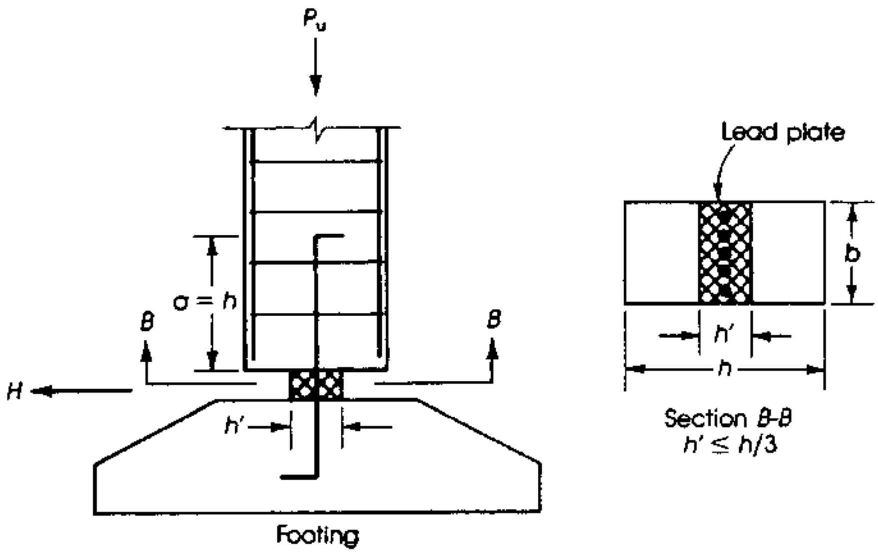


Figure 16.16 Lead hinge.

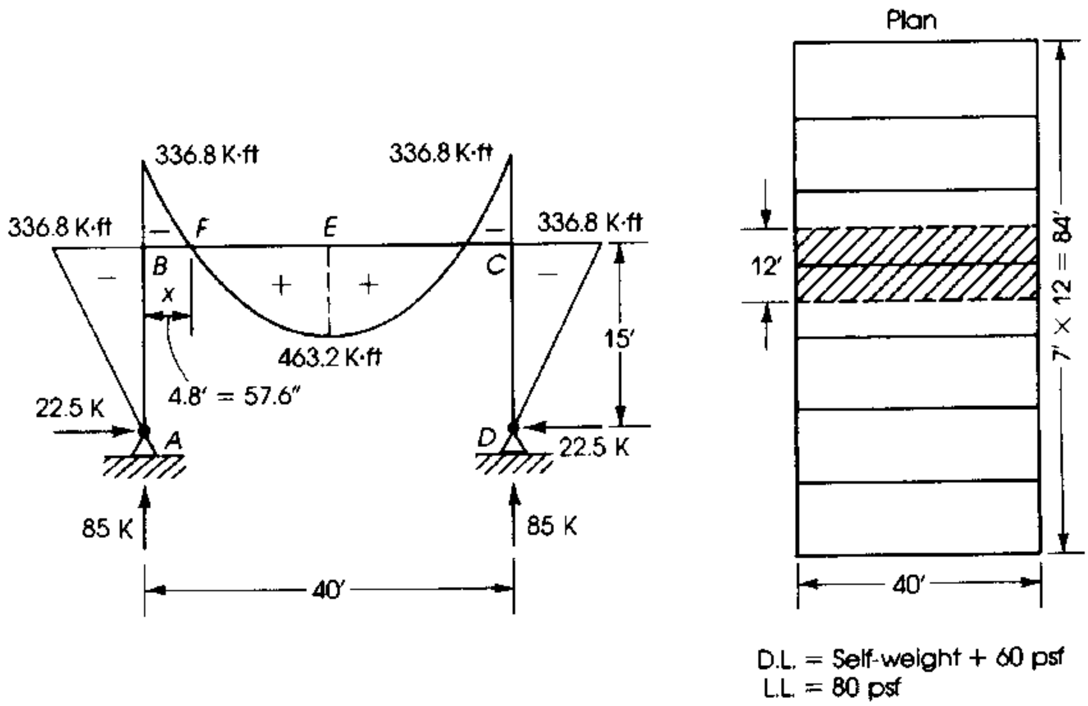


Figure 16.17 Example 16.2: design of portal frame.

1. One-way roof slab: The minimum thickness of the first slab is $L/30$, because one end is continuous and the other end is discontinuous (Table A.6 in Appendix A).

$$\text{Minimum depth} = \frac{12 \times 12}{30} = 4.8 \text{ in.}$$

Assume a slab thickness of 5.0 in. and design the slab following the steps of Example 9.5.

2. Analysis of an interior portal frame:

a. The loads on slabs are

$$\text{Dead load on slabs} = 75 + \left(\frac{5}{12} \times 150\right) = 137.5 \text{ psf}$$

$$\text{Factored load on slabs} = 1.2 \times 137.5 + 1.6 \times 85 = 301 \text{ psf}$$

b. Determine loads on frames: The interior frame carries a load from a 12-ft slab in addition to its own weight. Assume that the depth of the beam is $L/24 = (40 \times 12)/24 = 20$ in. Use a projection below the slab of 16 in., giving a total beam depth of 21 in.

$$\text{Dead load from self-weight of beam} = \left(\frac{16}{12}\right)^2 \times 150 = 267 \text{ lb/ft}$$

$$\text{Total factored load on frame} = 301 \times 12 + 1.2 \times 267$$

$$= 3932 \text{ lb/ft}$$

$$w_u = 4.0 \text{ K/ft}$$

c. Determine the moment of inertia of the beam and columns sections. The beam acts as a T-section. The effective width of slab acting with the beam is the smallest of span/4 = $40 \times 12/4 = 120$ in., $16h_s + b_w = 16 \times 5 + 16 = 96$, or 12 ft $\times$ 12 = 144 in. Use $b = 96$ in.

The centroid of the section from the top fibers is

$$y = \frac{96 \times 5 \times 2.5 + 16 \times 16 \times 13}{96 \times 5 + 16 \times 16} = 6.2 \text{ in.}$$

$$I_b(\text{beam}) = \left[\frac{96}{12}(5)^3 + 96 \times 5(3.7)^2\right] + \left[\frac{16}{12}(16)^3 + 16 \times 16(6.8)^2\right]$$

$$= 24{,}870 \text{ in.}^4$$

It is a common practice to consider an approximate moment of inertia of a **T**-beam as equal to twice the moment of inertia of a rectangular section having the total depth of the web and slab:

$$I_b(\text{beam}) = 2 \times \frac{16}{12}(21)^3 = 24{,}696 \text{ in.}^4$$

(For an edge beam, approximate $I = 1.5 \times bh_3/12$.) Assume a column section 16 by 20 in. (having the same width as the beam).

$$I_c(\text{column}) = \frac{16}{12}(20)^3 = 10{,}667 \text{ in.}^4$$

d. Let the factor

$$K = 3 + 2\left(\frac{I_b}{L} \times \frac{h}{I_c}\right) = 3 + 2\left(\frac{24{,}870}{40} \times \frac{15}{10{,}667}\right) = 4.75$$

Referring to Fig. 16.17 and for a uniform load $w_u = 4.0$ K/ft on BC,

$$M_B = M_C = -\frac{w_u L^2}{4K} = -\frac{4.0(40)^2}{4 \times 4.75} = -336.8 \text{ K·ft}$$

The maximum positive bending moment at midspan of BC equals

$$w_u \frac{L^2}{8} + M_B = \frac{4.0(40)^2}{8} - 336.8 = 463.2 \text{ K·ft}$$

The horizontal reaction at A is

$$H_a = H_D = \frac{M_B}{h} = \frac{336.8}{15} = 22.5 \text{ K}$$

The vertical reaction at A is

$$V_A = V_D = \frac{w_u L}{2} + \text{weight of column}$$

$$V_A = 4.0 \times \frac{40}{2} + \frac{20}{12} \times \frac{16}{12} \times 0.150 \times 15 \text{ ft} = 85.0 \text{ K}$$

The bending moment diagram is shown in Fig. 16.17.

e. To consider the sidesway effect on the frame, the live load is placed on half the beam BC, and the moments are calculated at the critical sections. This case is not critical in this example.

f. The maximum shear at the two ends of beam BC occurs when the beam is loaded with the factored load w_u, but the maximum shear at midspan occurs when the beam is loaded with half the live load and with the full dead load:

$$V_u \text{ at support} = 4.0 \times \frac{40}{2} = 80.0 \text{ K}$$

$$V_u \text{ at midspan} = W_l \frac{L}{8} = (1.7 \times 80 \times 12) \times \frac{40}{8}$$

$$= 8160 \text{ lb} = 8.16 \text{ K}$$

g. The axial force in each column is $V_A = V_D = 85.0\,\text{K}$.

h. Let the point of zero moment in BC be at a distance x from B; then

$$M_B = w_u L \frac{x}{2} - w_u \frac{x^2}{2}$$

$$336.8 = 4.0\left(\frac{40x}{2} - \frac{x^2}{2}\right) \quad \text{or} \quad x^2 - 40x + 168.4 = 0$$

$$x = 4.8 \text{ ft} = 57.6 \text{ in. from } B$$

3. Design of girder BC:

a. Design the critical section at midspan. $M_u = 463.2\,\text{K·ft}$, web width is flange width is $b_w = 16\,\text{in.}$, flange width is $b = 96\,\text{in.}$, and $d = 21 - 3.5 = 17.5\,\text{in.}$ (assuming two rows of steel bars). Check if the section acts as a rectangular section with effective $b = 96\,\text{in.}$ Assume $a = 1.0\,\text{in}$; then

$$A_s = \frac{M_u}{\phi f_y\left(d - \frac{a}{2}\right)} = \frac{463.2 \times 12}{0.9 \times 60\left(17.5 - \frac{1.0}{2}\right)} = 6.05 \text{ in.}^2$$

$$a = \frac{A_s f_y}{0.85\, f_c' b} = \frac{6.05 \times 60}{0.85 \times 4 \times 96} = 1.1 \text{ in.} < 5.0 \text{ in.}$$

The assumed a equals approximately the calculated a. The section acts as a rectangular section; therefore, use six no. 9 bars. Check $b_{\min}$ (to place bars in one row):

$$b_{\min} = 11\left(\frac{9}{8}\right) + 2\left(\frac{3}{8}\right) + 3 = 16.13 \text{ in.} > 16 \text{ in.}$$

Place bars in two rows, as shown in Fig. 16.18.

b. Design the critical section at joint B: $M_u = 336.8\,\text{K·ft}$, $b = 16\,\text{in.}$, and $d = 21 - 2.5 = 18.5\,\text{in.}$ (for one row of steel bars). The slab is under tension, and reinforcement bars are placed on top of the section.

$$R_u = \frac{M_u}{bd^2} = \frac{336.8 \times 12{,}000}{16(18.5)^2} = 738 \text{ psi}$$

From tables in Appendix A, $\rho = 0.016 < \rho_{\max} = 0.018$. (Tension-controlled section, $\phi = 0.9$)

$$A_s = 0.016 \times 16 \times 18.5 = 4.73 \text{ in.}^2$$

Use five no. 9 bars in one row.

4. Design the girder BC due to shear:

a. The critical section is at a distance d from the face of the column with a distance from the column centerline of $10 + 18.5 = 28.5\,\text{in.} = 2.4\,\text{ft}$. Thus,

$$V_u \text{ (at distance } d) = 80 - 4 \times 2.4 = 70.4 \text{ K}$$

b. The shear strength provided by concrete is

$$\phi V_c = \phi(2\lambda\sqrt{f_c'})b_w d$$

$$\phi V_c = \frac{0.75 \times 2 \times (1)}{1000} \times \sqrt{4000} \times 16 \times 18.5 = 28.1 \text{ K}$$

The shear force to be provided by web reinforcement is

$$\phi V_s = V_u - \phi V_c = 70.4 - 28.1 = 42.3 \text{ K}$$

$$V_s = \frac{42.3}{0.75} = 56.4 \text{ K}$$

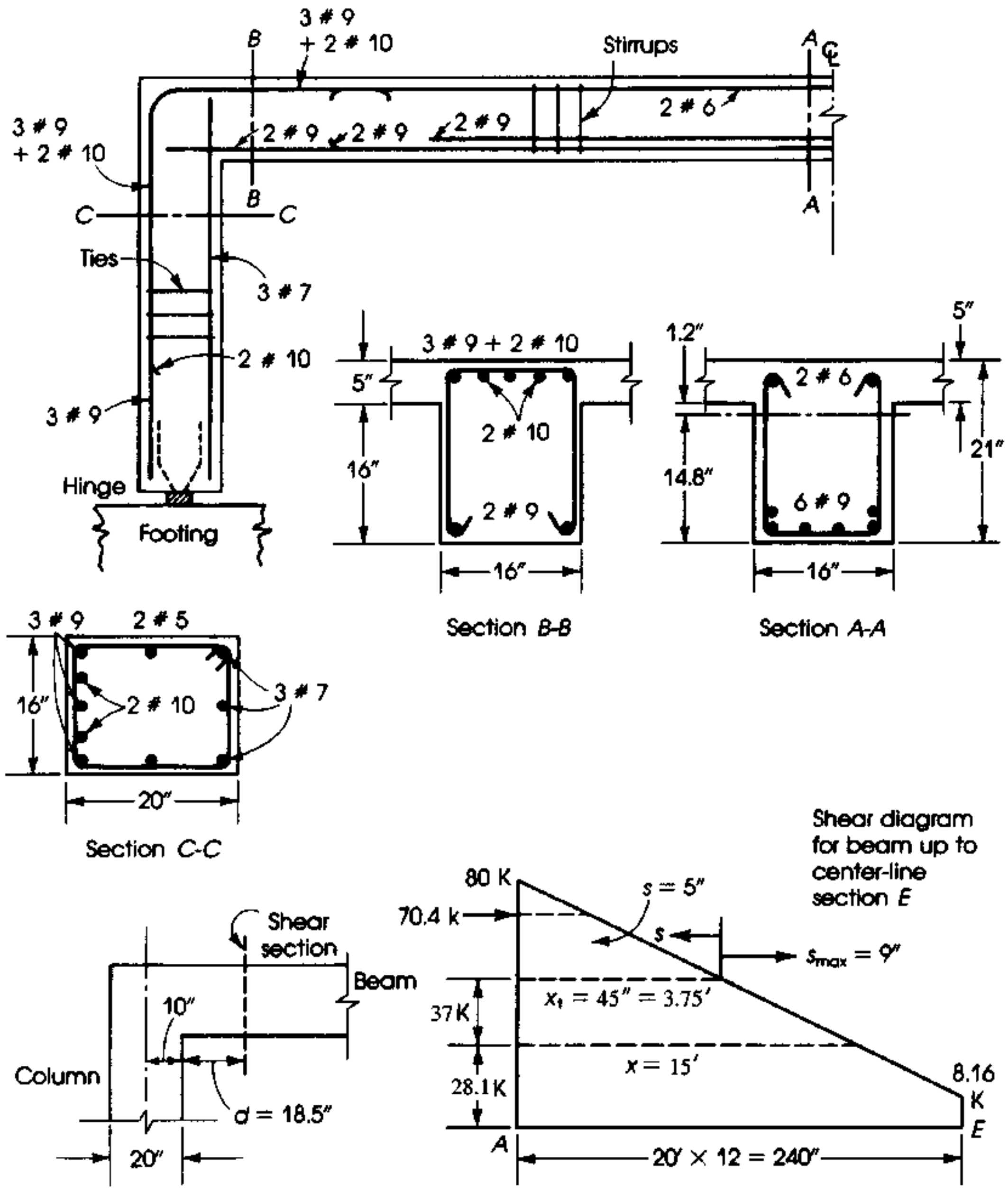


Figure 16.18 Example 16.2: reinforcement details of frame sections.

c. Choose no. 4 stirrups and $A_v = 2 \times 0.20 = 0.40\,\text{in.}^2$ Thus,

$$S = \frac{A_v f_y d}{V_s} = \frac{0.40 \times 60 \times 18.5}{56.4} = 7.8 \text{ in., say 7 in.}$$

d. Maximum spacing of no. 4 stirrups is

$$S_{\max} = \frac{d}{2} = \frac{18.5}{2} = 9.25 \text{ in.} \quad \text{say, 9 in.}$$

or

$$S_{\max} = \frac{A_v f_y}{50 b_w} = \frac{0.40 \times 60{,}000}{50 \times 16} = 30 \text{ in.}$$

Check for maximum spacing of $d/2$: $V_s \le 4\sqrt{f'_c} b_w d$ or

$$V_s \le 4\sqrt{4000} \times \frac{16 \times 18.5}{1000} = 74.9 \text{ K}$$

The value V_s of 56.4 is less than 74.9 K, so use $S_{max} = 9$ in.

$$V_s \text{ (for } S_{max} = 9 \text{ in.)} = \frac{A_v f_y d}{S} = \frac{0.40 \times 360 \times 18.5}{9} = 49.3 \text{ K}$$

$$\phi V_s = 0.75 \times 49.3 = 37 \text{ K}$$

The distance from the face of the column where $S_{max} = 9$ in. can be used is equal to 45 in. = 3.75 ft (from the triangle of shear forces).

e. Distribution of stirrups:

First stirrups at $S/2 = 3.0$ in.

7 stirrups at 7 in. = 49.0 in

19 stirrups at 9 in. = 171.0 in. (Total = 223 in.)

The distance from the face of the column to the centerline of the beam is 240 − 10 = 230 in. Use the same distribution for the second half of the beam, and place one stirrup at midspan.

5. Design the column section at joint B: $M_u = 336.8$ K·ft, $P_u = 80$ K, $b = 16$ in., and $h = 20$ in.

a. Assuming that the frame under the given loads will not be subjected to sidesway, then the effect of slenderness may be neglected, and the column can be designed as a short column when

$$\frac{Kl_u}{4} < 34 - \frac{12M_1}{M_2} \quad \text{(see Section 12.5)}$$

$$M_1 = 0 \quad \text{and} \quad M_2 = 336.8 \text{ K·ft}$$

Let $K = 0.8$ (Fig. 12.2), $L_u = 15 - 21/(2 \times 12) = 14.125$ ft, and $r = 0.3h = 0.3 \times 20 = 6$ in; then

$$\frac{KL_u}{r} = 0.8 \times \frac{14.125 \times 12}{6} = 22.6 < 34$$

If K is assumed equal to 1.0, then

$$\frac{KL_u}{r} = 28.25 < 34$$

Therefore, design the member as a short column.

b. The design procedure is similar to Examples 11.16 and 11.3.

$$\text{Eccentricity } (e) = \frac{M_u}{P_u} = \frac{336.8 \times 12}{80} = 50.5 \text{ in.}$$

This is a large eccentricity, and it will be assumed that the section is in the transition region, $\phi < 0.9$.

$$d = 20 - 2.5 = 17.5 \text{ in.}$$

c. Because $e = 50.5$ in. is much greater than d, determine approximate A_s and A'_s from the M_u only and then check the final section by statics, as was explained in Example 11.3. For $M_u = 336.8$ K·ft, $b = 16$ in., $h = 20$ in., and $d = 17.5$ in., $R_u = M_u/bd^2 = 336.8(12{,}000)/16(17.5)^2 = 825$ psi.

$$\rho = 0.0183 \quad \text{and} \quad A_s = \rho bd = 0.0183(16)(17.5) = 5.12 \text{ in.}^2$$

Choose three no. 9 and two no. 10 bars and $A_s = 5.53\,\text{in.}^2$ Choose $A'_s = A_s/3 = 5.13/3 = 1.7\,\text{in.}^2$ and three no. 7 bars ($A'_s = 1.8\,\text{in.}^2$) (Fig. 16.18). When the eccentricity, e, is quite large, it is a common practice to use $A'_s = A_s/3$ or $A_s/2$ instead of $A_s = A'_s$.

d. Check the load capacity of the final section using $A_s = 5.53\,\text{in.}^2$ and $A'_s = 1.8\,\text{in.}^2$, similar to Example 11.3, according to the following steps:

i. $P_n = C_c + C_s - T$

$$C_c = 0.85 f'_c ab = 0.85(4)(16)a = 54.4a$$

$$C_s = A'_s(f'_s - 0.85 f'_c) = 1.8(60 - 0.85 \times 4) = 101.8\ \text{K}$$

$$T = A_s f_y = 5.53(60) = 331.8\ \text{K}$$

$$P_n = 54.4a + 101.8 - 331.8 = (43.4a - 230) \qquad \text{(I)}$$

ii. Take moments about A_s:

$$P_n = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right]$$

$e' = e + d''$, where d'' is the distance from A_s to the plastic centroid of the section. The plastic centroid occurs at 11.1 in. from the extreme compression fibers and $d'' = d - x = 6.4$ in. (refer to Example 11.1).

$$e' = 50.5 + 6.4 = 56.9\ \text{in.}$$

$$P_n = \frac{1}{56.9}\left[54.4a\left(17.5 - \frac{a}{2}\right) + 101.8(15)\right]$$

$$= 16.73a - 0.478a^2 + 26.86 \qquad \text{(II)}$$

iii. Equate Eqs. I and II and solve to get $a = 6.313$ in. and $P_n = 113.5$ K. Check $f'_s = 87(c - d')/c \le f_y$: $c = a/0.85 = 7.43$ in. and $f'_s = 87(7.43 - 2.5)/7.43 = 58$ ksi, which is close to the 60 ksi assumed in the calculations. Choose no. 3 ties spaced at 16 in.

iv. Check ϕ: $d_t = 17.5$ in.

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right)0.003 = 0.00407$$

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.823$$

$$\phi P_n = 0.823(113.5) = 93.3\ \text{K} > 80\ \text{K}$$

The section is adequate.

6. Check the adequacy of the column section at midheight, 7.5 ft from A: $M_u = 336.8/2 = 168.4$ K·ft.

$$P_u = 80 + 2.5\ \text{(half the column weight)} = 82.5\ \text{K}$$

Use A_s = three no. 9 bars and A'_s = three no. 7 bars. In an approach similar to step 5, $\phi P_n = 122\,\text{K} > 82.5\,\text{K}$ (no. 10 bars can be terminated, and they have to be extended a development length below the midheight of the column).

7. Design the hinge at A: $M_u = 0$, $H = 22.5$ K, $P_u = 85$ K.

a. Choose a Mesnager hinge. Using Eqs. 16.3*a* and 16.3*b*, $R_1 = 72$ K and $R_2 = 27$ K. (Refer to Fig. 16.19 with $\theta = 30°$.)

$$A_{s1} = \frac{R_1}{0.55 f_y} = \frac{72}{0.55 \times 60} = 2.2\ \text{in.}^2$$

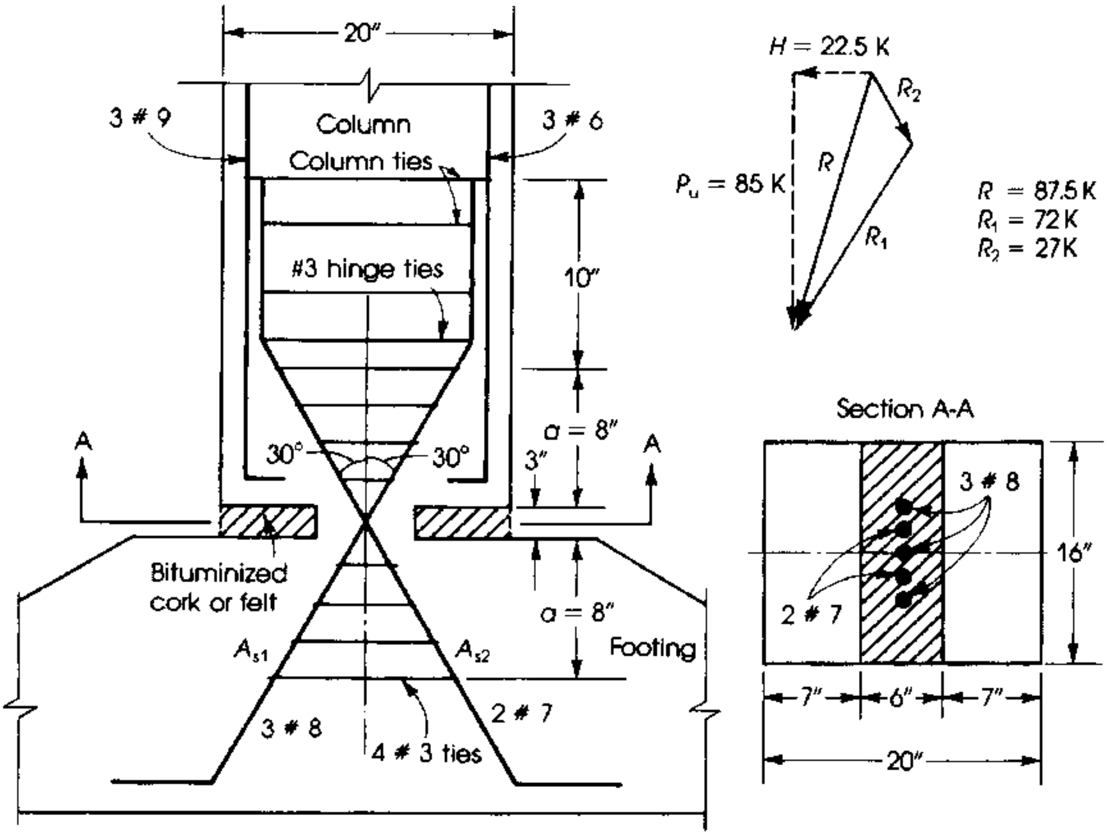


Figure 16.19 Example 16.2: hinge details.

Choose three no. 8 bars ($A_s = 2.35\,\text{in.}^2$).

$$A_{s_2} = \frac{R_2}{0.55 \times f_y} = \frac{27}{0.55 \times 60} = 0.82\ \text{in.}^2$$

Choose two no. 7 bars ($A_s = 1.2\,\text{in.}^2$). Arrange the crossing bars by placing one no. 8 bar and then one no. 7 bar, as shown in Fig. 16.19 (or use five no. 8 bars.)

b. Lateral ties should be placed along a distance $a = 8 -$ bar diameter $= 8.0\,\text{in.}$ within the column and footing. The bursting force is

$$F = \frac{P_u}{2}\tan\theta + \frac{Ha}{0.85d}$$

For $\theta = 30°$, $d = 17.5\,\text{in.}$, and $a = 8.0\,\text{in.}$,

$$F = \frac{85}{2}\tan 30° + \frac{22.5 \times 8}{0.85 \times 17.5} = 36.6\ \text{K}$$

$$\text{Area of ties} = \frac{36.6}{0.85 \times 60} = 0.72\ \text{in.}^2$$

If no. 3 closed ties (two branches) are chosen, then the area of one tie is $2 \times 0.11 = 0.22\,\text{in.}^2$ The number of ties is $0.72/0.22 = 3.27$, say, four ties spaced at $\frac{8}{3} = 2.7$ in., as shown in Fig. 16.19.

8. Design the footing: If the height of the footing is assumed to be h', then the forces acting on the footing are the axial load P and a moment $M = H/h'$. The soil pressure is calculated from

Eq. 13.14 of Chapter 13:

$$q = +\frac{P}{A} \pm \frac{Mc}{I} \leq \text{allowable soil pressure}$$

The design procedure of the footing is similar to that of Example 13.7.

16.7 INTRODUCTION TO LIMIT DESIGN

16.7.1 General

Limit state design of a structure falls into three distinct steps:

1. Determination of the factored design load, obtained by multiplying the dead and live loads by load factors. The ACI Code adopted the load factors given in Chapter 3.
2. Analysis of the structure under factored loads to determine the factored moments and forces at failure or collapse of the structure. This method of analysis has proved satisfactory for steel design; in reinforced concrete design, this type of analysis has not been fully adopted by the ACI Code because of the lack of ductility of reinforced concrete members. The Code allows only partial redistribution of moments in the structure based on an empirical percentage, as will be explained later in this chapter.
3. Design of each member of the structure to fail at the factored moments and forces determined from structural analysis. This method is fully established now for reinforced concrete design and the ACI Code permits the use of the strength design method, as was explained in earlier chapters.

16.7.2 Limit Design Concept

Limit design in reinforced concrete refers to the redistribution of moments that occurs throughout a structure as the steel reinforcement at a critical section reaches its yield strength. The ultimate strength of the structure can be increased as more sections reach their strength capacity. Although the yielding of the reinforcement introduces large deflections, which should be avoided under service loads, a statically indeterminate structure does not collapse when the reinforcement of the first section yields. Furthermore, a large reserve of strength is present between the initial yielding and the collapse of the structure.

In steel design, the term *plastic design* is used to indicate the change in the distribution of moments in the structure as the steel fibers, at a critical section, are stressed to their yield strength. The development of stresses along the depth of a steel section under increasing load is shown in Fig. 16.20. Limit analysis of reinforced concrete developed as a result of earlier research on steel structures and was based mainly on the investigations of Prager [4], Beedle [5], and J. B. Baker [6]. A. L. L. Baker [7] worked on the principles of limit design, whereas Cranston [8] tested portal frames to investigate the rotation capacity of reinforced concrete plastic hinges. However, more research work is needed before limit design can be adopted by the ACI Code.

16.7.3 Plastic Hinge Concept

The curvature, ϕ of a member increases with the applied bending moment M. For an underreinforced concrete beam, the typical moment–curvature and the load–deflection curves are shown

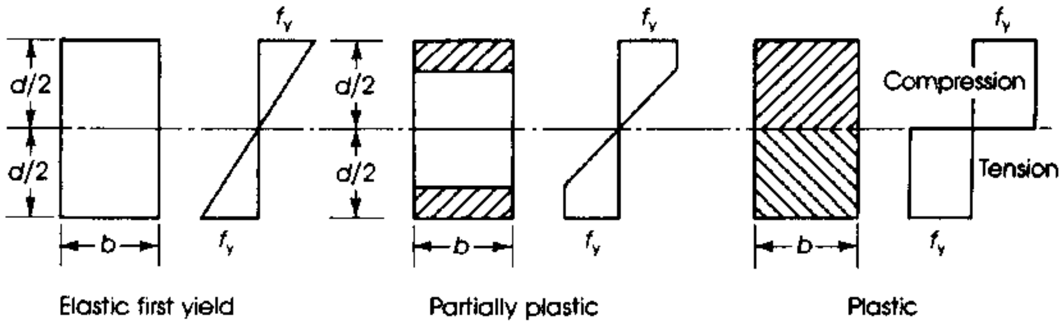


Figure 16.20 Distribution of yield stresses in a yielding steel rectangular section.

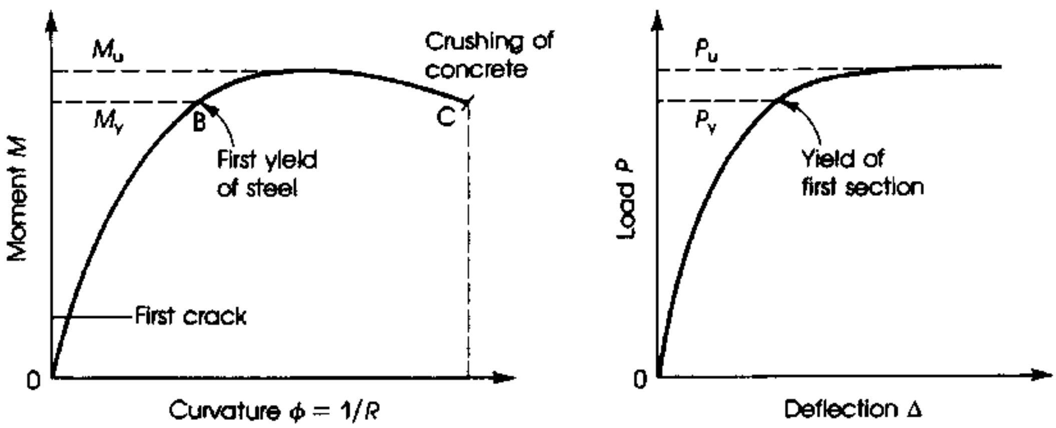


Figure 16.21 Yielding behavior of an under reinforced concrete beam.

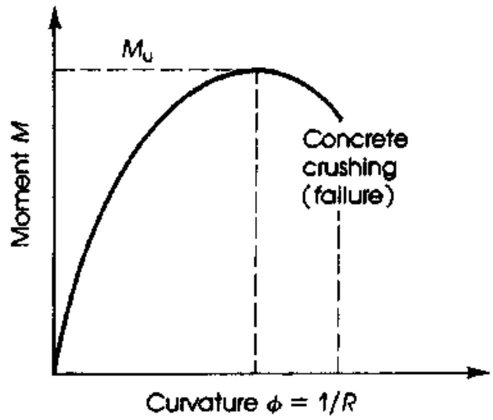


Figure 16.22 Yielding behavior of an overreinforced concrete beam.

in Fig. 16.21. A balanced or an overreinforced concrete beam is not permitted by the ACI Code, because it fails by the crushing of concrete and shows a small curvature range at factored moment (Fig. 16.22).

The significant part of the moment–curvature curve in Fig. 16.21 is that between B and C, in which M_u remains substantially constant for a wide range of values of ϕ. In limit analysis,

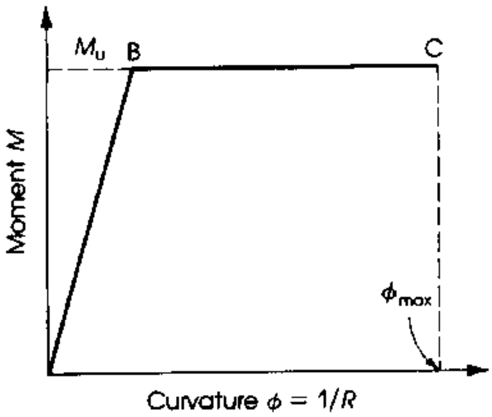


Figure 16.23 Idealized moment–curvature behavior of reinforced concrete beams.

the moment–curvature curve can be assumed to be of the idealized form shown in Fig. 16.23, where the curvature, ϕ, between B and C is assumed to be constant, forming a plastic hinge. Because concrete is a brittle material, there is usually considered to be a limit at which the member fails completely at maximum curvature at C.

Cranston [8] reported that in normally designed reinforced concrete frames, ample rotation capacity is available, and the maximum curvature at point C will not be reached until the failure or collapse of the frame. Therefore, when the member carries a moment equal to its factored moment, M_u, the curvature continues to increase between B and C without a change in the moment, producing a plastic hinge. The increase in curvature allows other parts of the statically indeterminate structure to carry additional loading.

16.8 THE COLLAPSEC MECHANISM

In limit design, the moment strength of a reinforced concrete member is reached when it is on the verge of collapse. The member collapses when there are sufficient numbers of plastic hinges to transform it into a mechanism. The required number of plastic hinges, n, depends upon the degree of redundancy, r, of the structure. The relation between n and r to develop a mechanism is

$$n = 1 + r \tag{16.8}$$

For example, in a simply supported beam no redundants exist, and $r = 0$. Therefore, the beam becomes unstable and collapses when one plastic hinge develops at the section of maximum moment, as shown in Fig. 16.24*a*. Applications to beams and frames are also shown in Fig. 16.24.

16.9 PRINCIPLES OF LIMIT DESIGN

Under working loads, the distribution of moments in a statically indeterminate structure is based on elastic theory, and the whole structure remains in the elastic range. In limit design, where factored loads are used, the distribution of moments at failure, when a mechanism is reached, is different from that distribution based on elastic theory. This change reflects moment redistribution.

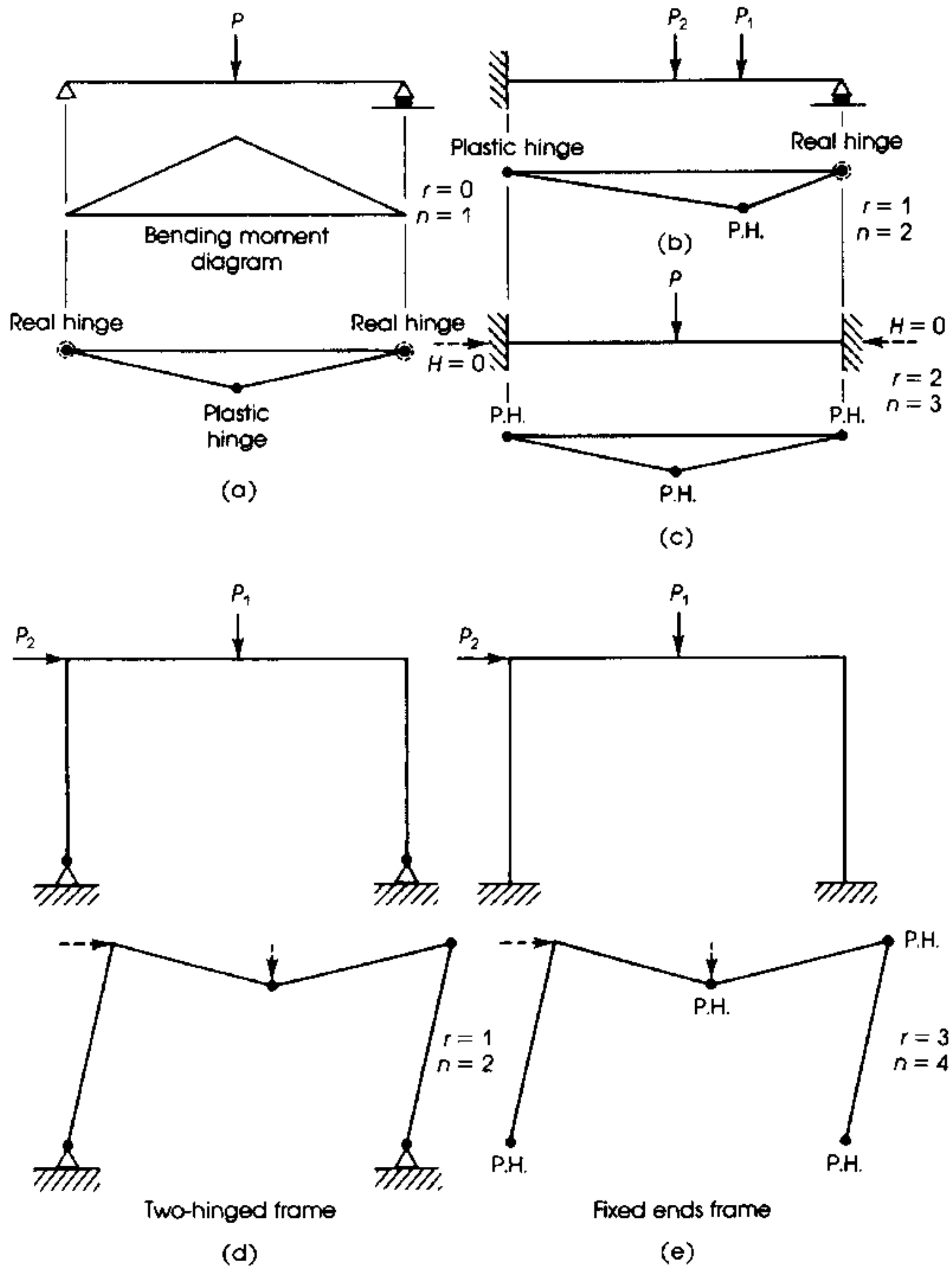


Figure 16.24 Development of plastic hinges (P.H.).

For limit design to be valid, four conditions must be satisfied.

1. *Mechanism condition:* Sufficient plastic hinges must be formed to transform the whole or part of the structure into a mechanism.
2. *Equilibrium condition:* The bending moment distribution must be in equilibrium with the applied loads.

3. *Yield condition:* The factored moment must not be exceeded at any point in the structure.
4. *Rotation condition:* Plastic hinges must have enough rotation capacity to permit the development of a mechanism.

Only the first three conditions apply to plastic design, because sufficient rotation capacity exists in ductile materials as steel. The fourth condition puts more limitations on the limit design of reinforced concrete members as compared to plastic design.

16.10 UPPER AND LOWER BOUNDS OF LOAD FACTORS

A structure on the verge of collapse must have developed the required number of plastic hinges to transform it into a mechanism. For arbitrary locations of the plastic hinges on the structure, the collapse loads can be calculated, which may be equal to or greater than the actual loads. Because the calculated loads cannot exceed the true collapse loads for the structure, then this approach indicates an upper or kinematic bound of the true collapse loads [10]. Therefore, if all possible mechanisms are investigated, the lowest M_u will be caused by the actual loads. Horne [11] explained the upper bound by assuming a mechanism and then calculating the external work, W_e, done by the applied loads and the internal work, W_i, done at the plastic hinges. If $W_e = W_i$, then the applied loads are either equal to or greater than the collapse loads.

If any arbitrary moment diagram is developed to satisfy the static equilibrium under the applied loads at failure, then the applied loads are either equal to or less than the true collapse loads. For different moment diagrams, different factored loads can be obtained. Higher values of the lower, or static, bound are obtained when the moments at several sections for the assumed moment diagram reach the collapse moment. Horne [11] explained the lower bound by assuming different moment distributions to obtain the one that is in equilibrium with the applied loads and satisfies the yield condition all over the structure. In this case, the applied loads are either equal to or less than the collapse loads.

16.11 LIMIT ANALYSIS

For the analysis of structures by the limit design procedure, two methods can be used, the virtual work method and the equilibrium method. In the virtual work method, the work done by the factored load, P_u (or w_u), to produce a given virtual deflection, Δ, is equated to the work absorbed at the plastic hinges. The external work done by loads is $W_e = \Sigma(w_u\,\Delta)$ or $\Sigma\,(P_u\Delta)$. The work absorbed by the plastic hinges is internal work $= Wi = \Sigma(M_u\theta)$.

Example 16.3

The beam shown in Fig. 16.25 carries a concentrated load at midspan. Calculate the collapse moment at the critical sections.

Solution

1. The beam is once statically indeterminate ($r = 1$), and the number of plastic hinges needed to transform the beam into a mechanism is $n = 1 + 1 = 2$ plastic hinges, at A and C. The first plastic hinge develops at A, and the beam acts as a simply supported member until a mechanism is reached.

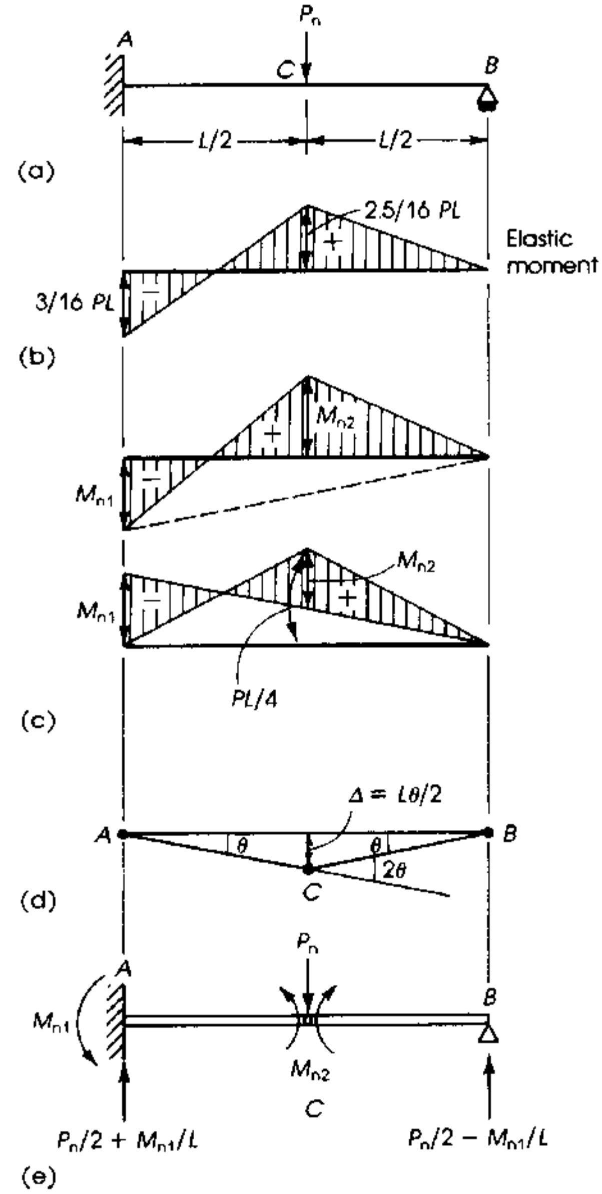


Figure 16.25 Example 16.3: $P_u = \phi P_n$ and $M_u = \phi M_n$.

2. If a rotation θ occurs at the plastic hinge at the fixed end, A, the rotation at the sagging hinge is $C = 2\theta$. The deflection of C under the load is $(L/2)\theta$ (Fig. 16.25).

$$W_e = \text{external work} = \sum P_u \Delta = P_u \left(\frac{L\theta}{2}\right)$$

$$W_i = \text{internal work} = \sum M_u \theta = M_{u_1}(\theta) + M_{u_2}(2\theta)$$

If the two sections at A and C have the same dimensions and reinforcement, then $M_{u_1} = M_{u_2} = M_u$, and $W_i = 3M_u\theta$. Equating W_e and W_i,

$$M_{u_1} + 2M_{u_2} = P_u \frac{L}{2} = 3M_u \quad \text{and} \quad M_u = \frac{P_u L}{6}$$

Example 16.4

Calculate the collapse moments at the critical sections for the beam shown in Fig. 16.26 due to a uniform load w_u.

Solution

1. The number of plastic hinges is two.
2. For a deflection at $C = 1.0$, the rotation at A, θ_A, is $1/a$; $\theta_B = 1/b$, and

$$\theta_c = \theta_A + \theta_B = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{L}{ab}$$

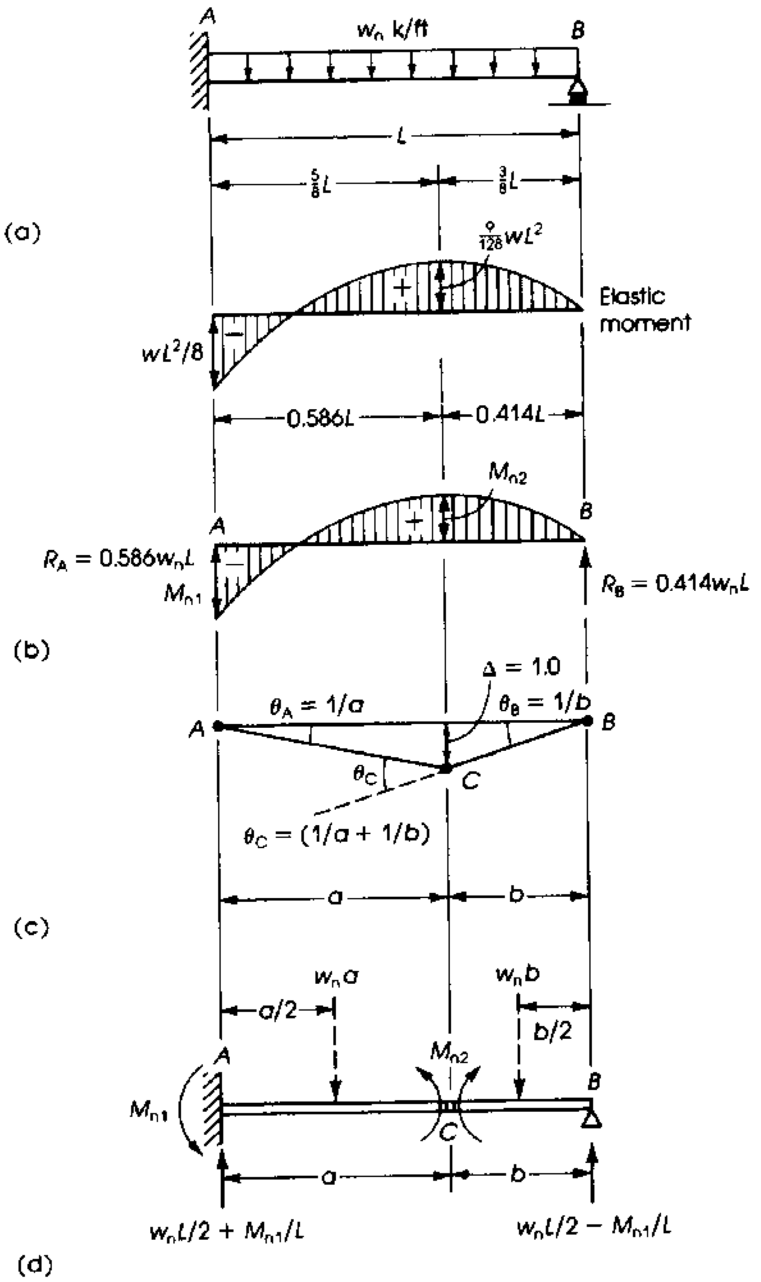


Figure 16.26 Example 16.4: $M_u = \phi M_n$ and $w_u = \phi w_n$.

3. External work is

$$W_e = \sum w_u \Delta = w_u \left(\frac{1 \times L}{2}\right) = \frac{w_u L}{2}$$

Internal work is

$$W_i = \sum \mathrm{m} M_u \theta = M_{u_1}\theta_A + M_{u_2}\theta_c$$

$$= M_{u_1}\left(\frac{1}{a}\right) + M_{u_2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Equating W_e and W_i,

$$w_u = \frac{2}{L}\left(\frac{M_{u_1}}{a} + \frac{M_{u_2}}{a} + \frac{M_{u_2}}{L-a}\right) \tag{16.9}$$

If both moments are equal, then

$$w_u = \frac{2M_u}{L}\left[\frac{2}{a} + \frac{1}{(L-a)}\right] = \frac{2M_u}{L}\left[\frac{(2L-a)}{a(L-a)}\right] \tag{16.10}$$

4. To determine the position of the plastic hinge at C that produces the minimum value of the collapse load w_u, differentiate Eq. 16.9 with respect to a and equate to 0:

$$\frac{\delta w_u}{\delta a} = 0 \quad -\left(\frac{M_{u_1}}{a^2} + \frac{M_{u_2}}{a^2} - \frac{M_{u_2}}{(L-a)^2}\right) = 0$$

If $M_{u_1} = M_{u_2} = M_u$, then

$$\frac{2}{a^2} = \frac{1}{(L-a)^2} \quad \text{or} \quad a = L(2-\sqrt{2}) = 0.586L$$

From Eq. 16.10, the collapse load is $w_u = 11.66\ (M_u/L^2)$, and the collapse moment is $M_u = 0.0858 w_u L^2$. The reaction at A is $0.586 w_u L$, and the reaction at B is $0.414 w_u L$.

In the equilibrium method, the equilibrium of the beam or of separate segments of the beam is studied under the forces present at collapse. To illustrate analysis by this method, the two previous examples are repeated here.

Example 16.5

For the beam shown in Fig. 16.25, calculate the collapse moments using the equilibrium method.

Solution

Two plastic hinges will develop at A and C. Referring to Fig. 16.25e, the reaction at A is $(P_u/2) + (M_{u_1} L)$ and the reaction at B is $(P_u/2) - (M_{u_1}/L)$.

Considering the equilibrium of beam BC and taking moments about C,

$$\left(\frac{P_u}{2} - \frac{M_{u_1}}{L}\right)\left(\frac{L}{2}\right) = M_{u_2}$$

$$M_{u_1} + 2M_{u_2} = P_u \frac{L}{2}$$

which is the same equation obtained in Example 16.3. When $M_{u_1} = M_{u_2} = M_u$, then

$$3M_u = P_u \frac{L}{2}$$

or $\quad M_u = P_u \dfrac{L}{6}$

Example 16.6

Calculate the collapse moments for the beam shown in Fig. 16.26 by the equilibrium method.

Solution

1. Two plastic hinges will develop in this beam at A and C. Referring to Fig. 16.26d, the reaction at $A = w_u(L/2) + (M_{u_1}/L)$ and the reaction at $B = w_u(L/2) - (M_{u_1}/L)$. The load on BC is $w_u b$ acting at $b/2$ from B, and $b = (L - a)$. Considering the equilibrium of segment BC and taking moments about C,

$$\left(w_u\frac{L}{2} - \frac{M_{u_1}}{L}\right)b - (w_u b)\frac{b}{2} = M_{u_2}$$

If $M_{u_1} = M_{u_2} = M_u$, then

$$w_u\frac{b}{2}(L - b) = M_u\left(1 + \frac{b}{L}\right) = \frac{M_u}{L}(2L - a)$$

$$w_u = \frac{2M_u}{L} \times \frac{(2L - a)}{a(L - a)}$$

which is similar to the results obtained in Example 16.4.

$$M_u = \frac{w_u L}{2} \times \frac{a(L - a)}{(2L - a)}$$

2. The position of a can be determined as before, where $a = 0.586L$, $M_u = 0.0858 w_u L_2$, and $w_u = 11.66(M_u/L^2)$.

16.12 ROTATION OF PLASTIC HINGES

16.12.1 Plastic Hinge Length

The assumption that the inelastic rotation of concrete occurs at the point of maximum moment while other portions of the member act elastically is a theoretical one; in fact, the plastic rotation occurs on both sides of the maximum moment section over a finite length. This length is called the plastic hinge length, l_p. The hinge length, l_p, is a function of the effective depth d, and the distance from the section of highest moment to the point of contraflexure (zero moment).

Referring to Fig. 16.27a, the length $L_p/2$ represents the plastic hinge length on one side of the center of support. M_u and ϕ_u indicate the factored moment and ultimate curvature at the critical section, whereas M_y and ϕ_y indicate the moment and curvature at first yield. The plastic curvature at the critical section ϕ_p is equal to $\phi_u - \phi_y$) and the rotation capacity is equal to $(\phi_p l_p)$.

The estimated length of the plastic hinge was reported by many investigators. A. L. L. Baker [7] assumed that the length of the plastic hinge is approximately equal to the effective depth d. Corley [12] proposed the following expression for the equivalent length of the plastic hinge:

$$l_p = 0.5d + 0.2\sqrt{d}\left(\frac{z}{d}\right) \qquad (16.11)$$

where z = distance of the critical section to the point of contraflexure and d = effective depth of the section. Mattock [13] suggested a simpler form:

$$l_p = 0.5d + 0.05z \qquad (16.12)$$

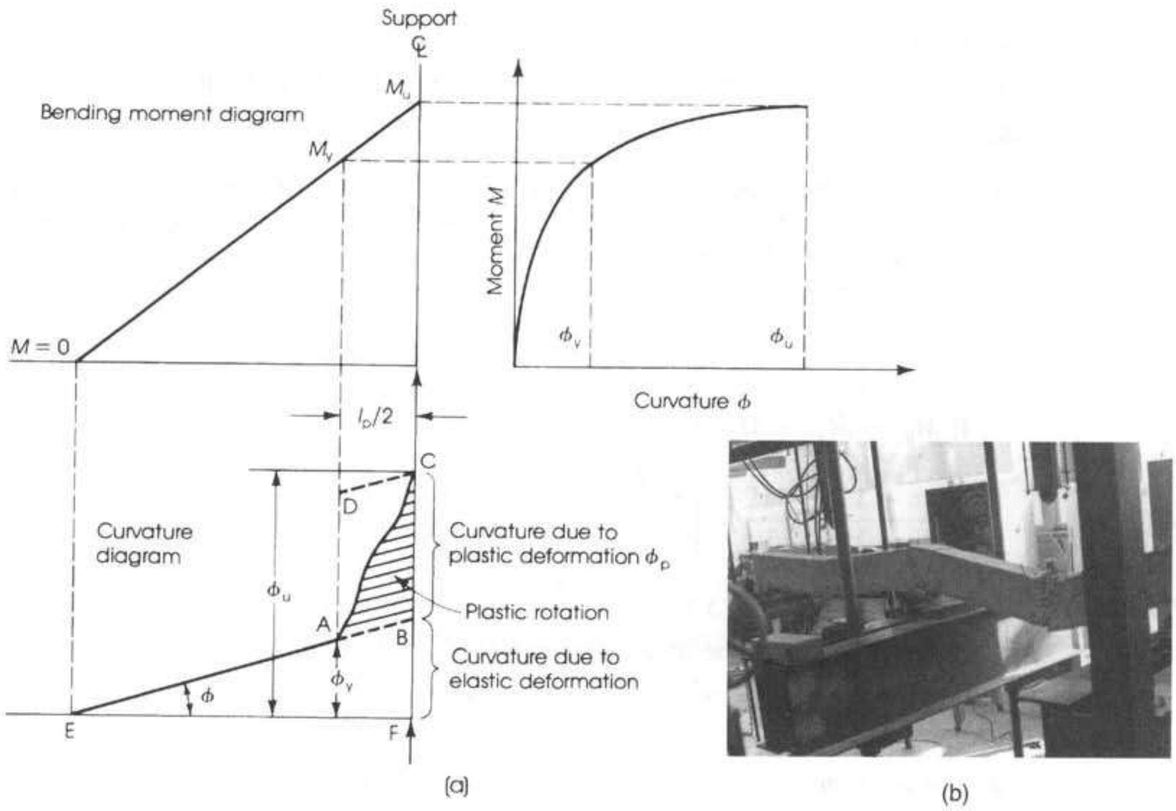


Figure 16.27 (*a*) Plastic rotation from moment-curvature and moment gradient and (*b*) development of plastic hinges in a reinforced concrete continuous beam.

Tests [14] on reinforced concrete continuous beams showed that l_p can be assumed equal to $1.06d$. They also showed that the length of the plastic hinge, in reinforced concrete continuous beams containing hooked-end steel fibers, increases with the increase in the amount of the steel fibers and the main reinforcing steel according to the following expression:

$$l_p = (1.06 + 0.13\rho\rho_s)d \tag{16.13}$$

where ρ = percentage of main steel in the section and ρ_s = percentage of steel fibers by volume, $0 \leq \rho_s \leq 1.2$. For example, if $\rho = 1.0\%$ and $\rho_s = 0.8\%$, then $l_p = 1.164d$.

16.12.2 Curvature Distribution Factor

Another important factor involving the calculation of plastic rotations is the curvature distribution factor, β. The curvature along the plastic hinge varies significantly, and in most rotation estimations this factor is ignored, which leads to an overestimation of the plastic rotations. Referring to Fig. 16.27, the shaded area, *ABC*, represents the inelastic rotation that can occur at the plastic hinge, whereas the unshaded area, *EBF*, represents the elastic contribution to the rotation over the length of the member. The shaded area *ABC* can be assumed to be equal to β times the total area *ABCD* within the plastic hinge length, $l_p/2$, on one side of the critical section. The curvature distribution factor, β, represents the ratio of the actual plastic rotation, θ_{pc}, to ϕl_p, where ϕ is the curvature and l_p is the length of the plastic hinge. The value of β was reported to vary between 0.5 and 0.6. Tests [14] have showed that β can be assumed to be equal to 0.56.

When hooked-end steel fibers were used in concrete beams, the value of β decreased according to the following expression:

$$\beta = 0.56 - 0.16\rho_s \tag{16.14}$$

where ρ_s is the percentage of steel fibers, $0 \leq \rho_s \leq 1.2\%$. The reduction of the curvature distribution factor of fibrous concrete does not imply that the rotation capacity is reduced: The plastic curvature of fibrous concrete is substantially higher than that of concrete without fibers. Figure 16.28 shows the distribution of the curvature along the plastic hinge length. The area ABC_1 represents the plastic rotation for concrete that does not contain steel fibers, $\beta = 0.56$, whereas the areas ABC_2 and ABC_3 represent the plastic rotation for concretes containing 0.8% and 1.2% steel fibers, respectively.

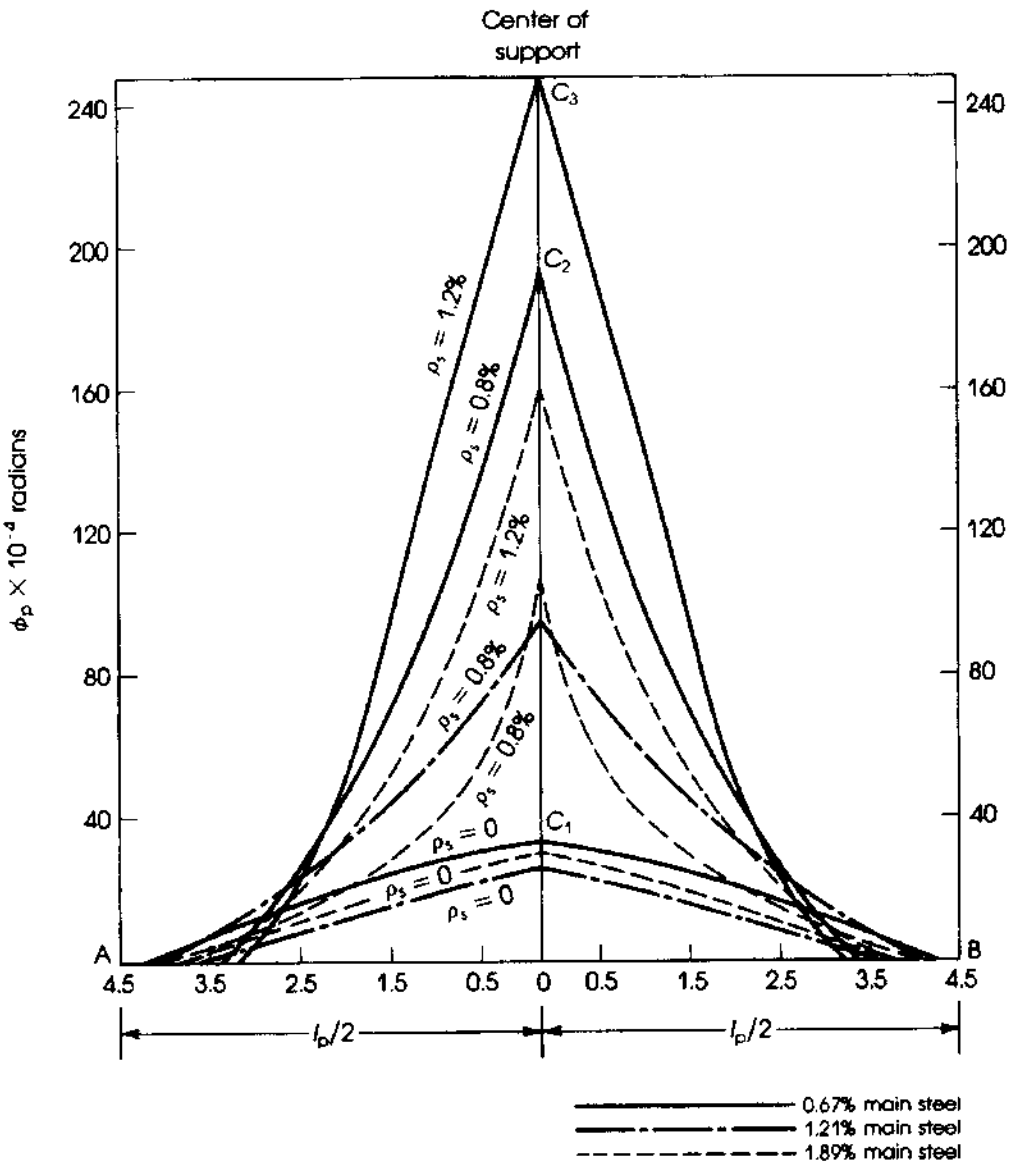


Figure 16.28 Curvature distribution along the plastic hinge.

16.12.3 Ductility Index

The ratio of ultimate to first-yield curvature is called the *ductility index*, $\mu = \phi_u/\phi_y$. The ductility index of reinforced concrete beams was reported [15] to vary between 4 and 6. If steel fibers are used in concrete beams, the ductility index increases according to the following expression [14]:

$$\mu' = (1.0 + 3.8\rho_s)\mu \tag{16.15}$$

where

μ = the ratio of ultimate to first-yield curvature
μ' = ductility index of the fibrous concrete
ρ_s = percentage of steel fibers by volume, $0 \le \rho_s \le 1.2\%$.

16.12.4 Required Rotation

The rotation of a plastic hinge in a reinforced concrete indeterminate structure is required to allow other plastic hinges to develop, and the structure to reach a mechanism can be determined by slope deflection from the following expression [7,20]. For a segment AB between two plastic hinges, the rotation at A is

$$\theta_A = \frac{L}{6E_cI}[2(M_A - M_{FA}) + (M_B - M_{FB})] \tag{16.16}$$

where

M_A and M_B = factored moments at A and B, respectively
M_{FA} and M_{FB} = elastic fixed-end moments at A and B
E_c = modulus of elasticity of concrete = $33w^{1.5}\sqrt{f'_c}$
I = moment of inertia of a cracked section (Chapter 5)

16.12.5 Rotation Capacity Provided

Typical tensile plastic hinges at the support and midspan sections of a frame are shown in Fig. 16.29. The rotation capacity depends mainly on the following:

1. The ultimate strain capacity of concrete, ε'_c, which may be assumed to be 0.003 or 0.0035, as used by Baker [7].
2. The length, l_p, over which yielding occurs at the plastic hinge, which can be assumed to be approximately equal to the effective depth of the section where the plastic hinge developed ($l_p = d$).
3. The depth of the compressive block c in concrete at failure at the section of the plastic hinge. Baker [7] estimated the angle of rotation, θ, of a tensile plastic hinge as follows:

$$\theta = \frac{\varepsilon_p l_p}{c} \tag{16.17}$$

where ε_p is the increase in the strain in the concrete measured from the initial yielding of steel reinforcement in the section (see Fig. 16.29c):

$$\varepsilon_p = \varepsilon'_c - \varepsilon_{c_1} = 0.0035 - \varepsilon_{c_1}$$

If $l_p = d$ and the ratio c/d equals $\lambda \le 0.5$,

$$\theta = \frac{(0.0035 - \varepsilon_{c_1})d}{\lambda d} = \frac{0.0035 - \varepsilon_{c_1}}{\lambda}$$

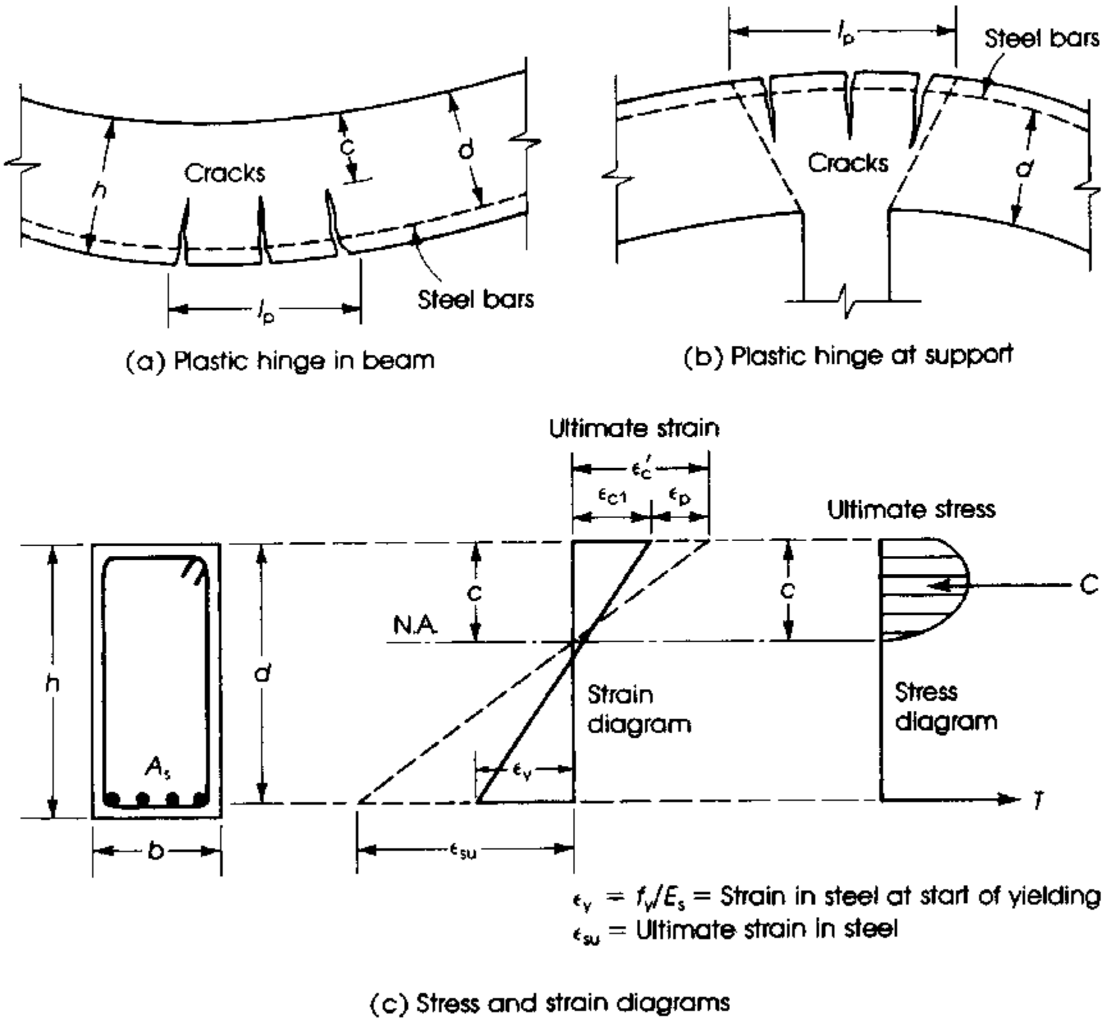


Figure 16.29 Plastic hinge and typical stress and strain distribution [2].

From strain triangles (Fig. 16.29),

$$\varepsilon_{c_1} = \varepsilon_y \left(\frac{c}{d-c}\right) = \frac{f_y}{E_s}\left(\frac{\lambda d}{d-\lambda d}\right) = \frac{f_y}{E_s}\left(\frac{\lambda}{1-\lambda}\right)$$

where f_y = yield strength of steel bars and E_s = modulus of elasticity of steel = 29×10^6 psi. Therefore,

$$\theta = \frac{0.0035}{\lambda} - \frac{\varepsilon_{c_1}}{\lambda} = \frac{0.0035}{\lambda} - \frac{f_y}{E_s(1-\lambda)} \qquad (16.18)$$

For grade 40 steel, f_y = 40 ksi, and using a maximum value of λ of 0.50, then

$$\theta_{\min} = \frac{0.0035}{0.50} - \frac{40}{29{,}000 \times (1-0.50)} = 0.00424 \text{ rad}$$

For grade 60 steel, f_y = 60 ksi and $\lambda_{\max} = 0.44$;

$$\theta_{\min} = \frac{0.0035}{0.44} - \frac{60}{29{,}000(1-0.44)} = 0.00426 \text{ rad}$$

The θ_{min} calculated here is from one side only, and the total permissible rotation at the plastic hinge equals 2θ or $2\theta_{min}$. The actual λ can be calculated as follows, given $\alpha = \beta_1 c$ and $\beta_1 = 0.85$ for $f'_c \leq 4$ ksi:

$$c = \frac{a}{0.85} = \frac{A_s f_y}{(0.85)^2 f'_c b}$$

$$\lambda = \frac{c}{d} = \frac{A_s f_y}{0.72 f'_c bd} = \frac{\rho f_y}{0.72 f'_c} \leq 0.5 \tag{16.19}$$

where $\rho = A_s/bd$. (λ_{max} is obtained when ρ_{max} is used.)

If the rotation provided is not adequate, one can increase the section dimensions or reduce the percentage of steel reinforcement to obtain a smaller c, a smaller λ, and greater θ. A. L. L. Baker [3] indicated that if special binding or spirals are used, the ultimate crushing strain in bound concrete may be as high as 0.012.

For a compression plastic hinge (as in columns),

$$\theta = \frac{\varepsilon_p l_p}{h} \tag{16.20}$$

where h = overall depth of the section and l_p = length over which yielding occurs. In compression hinges, l_p varies between $0.5h$ and h.

At a concrete ultimate stress of f'_c, $\varepsilon_c = 0.002$; thus, $\varepsilon_p = \varepsilon'_c - 0.002 = 0.0035 - 0.002 = 0.0015$ is the minimum angle of rotation on one side. Therefore,

$$\theta_{min} = \frac{0.0015 \times 0.5h}{h} = 0.00075 \text{ rad}$$

With special binding or spirals, θ may be increased to

$$\theta_{max} = (0.012 - 0.002) \times \frac{0.5h}{h} = 0.005 \text{ rad}$$

The extreme value of $\varepsilon'_c = 0.012$ is quite high, and a smaller value may be used with proper spirals; otherwise a different section must be adopted.

In reinforced concrete continuous beams containing steel fibers, the plastic rotation may be estimated as follows [14]:

$$\theta_p = \lambda\beta\left(\frac{0.0035}{\lambda} - \frac{f_y}{E_s(1-\lambda)}\right) \tag{16.21}$$

where

$$\lambda = (4.3 + 2.24\rho_s - 0.043 f_y + 4.17\rho\rho_s) \tag{16.22}$$

$$\beta = 0.56 - 0.16\rho_s \tag{16.14}$$

f_y = yield strength of steel, ksi

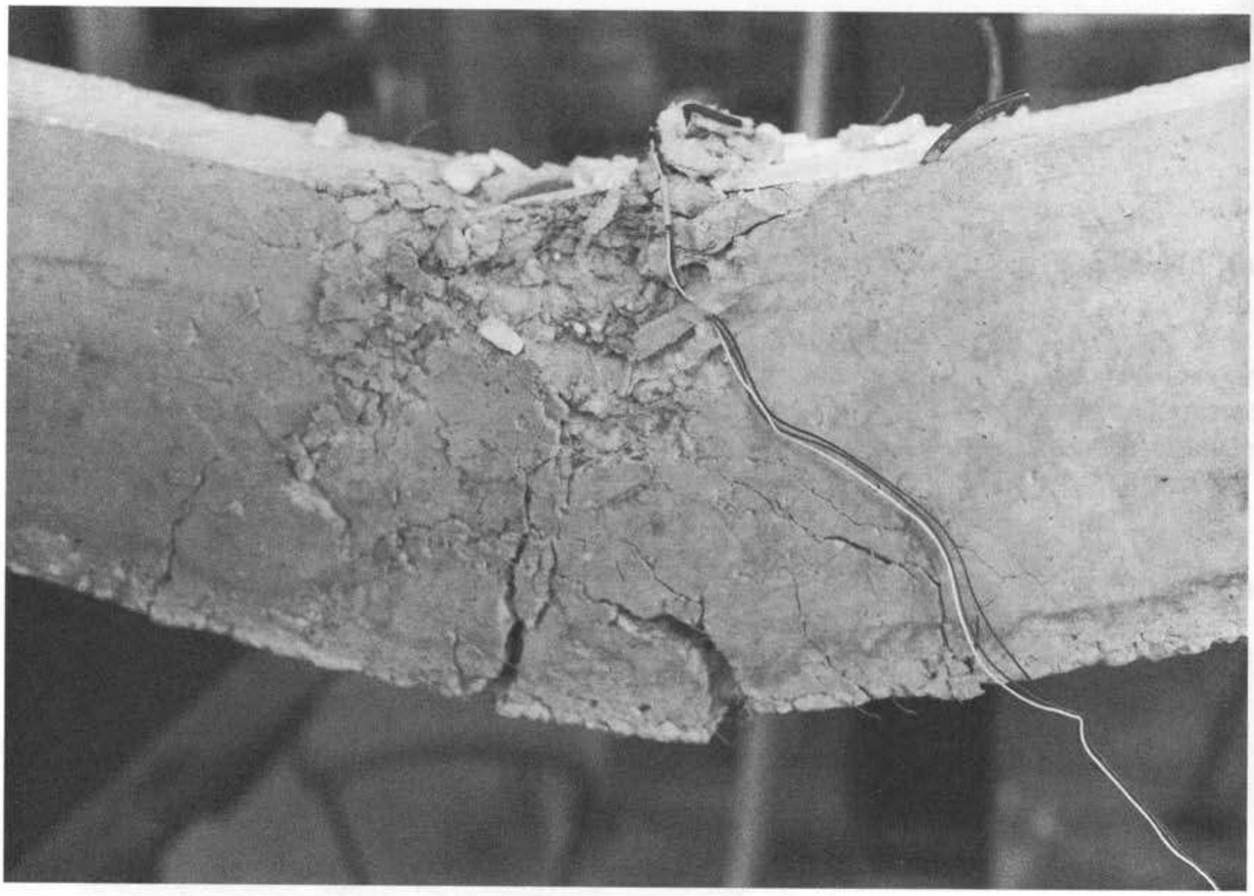
E_s = modulus of elasticity of main steel

ρ = percentage of main steel

ρ_s = percentage of steel fibers

From Eq. 16.21, it is obvious that the plastic rotation of fibrous reinforced concrete is dependent upon the percentage of steel fibers and percentage of the main steel and its yield strength. Raising the yield strength of the main steel reduces the plastic rotation. Equation 16.21 also includes the effect of the plastic hinge length on rotation.

Plastic hinge in the maximum negative moment region.



Plastic hinge in the maximum positive moment region.

A simplified form can be presented [14]:

$$\theta_p = \lambda\beta\left(\frac{0.003}{\lambda}\right) \tag{16.23}$$

For example, if $\rho_s = 0$ and $f_y = 60\,\text{ksi}$, then $\theta_{p_1} = 0.00289/\lambda$, and if $\rho_s = 1.0\%$, $\rho = 1.5\%$, and $f_y = 60\,\text{ksi}$, then $\theta_{p_2} = 0.01222/\lambda$. This means that the rotation capacity of a concrete beam may be increased by about four times if 1% of steel fibers is used.

16.13 SUMMARY OF LIMIT DESIGN PROCEDURE

1. Compute the factored loads using the load factors given in Chapter 3:

$$w_u = 1.2D + 1.6L$$

2. Determine the mechanism, plastic hinges, and factored moments M_u.
3. Design the critical sections using the strength design method.
4. Determine the required rotation of plastic hinges.
5. Calculate the rotation capacity provided at the sections of plastic hinges. The rotation capacity must exceed that required.

6. Check the factor against yielding of steel and excessive cracking, that is, ϕM_u/elastic moment at service load.
7. Check deflection and cracking under service loads.
8. Check that adequate shear reinforcement is provided at all sections.

For more details, see Ref. 21.

Example 16.7

The beam shown in Fig. 16.30 is fixed at both ends and carries a uniform factored load of 5.5 K/ft, and a concentrated factored load of 48 K. Design the beam using the limit design procedure. Use $b = 14$ in., $f'_c = 3$ ksi, and $f_y = 40$ ksi.

Solution

1. Factored uniform load $w_u = 5.5$ K/ft. Factored concentrated load $P_u = 48$ K.
2. The plastic hinges will develop at A, B, and C, causing the mechanism shown in Fig. 16.30. Using the virtual work method of analysis and assuming a unit deflection at C, then the external work is equal to

$$W_e = 48 \times 1 + 5.5\left(24 \times \frac{1}{2}\right) = 114 \text{ K·ft}$$

The internal work absorbed by the plastic hinges is

$$W_i = M_u\theta(\text{at } A) + M_u\theta(\text{at } B) + M_u(2\theta) \text{ at } C$$

$$= 4M_u\theta = 4M_u\left(\frac{1}{12}\right) = \frac{M_u}{3}$$

Equating W_e and W_i gives $M_u = 342$ K·ft. The general analysis gives directly

$$M_u = \frac{w_u L^2}{16} + P_u\frac{L}{8} = \frac{5.5}{16}(24)^2 + 48\frac{24}{8} = 342 \text{ K·ft}$$

3. Design the critical sections at A, B, and C for $M_u = 342$ K·ft. From tables in Appendix A and for $f'_c = 3$ ksi, $f_y = 40$ ksi, and a steel percentage $\rho = 0.013$, $R_u = 420$ psi ($\rho_{\max} = 0.0203$).

$$M_u = R_u bd^2$$

$$342 \times 12 = 0.42 \times 14(d)^2$$

$d = 26.4$ in. and the total depth is $h = 26.4 + 2.5 = 28.9$ in., say, 29 in.

$$A_s = \rho bd = 0.013 \times 14 \times 26.4 = 4.8 \text{ in.}^2$$

Use five no. 9 bars in one row; A_s provided $= 5.0$ in.^{2}, $b_{\min} = 13.875$ in. < 14 in.

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{5.0 \times 40}{0.85 \times 3 \times 14} = 5.6 \text{ in.}$$

$$c = \frac{a}{0.85} = 6.6 \text{ in.} \quad \lambda = \frac{c}{d} = \frac{6.6}{26.4} = 0.25$$

4. The required rotation of plastic hinges is as follows:

a.

$$\theta_A = \frac{L}{6E_c I}[2(M_A - M_{\text{FZ}}) + (M_B - M_{\text{FB}})]$$

$$E_c = 57{,}400\sqrt{f'_c} = 3.144 \times 10^6 \text{ psi}$$

$$E_s = 29 \times 10^6 \text{ psi} \quad \text{and} \quad n = \frac{E_s}{E_c} = 9.2$$

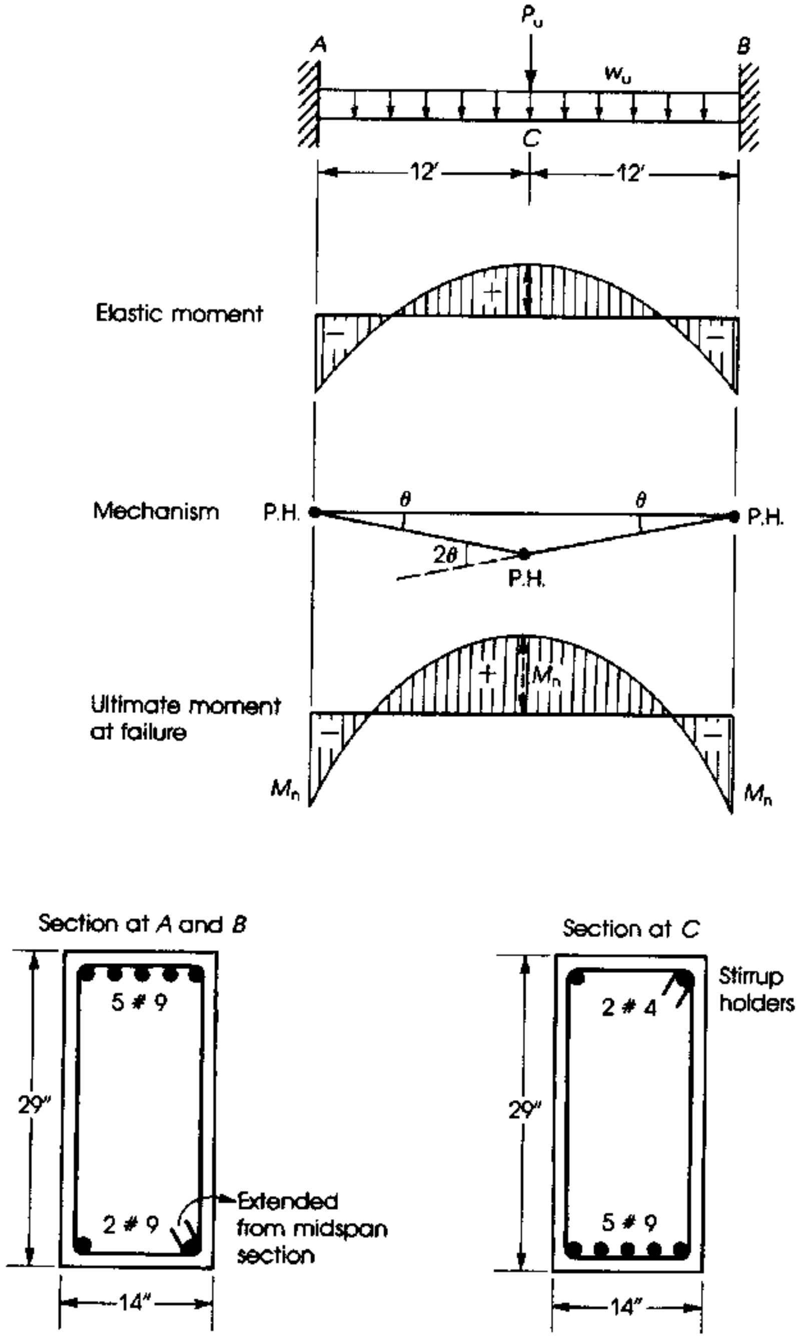


Figure 16.30 Example 16.7.

b. Determine the fixed end moments at A and B using factored loads:

$$M_{FA} = M_{FB} = \frac{w_u L^2}{12} \text{ (uniform load)} + \frac{P_u L}{8} \text{ (concentrated load)}$$

$$= 5.5\frac{(24)^2}{12} + 48 \times \frac{24}{8} = 408 \text{ K·ft}$$

Plastic M_A = plastic M_B = 342 K·ft

c. The cracked moment of inertia can be calculated from

$$I_{cr} = b\frac{x^3}{3} + nA_s(d - x)^2$$

where x is the distance from compression fibers to the neutral axis (kd). To determine x (see Chapter 6), $x = 10.3$ in. and $I_{cr} = 17{,}172$ in.^{4}

d. Required minimum rotation: Considering all moments at supports A and B are negative, then

$$\theta_A = \frac{24 \times 12}{6 \times 3.144 \times 10^6 \times 17{,}172}[2(-342 + 408) + (-342 + 408)](12{,}000) = 0.00211 \text{ rad}$$

5. The rotation capacity provided is

$$\theta_A = \frac{0.0035}{\lambda} - \frac{f_y}{E_s(1 - \lambda)} = \frac{0.0035}{0.25} - \frac{40}{29{,}000(1 - 0.25)}$$
$$= 0.0122 \text{ rad} > 0.00211 \quad \text{required}$$

The rotation capacity provided is about 5.5 times that required, indicating that the section is adequate.

6. Check the ratio of factored to elastic moment at service load:

$$M_A = M_B = \frac{wL^2}{12} + \frac{PL}{8}$$
$$= 3.5\frac{(24)^2}{12} + \frac{30 \times 24}{8} = 258 \text{ K·ft}$$

Actual $\phi M_n = \phi A_s f_y[d - (a/2)] = 0.9 \times 5 \times 40[26.5 - (5.6/2)]/12 = 356$ K·ft. The ratio is $356/258 = 1.38$, which represents the factor of safety against the yielding of steel bars at the support.

7. Check maximum deflection due to service load (at midspan): Let the uniform service load (w) $= 3.5$ K/ft, and $P = 30$ K then:

$$\Delta_1 = \frac{wL^4}{384EI}$$

For a concentrated load at midspan,

$$\Delta_2 = \frac{PL^3}{192EI}$$

and total deflection is

$$\Delta = \frac{(3500/12)(24 \times 12)^4}{384(17{,}172)(3.144 \times 10^6)} + \frac{30{,}000(24 \times 12)^3}{192(17{,}172)(3.144 \times 10^6)} = 0.166 \text{ in.}$$

$$\frac{\Delta}{L} = \frac{0.166}{24 \times 12} = \frac{1}{1735}$$

which is a very small ratio.

8. Adequate shear reinforcement must be provided to avoid any possible shear failure.

16.14 MOMENT REDISTRIBUTION OF MAXIMUM NEGATIVE OR POSITIVE MOMENTS IN CONTINUOUS BEAMS

Moment redistribution of maximum positive or negative moments in continuous flexural members is based on the net tensile strain (NTS), ε_t, for both reinforced and prestressed concrete members. Figure 16.31 shows the permissible limits on moment redistribution. It indicates that the percentage decrease in the negative moments at supports and positive moments between supports of continuous beam, q', calculated by the elastic theory, must not exceed $1000\varepsilon_t\%$, with maximum of 20%. Moment redistribution is allowed only when $\varepsilon_t \geq 0.0075$, indicating adequate ductility is available at the section at which moment is reduced. When $\varepsilon_t < 0.0075$, no moment redistribution is allowed. The modified negative moments must be used to calculate the modified positive moments within the span, ACI Code, Section 8.4. Moment redistribution does not apply to members designed by the direct design method for slab systems. (Refer to Chapter 17.)

In summary, the percentage of decrease in maximum negative or positive moments in continuous beams is as follows:

1. When $\varepsilon_t \geq 0.0075$, moment redistribution is allowed. $(\rho/\rho_b > 0.476)$
2. When $\varepsilon_t = 0.0075$, the percentage of moment redistribution is 75% $(\rho/\rho_b = 0.476)$.
3. When $\varepsilon_t \geq 0.020$, the percentage of moment redistribution is 20% $(\rho/\rho_b = 0.217)$.
4. When $0.0075 < \varepsilon_t < 0.020$, the percentage of moment redistribution is:

$$q' = 1000\varepsilon_t \tag{16.24}$$

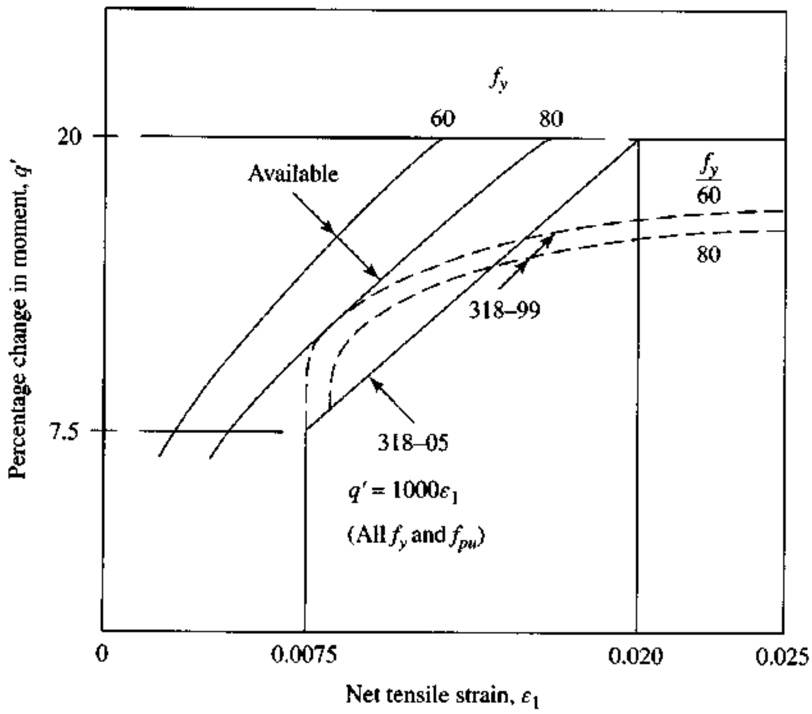


Figure 16.31 Permissible moment redistribution for minimum rotation capacity [22]. Courtesy of ACI-PCA.

Table 16.1 Percentage Change in Moment Redistribution (q'), $f_y = 60$ ksi

ε_t	**0.0075**	**0.0100**	**0.0125**	**0.0150**	**0.0175**	**0.0200**	**0.0225**
ρ/ρ_b	0.476	0.385	0.323	0.278	0.244	0.217	0.196
q' %	7.5	10.0	12.5	15.0	17.5	20.0	20.0

For example, if $\varepsilon_t = 0.010$, then the percentage of moment redistribution is 10%. The relationship between the steel percentage, ρ, in the section and the net tensile strain, ε_t, is as follows (refer to Section 3.10):

$$\varepsilon_t = \left[\frac{\left(0.003 + \frac{f_y}{E_s}\right)}{\left(\frac{\rho}{\rho_b}\right)}\right] - 0.003 \tag{3.24}$$

For grade 60 steel, $f_y = 60$ ksi and $E_s = 29{,}000$ ksi. Assuming $f_y/E_s = 0.002$, then

$$\varepsilon_t = \left[\frac{0.005}{\left(\frac{\rho}{\rho_b}\right)}\right] - 0.003 \tag{3.25}$$

For $\varepsilon_t = 0.0075$, the ductility limit $\varepsilon_t/\varepsilon_y = 0.0075/0.002 = 3.75$. The percentage change in moment redistribution according to these limitations and for $f_y = 60$ ksi given in Tables 16.1 and 16.2.

Whatever percentage of moment redistribution is used, it is essential to ensure that no sections is likely to suffer local damage or excessive cracking at service loads and that adequate rotation capacity is maintained at every critical section in the structure. The redistribution of moments in a statically indeterminate structure will result in a reduction in the negative moments at the supports and in the positive moments within the spans. This reduction will not imply that the safety of the structure has been reduced or jeopardized as compared with determinate structures. In fact, continuity in structures provides additional strength, stability and economy in the design.

Moment redistribution factor, q, based on the ACI Code 318-02 is calculated as follows:

$$q = 20\left[1 - \frac{(\rho - \rho')}{\rho_b}\right] \tag{16.25}$$

In Eq. 16.25, the steel ratio ρ or $(\rho - \rho')$, at the section where the moment is limited to a maximum ratio of $0.5\rho_b$. The minimum steel ratio in the section, for flexural design is limited

Table 16.2 Percentage Change in Moment Redistribution (q') for a Given ρ/ρ_b Ratio

ρ/ρ_b	0.48	0.45	0.40	0.35	0.30	0.25	0.20
ε_t	0.0074	0.0081	0.0095	0.0113	0.0137	0.017	0.022
q' %	0.0	8.1	9.5	11.3	13.7	17.0	20.0

Table 16.3 Maximum and Minimum Moment Redistribution q (Eq. 16.25)

f'_c (ksi)	f_y (ksi)	ρ_b	ρ_{min}	q_{max} % (for ρ_{min})	q_{min} % (for 0.5 ρ_b)
3	60	0.0215	0.0033	16.9	10
4	60	0.0285	0.0033	17.7	10
5	60	0.339	0.0035	17.9	10

to $3\sqrt{f'_c}/f_y \geq 200/f_y$. Using these extreme limitations, the maximum and minimum moment redistribution percentages are shown in Table 16.3.

Example 16.8

Determine the maximum elastic moments at the supports and midspans of the continuous beam of four equal spans shown in Fig. 16.32*a*. The beam has a uniform section and carries a uniform dead load of 8 K/ft and a live load of 6 K/ft. Assume 10% maximum redistribution of moments and consider the following two cases: (1) When the live load is placed on alternate spans, calculate the maximum positive moments within the spans, and (2) when the live load is placed on adjacent spans, calculate the maximum negative moments at the supports.

Solution

1. The beam has a uniform moment of inertia I and has the same E; thus, EI is constant. The three-moment equation to analyze the beam and for a constant EI is

$$M_A L_1 + 2M_b(L_1 + L_2) + M_c L_2 = -\frac{w_1 L_1^3}{4} - \frac{w_2 L_2^3}{4}$$

Because the spans are equal,

$$M_A + 4M_B + M_C = -\frac{L^2}{4}(w_1 + w_2) \tag{16.26}$$

In this example $M_A = M_E = 0$. Six different cases of loading will be considered, as shown in Fig. 16.31:

Case 1. Dead load is placed on the whole beam $ABCDE$ (Fig. 16.32*b*).

Case 2. Live load is placed on AB and CD for maximum positive moments within AB and CD (Fig. 16.32*c*).

Case 3. Similar to Case 2 for beams BC and DE (Fig. 16.32*d*).

Case 4. Live load is placed on AB, BC, and DE for a maximum negative moment at B (Fig. 16.32*e*).

Case 5. Live load is placed on spans CD and DE (Fig. 16.32*f*).

Case 6. Live load is placed on BC and CD for a maximum negative moment at C (Fig. 16.32*g*).

2. Case 1. Apply Eq. 16.26 to the beam segments ABC, BCD, and CDE, respectively:

$$4M_B + M_C = -\frac{(20)^2}{4}(8 + 8) = -1600 \text{ K·ft}$$
$$M_B + 4M_C + M_D = -1600 \text{ K·ft}$$
$$M_C + 4M_D = -1600 \text{ K·ft}$$

Solve the three equations to get

$$M_B = M_D = -342.8 \text{ K·ft} \quad \text{and} \quad M_C = -228.6 \text{ K·ft}$$

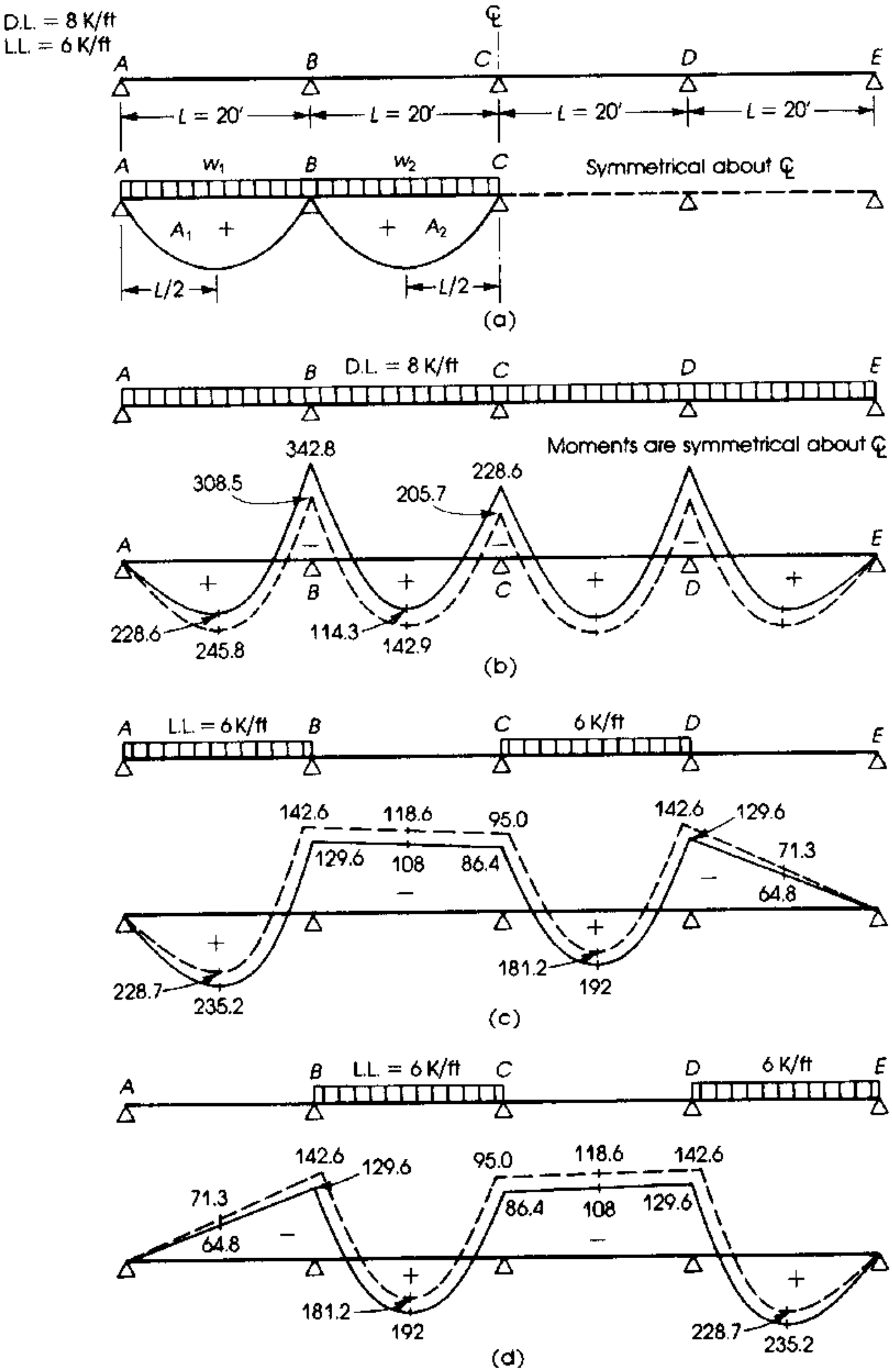


Figure 16.32 Example 16.8: Bending moments are drawn on the tension side.

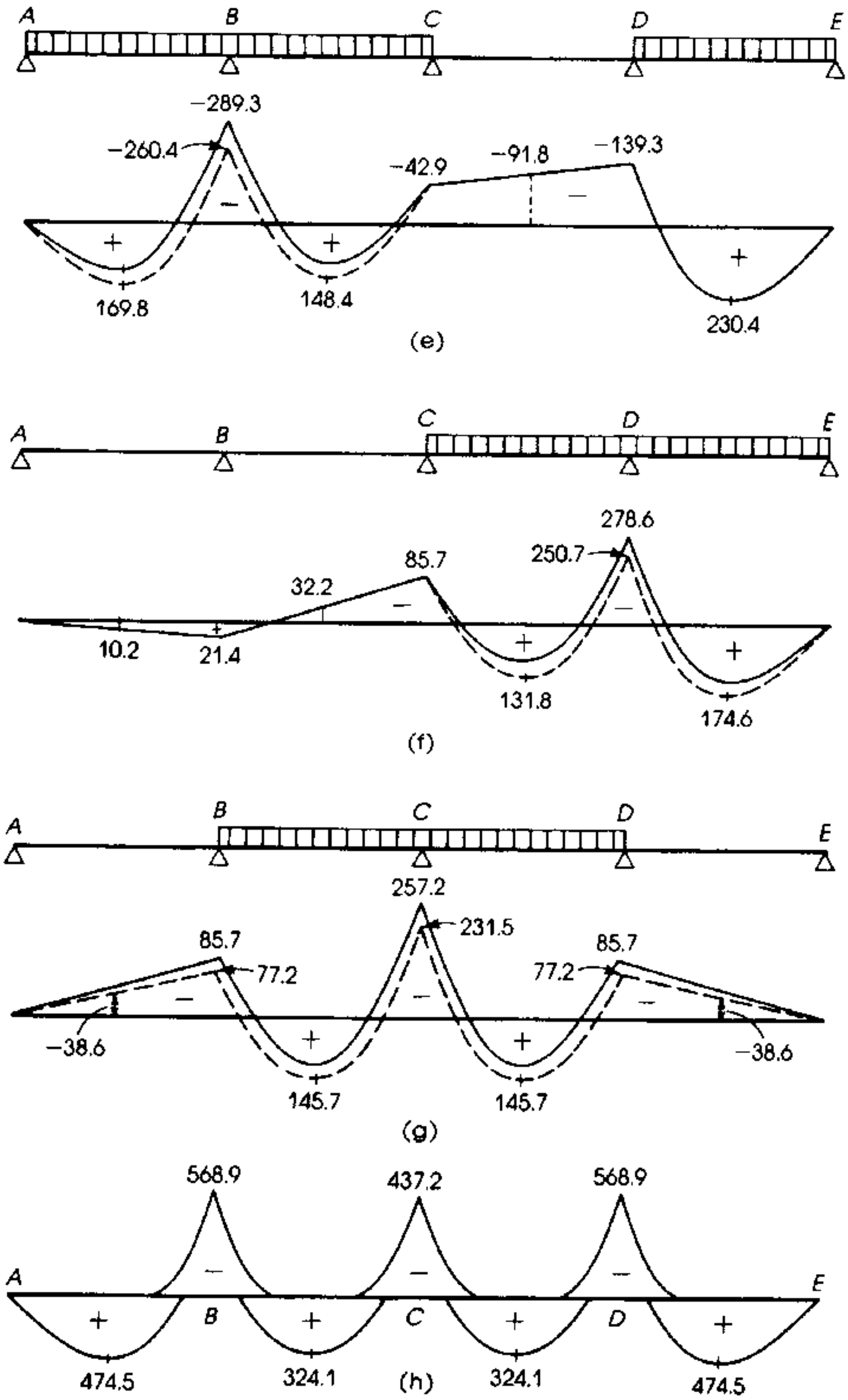


Figure 16.32 (*continued*)

For a 10% reduction in moments,

$$M'_B = M'_D = 0.9(-342.8) = -308.5 \text{ K·ft}$$

$$M'_C = 0.9(-228.6) = -205.7 \text{ K·ft}$$

The corresponding midspan moments are

$$\text{Span } AB = DE = \frac{w_D L^2}{8} + \frac{1}{2}M_B = \frac{8(20)^2}{8} - \frac{1}{2} \times 308.5 = 245.8 \text{ K·ft}$$

$$\text{Span } BC = CD = \frac{w_D L^2}{8} - \frac{1}{2}(308.5 + 205.7) = \frac{8(20)^2}{8} - 257.1 = 142.9 \text{ K·ft}$$

3. Case 2. Apply Eq. 16.26 to *ABC*, *BCD*, and *CDE*, respectively:

$$4M_B + M_C = -\frac{(20)^2}{4}(6) = -600 \text{ K·ft}$$

$$M_B + 4M_C + M_D = -600 \text{ K·ft}$$

$$M_C + 4M_D = -600 \text{ K·ft}$$

Solve the three equations to get

$$M_B = M_D = -129.6 \text{ K·ft} \qquad M_C = -86.4 \text{ K·ft}$$

The corresponding elastic midspan moments are

$$\text{Beam } AB = \frac{w_L L^2}{8} + \frac{M_B}{2} = \frac{6(20)^2}{8} - \frac{129.6}{2} = +235.2 \text{ K·ft}$$

$$BC = 0 - \frac{1}{2}(129.6 + 86.4) = -108 \text{ K·ft}$$

$$CD = \frac{w_L L^2}{8} - \frac{1}{2}(129.6 + 86.4) = \frac{6(20)^2}{8} - 108 = +192 \text{ K·ft}$$

$$DE = 0 - \frac{1}{2} \times 129.6 = -64.8 \text{ K·ft}$$

To reduce the positive span moment, increase the support moments by 10% and calculate the corresponding positive span moments. The resulting positive moment must be at least 90% of the first calculated moments given previously.

$$M'_B = M'_D = 1.1(-129.6) = -142.6 \text{ K·ft}$$

$$M'_C = 1.1(-86.4) = -95.0 \text{ K·ft}$$

The corresponding midspan moments are

$$\text{Beam } AB = \frac{w_L L^2}{8} + \frac{M'_B}{2} = \frac{6(20)^2}{8} - \frac{142.6}{2} = +228.7 \text{ K·ft}$$

$$BC = -\frac{1}{2}(142.6 + 95) = -118.8 \text{ K·ft}$$

$$CD = \frac{w_L L^2}{8} + \frac{1}{2}(M'_C + M'_D) = \frac{6(20)^2}{8} - \frac{1}{2}(95 + 142.6) = 181.2 \text{ K·ft}$$

$$DE = -\frac{1}{2} \times 142.6 = -71.3 \text{ K·ft}$$

4. Case 3. This case is similar to Case 2, and the moments are shown in Fig. 16.32*d*.

5. Case 4. Consider the spans *AB, BC*, and *DE* loaded with live load to determine the maximum negative moment at support *B*:

$$4M_B + M_C = -\frac{w_L L^2}{2} = -\frac{6(20)^2}{2} = -1200 \text{ K·ft}$$

$$M_B + 4M_C + M_D = -\frac{w_L L^2}{4} = -\frac{6(20)^2}{4} = -600 \text{ K·ft}$$

$$M_C + 4M_D = -\frac{6(20)^2}{4} = -600 \text{ K·ft}$$

Solve the three equations to get

$$M_C = -42.9 \text{ K·ft}$$

$$M_B = -289.3 \text{ K·ft}$$

$$M_D = -139.3 \text{ K·ft}$$

For 10% reduction in moment at support *B*,

$$M'_B = 0.9 \times (-289.3) = -260.4 \text{ K·ft}$$

The corresponding midspan moments are

$$\text{Beam } AB = \frac{w_L L^2}{8} + \frac{M_B}{2} = \frac{6(20)^2}{8} - \frac{260.4}{2} = 169.8 \text{ K·ft}$$

$$BC = \frac{w_L L^2}{8} - \frac{1}{2}(260.4 + 42.9) = 148.4 \text{ K·ft}$$

$$CD = -\frac{1}{2}(42.9 + 139.3) = -91.1 \text{ K·ft}$$

$$DE = 300 - \frac{1}{2} \times 139.3 = +230.4 \text{ K·ft}$$

6. Case 5. This is similar to Case 4, except that one end span is not loaded to produce maximum positive moment at support *B* (or support *D* for similar loading). The bending moment diagrams are shown in Fig. 16.32*f*.

7. Case 6. Consider the spans *BC* and *CD* loaded with live load to determine the maximum negative moment at support *C*:

$$4M_B + M_C = \frac{w_L L^2}{4} = -600 \text{ K·ft}$$

$$M_B + 4M_C + M_D = -\frac{w_L L^2}{2} = -1200 \text{ K·ft}$$

$$M_C + 4M_D = -\frac{w_L L^2}{4} = -600 \text{ K·ft}$$

Solve the three equations to get

$$M_C = -257.2 \text{ K·ft}$$

$$M_B = M_D = -85.7 \text{ K·ft}$$

For 10% reduction in support moments,

$$M'_C = 0.9 \times (-257.2) = -231.5 \text{ K·ft}$$

$$M'_B = M'_C = 0.9 \times (-85.7) = -77.2 \text{ K·ft}$$

Table 16.4 Final Moments of Example 16.8 after Moment Redistribution

Case Section Location	1 D.L. Moments	2 L.L. Maximum Negative	3 L.L. Maximum Positive	4 D.L. + L.L. (1) + (2) Maximum Negative	5 D.L. + L.L. (1) + (3) Maximum Positive
			Support		
A	0	0	0	0	0
B	−308.5	−260.4	+21.4	−568.9*	−287.1
C	−205.7	−231.5	—	−437.2*	−205.7
D	−308.5	−260.4	+21.4	−568.9*	−287.1
E	0	0	0	0	0
			Midspan		
AB	245.8	−71.3	228.7	±174.5	±474.5*
BC	142.9	−118.6	181.2	±24.3	±324.1*
CD	142.9	−118.6	181.2	±24.3	±324.1*
DE	245.8	−71.3	228.7	±174.5	±474.5*

*Final maximum and minimum design moments.

The corresponding midspan moments are

$$\text{Beam } AB = DE = -\frac{77.2}{2} = -38.6 \text{ K·ft}$$

$$BC = CD = \frac{w_L L^2}{8} - \frac{1}{2}(231.5 + 77.2) = \frac{6(20)^2}{8} - 154.3 = 145.7 \text{ K·ft}$$

8. The final maximum and minimum moments after moment redistribution are shown in Table 16.4. The moment envelope is shown in Fig. 16.32*h*.
9. In this example, the midspan sections are used for simplicity: The midspan moments are not necessarily the maximum positive moments. In the case of the end spans *AB* and *DE*, the maximum moment after 10% moment redistribution is equal to $(w_D L_2)/12.2$ and occurs at $0.4L$ from *A* and *D*.

Example 16.9

Determine the permissible redistribution of negative moments at supports *B*, *C*, *D*, and *E* of the continuous beam *ABCDEF* shown in Fig. 16.33. The beam has a rectangular section, $b = 12$ in., $h = 22$ in., and $d = 19.5$ in., and it is reinforced as shown in the following table ($f'_c = 4$ ksi and $f_y = 60$ ksi).

1. Use Appendix B.
2. Use the ACI Code limitations.

Solution

1. For $f'_c = 4$ ksi and $f_y = 60$ ksi, $\rho_b = 0.0285$. The ACI Code redistribution factor was given as follows:

$$q = 20\left[1 - \frac{\rho - \rho'}{\rho_b}\right] \tag{16.25}$$

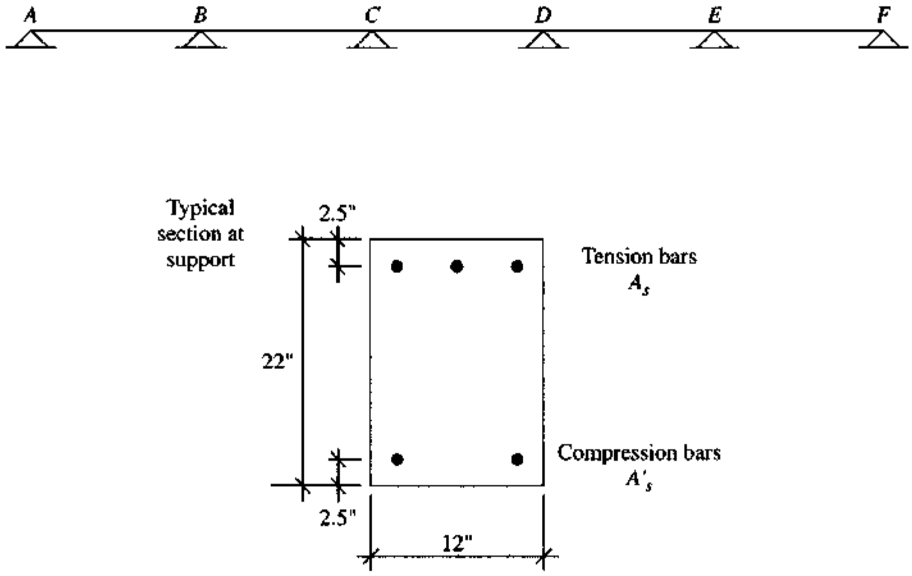


Figure 16.33 Example16.9.

2. The ACI Code redistribution factor is a function of the net tensile strain, ε_t, and varies between 7.5% and 20%, as shown in Fig. 16.31.

$$q' = 1000\varepsilon_t \tag{16.24}$$

$$\varepsilon_t = \frac{0.003 + \dfrac{f_y}{E_s}}{\dfrac{\rho}{\rho_b}} - 0.003$$

and

$$\varepsilon_t = \frac{0.005}{\dfrac{\rho}{\rho_b}} - 0.003 \quad (\text{for } f_y = 60 \text{ ksi})$$

The table shows the values of q and q', which are not compatible.

Support	Tension Bars (A_s)	ρ	Compression Bars (A'_s)	ρ'	$\dfrac{\rho - \rho'}{\rho_b}$	q%	ε_t	q'
B	3 no. 9	0.01282	0	0	0.45	11.0	0.0113	11.3
C	3 no. 10	0.0160	0	0	0.56	8.8	0.006	0
D	3 no. 6	0.00564	0	0	0.198	16.0	0.0226	20
E	4 no. 8	0.01342	3 no. 6	0.0056	0.273	14.5	0.0153	15.3

SUMMARY

Sections 16.1–16.3

In continuous beams, the maximum and minimum moments are obtained by considering the dead load acting on all spans, whereas pattern loading is considered for live or moving loads, as shown in Figs. 16.2 and 16.3. The ACI moment coefficients given in Chapter 9 may be used to compute approximate values for the maximum and minimum moments and shears.

Sections 16.4–16.5

A frame subjected to a system of loads may be analyzed by the equivalent frame method. Frames may be statically determinate or indeterminate.

Section 16.6

There are several types of frame hinges: Mesnager, Considére, lead, and concrete hinges. The steel for a Mesnager hinge is calculated as follows:

$$A_{s_1} = \frac{R_1}{0.55 f_y} \quad \text{and} \quad A_{s_2} = \frac{R_2}{0.55 f_y} \tag{16.2}$$

$$\text{Burst force:} \quad F = \frac{P_u}{2}\tan\theta + \frac{Ha}{0.85d} \tag{16.4}$$

$$\text{Stress in ties} \quad f_s = \frac{F}{0.005ab + A_{st}\ \text{(ties)}} \leq 0.85\ f_y \tag{16.6}$$

Sections 16.7–16.8

Limit design in reinforced concrete refers to redistribution of moments, which occurs throughout the structure as steel reinforcement reaches its yield strength. Ultimate strength is reached when the structure is on the verge of collapse. This case occurs when a number of plastic hinges, n, develop in a structure with redundants, r, such that $n = 1 + r$.

Sections 16.9–16.11

For limit design to be valid, four conditions must be satisfied: mechanism, equilibrium, yield, and rotation. Two methods of analysis may be used: the virtual work method and the equilibrium method, which are both explained in Examples 16.3 through 16.6.

Sections 16.12–16.13

The plastic hinge length, l_p, can be considered equal to the effective depth, d. In fibrous concrete,

$$l_p = (1.06 + 0.13\rho\rho_s)d \tag{16.13}$$

$$\text{Ductility index } \mu = \frac{\phi_u}{\phi_y}$$

For fibrous concrete,

$$\mu' = (1.0 + 3.8\rho_s)\mu \tag{16.15}$$

$$\text{Angle of rotation } \theta = \frac{0.0035}{\lambda} - \frac{f_y}{E_s(1-\lambda)} \tag{16.18}$$

$$\lambda = \frac{\rho f_y}{0.72 f_c'} \leq 0.5 \tag{16.19}$$

A summary of the limit design procedure is given in Section 16.14.

Section 16.14

Moment redistribution may be taken into account in the analysis of statically indeterminate structures. In this case, the maximum negative moments calculated by the elastic theory may be increased or decreased by not more than the ratio q', where

$$q' = 1000\varepsilon_t \tag{16.24}$$

Table 16.1 gives the different values of q. Moment redistribution is explained in detail in Example 16.8.

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PROBLEMS

16.1 The slab-beam floor system shown in Fig. 16.34 carries a uniformly distributed dead load (excluding weight of slab and beam) of 40 psf and a live load of 100 psf. Using the ACI Code coefficients, design the interior continuous beam *ABCD* and draw detailed sections. Given: $f_c' = 4$ ksi, $f_y = 60$ ksi, width of beam web = 12 in., slab thickness = 4.0 in., and column dimensions = 14 by 14 in.

16.2 Repeat Problem 16.1 using span lengths of the beams shown in Fig. 16.32 as follows:

$$L_1 = 20 \text{ ft} \quad L_2 = 24 \text{ ft}$$

$$L_3 = 20 \text{ ft} \quad L_4 = 10 \text{ ft}$$

16.3 For the beam shown in Fig. 16.35, compute the reactions at *A*, *B*, and *C* using constant *EI*. Draw the shear and bending moment diagrams and design all critical sections, using $b = 14$ in, $h = 25$ in., $f_c' = 4$ ksi, $f_y = 60$ ksi, and a load factor = 1.6.

16.4 Repeat Problem 16.3 using span lengths of beams as follows: span $AB = 20$ ft and span $BC = 16$ ft.

16.5 The two-hinged portal frame *ABCD* shown in Fig. 16.36 carries a uniform dead load (excluding self-weight) = 2.6 K/ft and a uniform live load of 1.8 K/ft. Design the frame *ABCD*, the hinges, and footings using $f_c' = 4$ ksi, $f_y = 60$ ksi, and a beam width of $b = 16$ in. The footing is placed 5 ft below ground level and the allowable bearing soil pressure is 5 ksf. Use a slab thickness of 6 in.

16.6 Design the portal frame *ABCD* of Problem 16.5 if the frame ends at *A* and *D* are fixed.

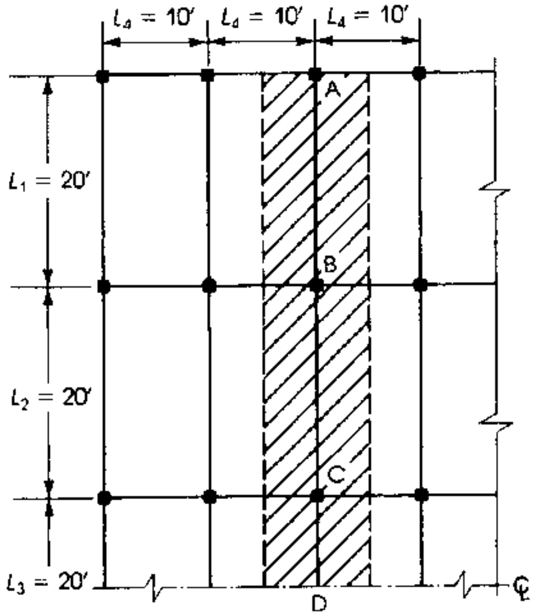


Figure 16.34 Problem 16.1.

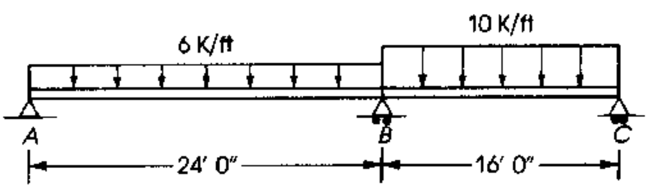


Figure 16.35 Problem 16.3.

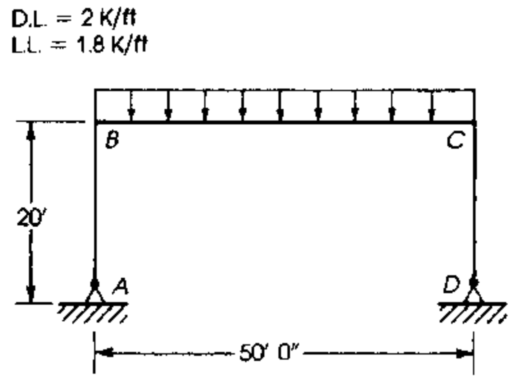


Figure 16.36 Problem 16.5.

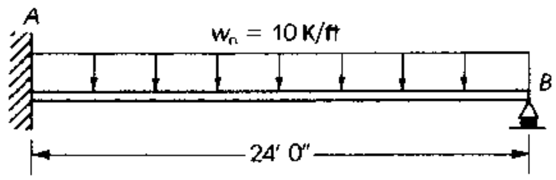


Figure 16.37 Problem 16.7.

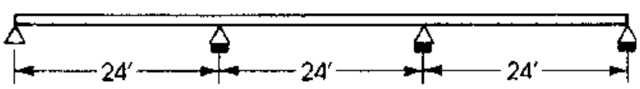


Figure 16.38 Problem 16.8.

16.7 Calculate the collapse moments at the critical sections of the beams shown in Fig. 16.37.

16.8 Repeat Problem 16.7 for Fig. 16.38.

16.9 If the beam shown in Fig. 16.36 carries a uniform dead load of 2.5 K/ft and a live load of 2.4 K/ft, design the beam using the limit design procedure. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and a beam width of $b = 14$ in.

16.10 Determine the maximum and minimum elastic moments at the supports and midspans of the three-span continuous beam shown in Fig. 16.37. The beam has a uniform rectangular section and carries a uniform dead load of 6 K/ft and a live load of 5 K/ft. Assuming 10% maximum redistribution of moments, recalculate the maximum and minimum moments at the supports and midspans of the beam *ABC*. *Note:* Place the live load on alternate spans to calculate maximum positive moments and on adjacent spans to calculate the maximum negative (minimum) moments (Example 16.8).

16.11 Repeat Problem 16.10 if the beam consists of four equal spans, each 24 ft in length (Fig. 16.39).

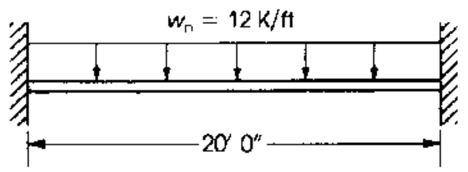


Figure 16.39 Problem 16.11.

CHAPTER 17

DESIGN OF TWO-WAY SLABS

The Bonaventure Complex and the Bonaventure Hilton Hotel, Montreal, Canada.

17.1 INTRODUCTION

Slabs can be considered as structural members whose depth, h, is small as compared to their length, L, and width, S. The simplest form of a slab is one supported on two opposite sides, which primarily deflects in one direction and is referred to as a *one-way slab*. The design of one-way slabs was discussed in Chapter 9.

When the slab is supported on all four sides and the length, L, is less than twice the width, S, the slab will deflect in two directions, and the loads on the slab are transferred to all four supports. This slab is referred to as a *two-way slab*. The bending moments and deflections in such slabs are less than those in one-way slabs; thus, the same slab can carry more load when supported on four sides. The load in this case is carried in two directions, and the bending moment in each direction is much less than the bending moment in the slab if the load were carried in one direction only. Typical slab-beam-girder arrangements of one-way and two-way slabs are shown in Fig. 17.1.

17.2 TYPES OF TWO-WAY SLABS

Structural two-way concrete slabs may be classified as follows:

1. *Two-Way Slabs on Beams:* This case occurs when the two-way slab is supported by beams on all four sides (Fig. 17.1). The loads from the slab are transferred to all four supporting beams, which, in turn, transfer the loads to the columns.
2. *Flat Slabs:* A flat slab is a two-way slab reinforced in two directions that usually does not have beams or girders, and the loads are transferred directly to the supporting columns. The column tends to punch through the slab, which can be treated by three methods (refer to Figs. 17.2 and 17.3):
   a. Using a drop panel and a column capital.
   b. Using a drop panel without a column capital. The concrete panel around the column capital should be thick enough to withstand the diagonal tensile stresses arising from the punching shear.

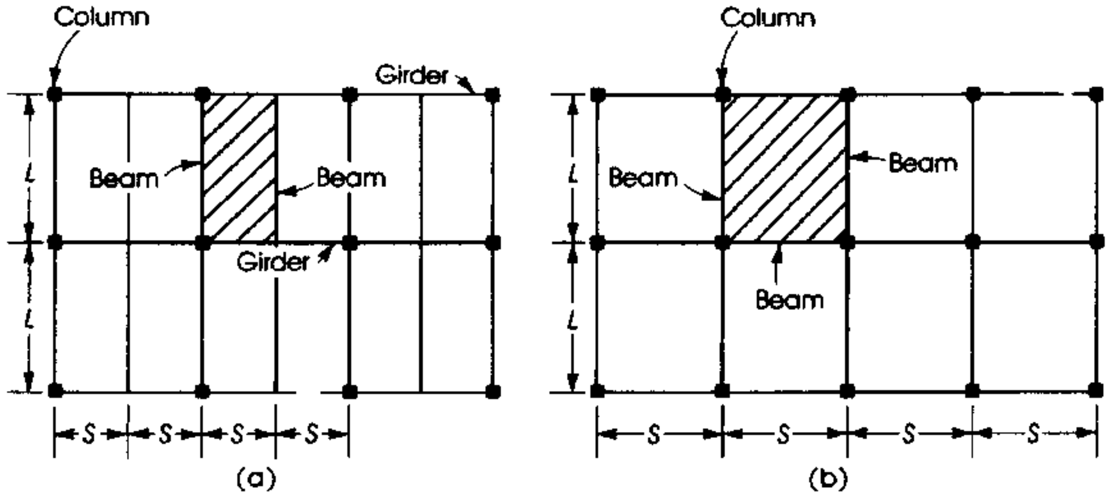


Figure 17.1 (*a*) One-way slab, $L/S > 2$, and (*b*) two-way slab, $L/S \leq 2$.

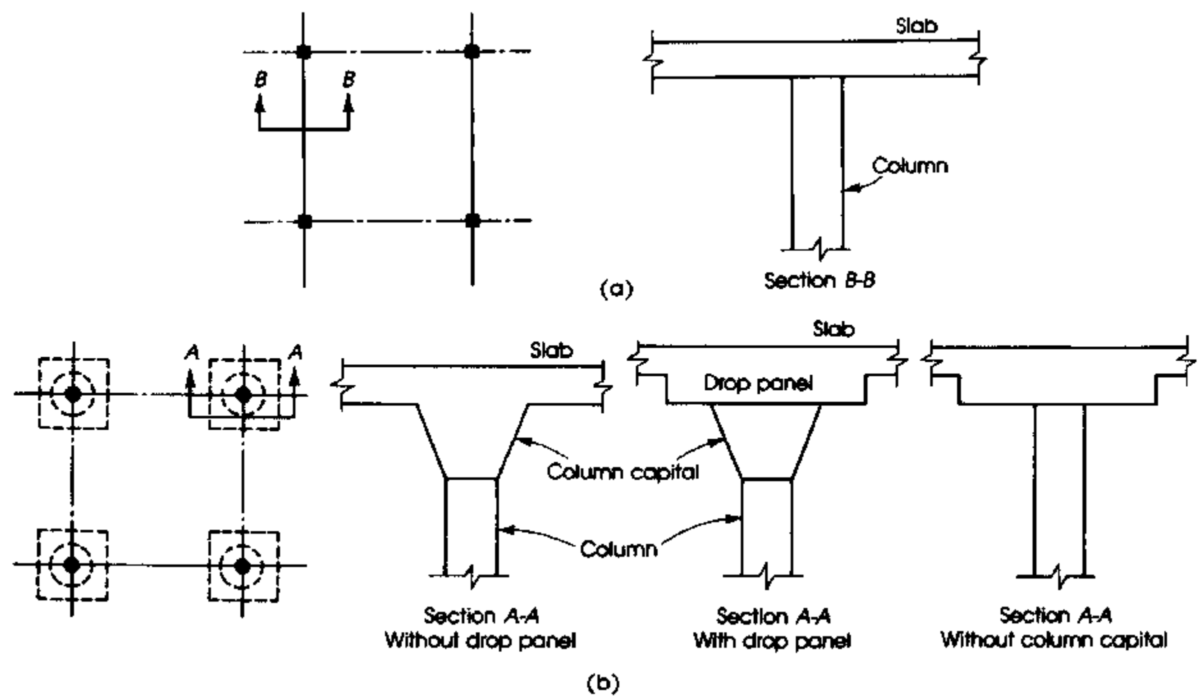
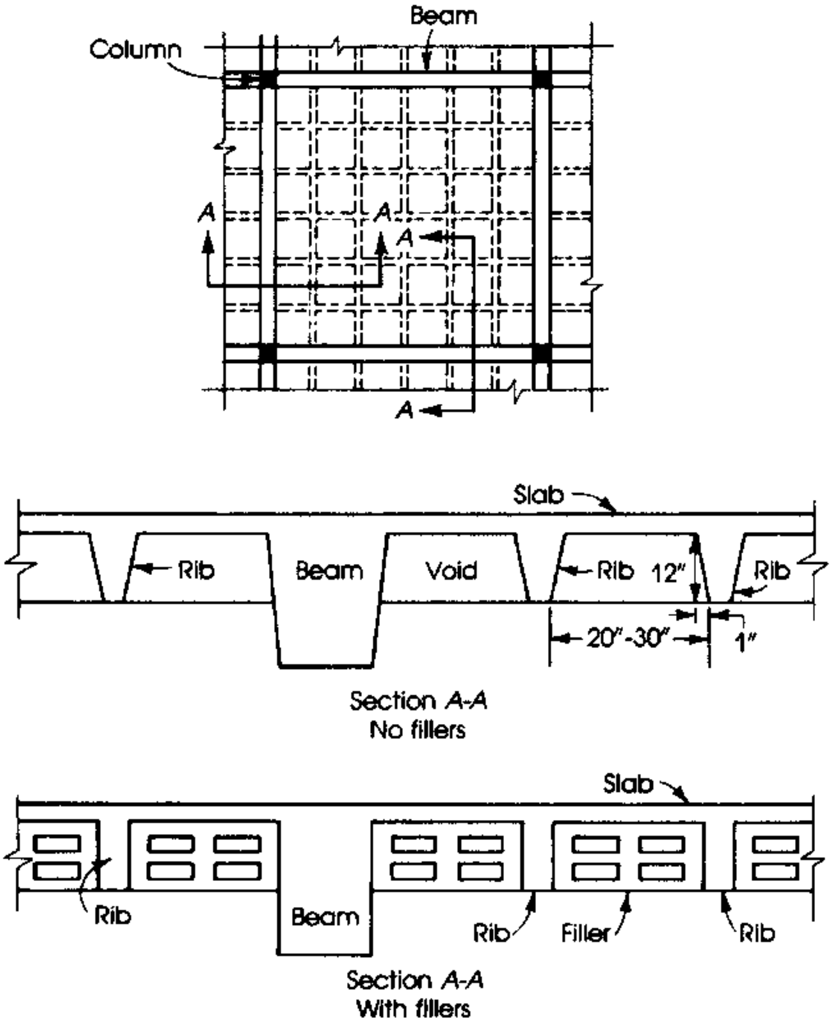


Figure 17.2 Two-way slabs without beams: (*a*) flat plate floor and section; (*b*) flat slab floor and sections; (*c*) ribbed slab and sections.

c. Using a column capital without drop panel, which is not common.

3. *Flat-Plate Floors:* A flat-plate floor is a two-way slab system consisting of a uniform slab that rests directly on columns and does not have beams or column capitals (Fig. 17.2*a*). In this case the column tends to punch through the slab, producing diagonal tensile stresses. Therefore, a general increase in the slab thickness is required or special reinforcement is used.



(c)

Figure 17.2 (*continued*)

4. *Two-Way Ribbed Slabs and the Waffle Slab System:* This type of slab consists of a floor slab with a length-to-width ratio less than 2. The thickness of the slab is usually 2 to 4 in. and is supported by ribs (or joists) in two directions. The ribs are arranged in each direction at spacings of about 20 to 30 in., producing square or rectangular shapes (Fig. 17.2*c*). The ribs can also be arranged at 45° or 60° from the centerline of slabs, producing architectural shapes at the soffit of the slab. In two-way ribbed slabs, different systems can be adopted:
   a. A two-way rib system with voids between the ribs, obtained by using special removable and usable forms (pans) that are normally square in shape. The ribs are supported on four sides by girders that rest on columns. This type is called a *two-way ribbed (joist) slab system*.
   b. A two-way rib system with permanent fillers between ribs that produce horizontal slab soffits. The fillers may be of hollow, lightweight or normal-weight concrete or any other lightweight material. The ribs are supported by girders on four sides, which in turn are supported by columns. This type is also called a *two-way ribbed (joist) slab system* or a *hollow-block two-way ribbed system*.

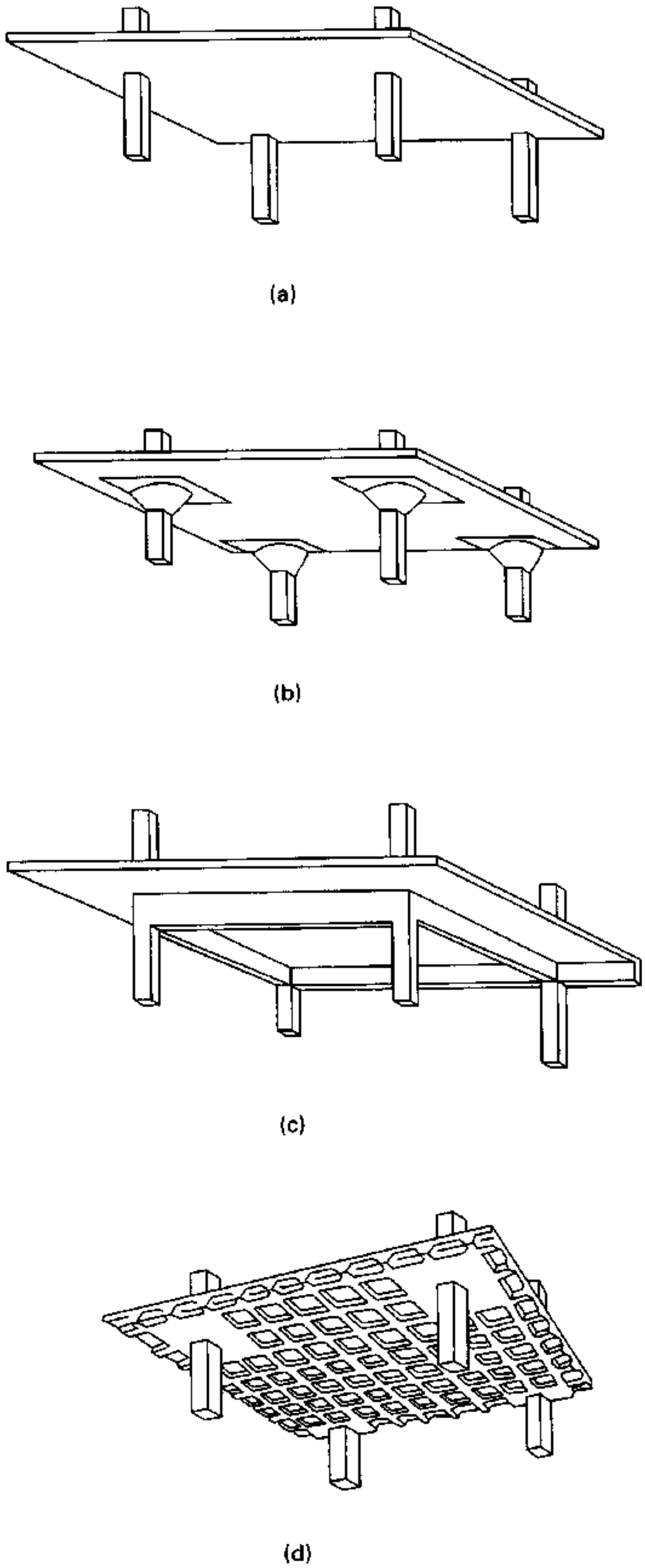


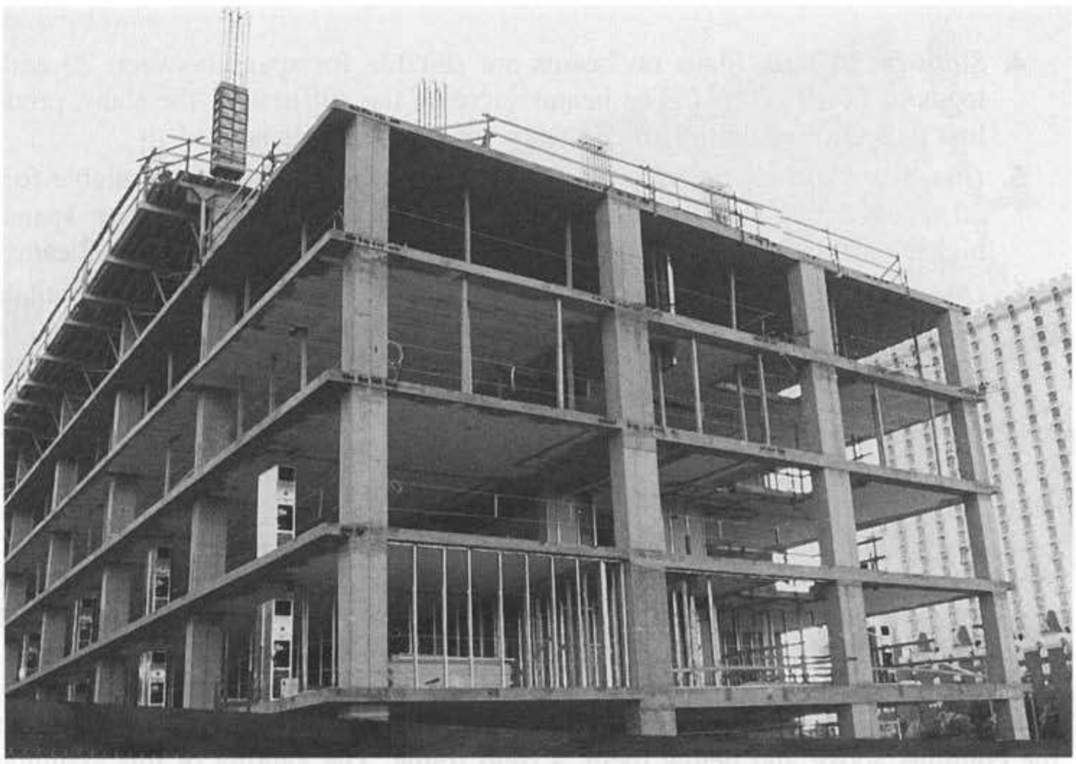
Figure 17.3 Types of two-way slab systems: (*a*) flat plate, (*b*) flat slab, (*c*) slab on beams, and (*d*) waffle slab.

c. A two-way rib system with voids between the ribs with the ribs continuing in both directions without supporting beams and resting directly on columns through solid panels above the columns. This type is called a *waffle slab system*.

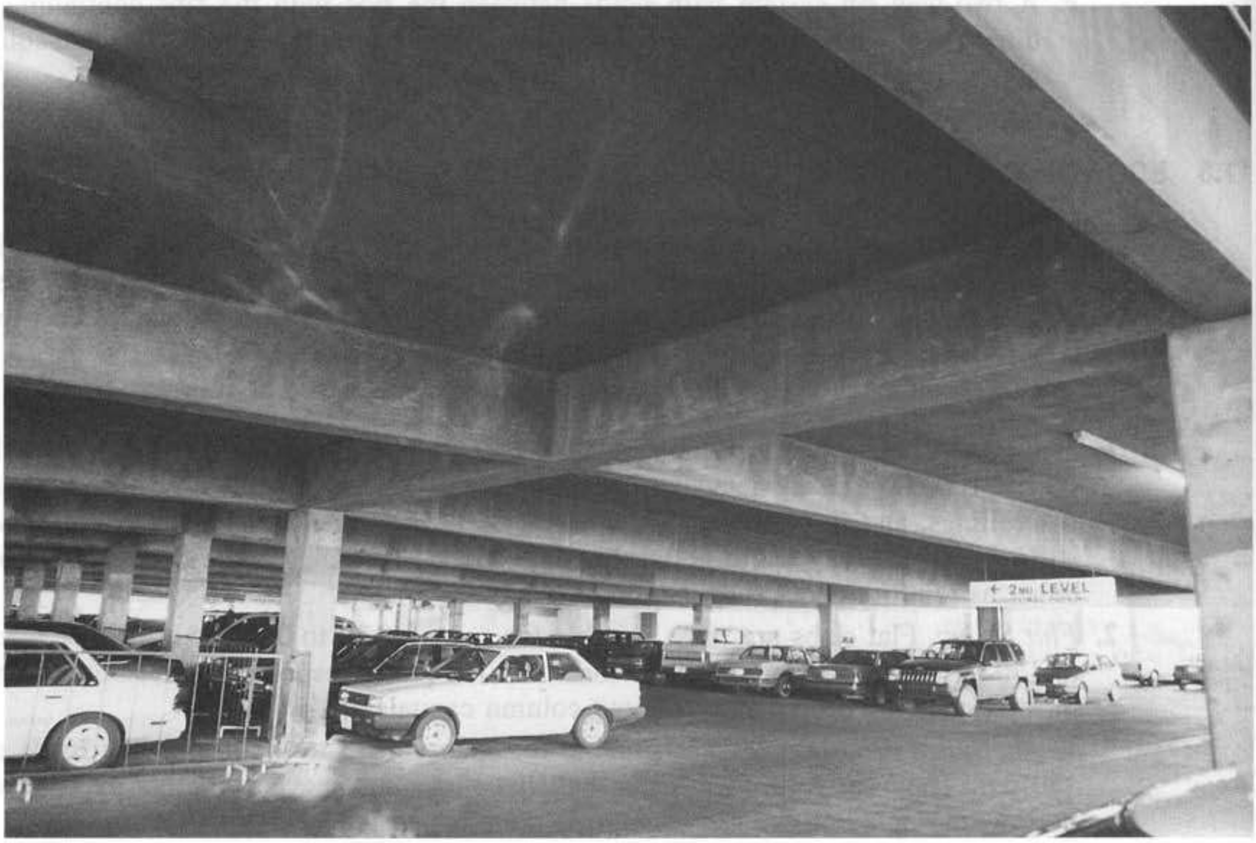
17.3 ECONOMICAL CHOICE OF CONCRETE FLOOR SYSTEMS

Various types of floor systems can be used for general buildings, such as residential, office, and institutional buildings. The choice of an adequate and economic floor system depends on the type of building, architectural layout, aesthetic features, and the span length between columns. In general, the superimposed live load on buildings varies between 80 and 150 psf. A general guide for the economical use of floor systems can be summarized as follows:

1. *Flat Plates:* Flat plates are most suitable for spans of 20 to 25 ft and live loads between 60 and 100 psf. The advantages of adopting flat plates include low-cost formwork, exposed flat ceilings, and fast construction. Flat plates have low shear capacity and relatively low stiffness, which may cause noticeable deflection. Flat plates are widely used in buildings either as reinforced or prestressed concrete slabs.
2. *Flat Slabs:* Flat slabs are most suitable for spans of 20 to 30 ft and for live loads of 80 to 150 psf. They need more formwork than flat plates, especially for column capitals. In most cases, only drop panels without column capitals are used.
3. *Waffle Slabs:* Waffle slabs are suitable for spans of 30 to 48 ft and live loads of 80 to 150 psf. They carry heavier loads than flat plates and have attractive exposed ceilings. Formwork, including the use of pans, is quite expensive.



Flat-plate floor system.



Slab on beams.

4. *Slabs on Beams:* Slabs on beams are suitable for spans between 20 and 30 ft and live loads of 60 to 120 psf. The beams increase the stiffness of the slabs, producing relatively low deflection. Additional formwork for the beams is needed.
5. *One-Way Slabs on Beams:* One-way slabs on beams are most suitable for spans of 10 to 20 ft and a live load of 60 to 100 psf. They can be used for larger spans with relatively higher cost and higher slab deflection. Additional formwork for the beams is needed.
6. *One-Way Joist Floor System:* A one-way joist floor system is most suitable for spans of 20 to 30 ft and live loads of 80 to 120 psf. Because of the deep ribs, the concrete and steel quantities are relatively low, but expensive formwork is expected. The exposed ceiling of the slabs may look attractive.

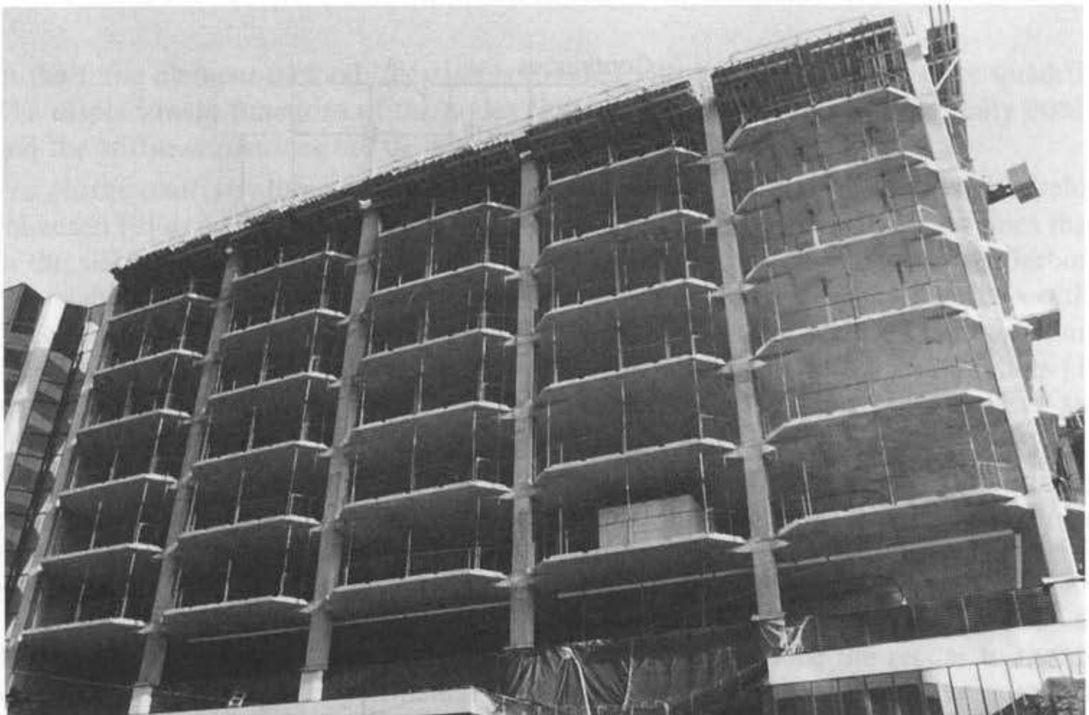
17.4 DESIGN CONCEPTS

An exact analysis of forces and displacements in a two-way slab is complex, due to its highly indeterminate nature; this is true even when the effects of creep and nonlinear behavior of the concrete are neglected. Numerical methods such as finite elements can be used, but simplified methods such as those presented by the ACI Code are more suitable for practical design. The ACI Code, Chapter 13, assumes that the slabs behave as wide, shallow beams that form, with the columns above and below them, a rigid frame. The validity of this assumption of dividing the structure into equivalent frames has been verified by analytical [1,2] and experimental [3,4] research. It is also established [3,5] that factored load capacity of two-way slabs with restrained boundaries is about twice that calculated by theoretical analysis, because a great deal of moment redistribution occurs in the slab before failure. At high loads, large deformations and deflections are expected; thus, a minimum slab thickness is required to maintain adequate deflection and cracking conditions under service loads.

The ACI Code specifies two methods for the design of two-way slabs:

1. The direct design method, DDM (ACI Code, Section 13.6), is an approximate procedure for the analysis and design of two-way slabs. It is limited to slab systems subjected to uniformly distributed loads and supported on equally or nearly equally spaced columns. The method uses a set of coefficients to determine the design moments at critical sections. Two-way slab systems that do not meet the limitations of the ACI Code, Section 13.6.1, must be analyzed by more accurate procedures.
2. The equivalent frame method, EFM (ACI Code, Section 13.7), is one in which a three-dimensional building is divided into a series of two-dimensional equivalent frames by cutting the building along lines midway between columns. The resulting frames are considered separately in the longitudinal and transverse directions of the building and treated floor by floor, as shown in Fig. 17.4.

Two ACI Code procedures are based on the results of elastic analysis of the structure as a whole using factored loads. A modified approach to the direct design method was presented in the commentary of the 1989 Code as the modified stiffness method, or MSM. It is based on specific distribution factors introduced as a function of the stiffness ratio, α_{ec}, for proportioning the total static moment in an end span. This method is explained later.



Flat slab system with drop panels (no column capitals).

In addition to the ACI Code procedures, a number of other alternatives are available for the analysis and design of slabs. The resulting slabs may have a greater or lesser amount of reinforcement. The analytical methods may be classified in terms of the basic relationship between load and deformation as elastic, plastic, and nonlinear.

1. In *elastic analysis*, a concrete slab may be treated as an elastic plate. The flexure, shear, and deflection may be calculated by the fourth differential equation relating load to deflection for thin plates with small displacements, as presented by Timoshenko [6]. Finite difference as well as finite element solutions have been proposed to analyze slabs and plates [7,8].

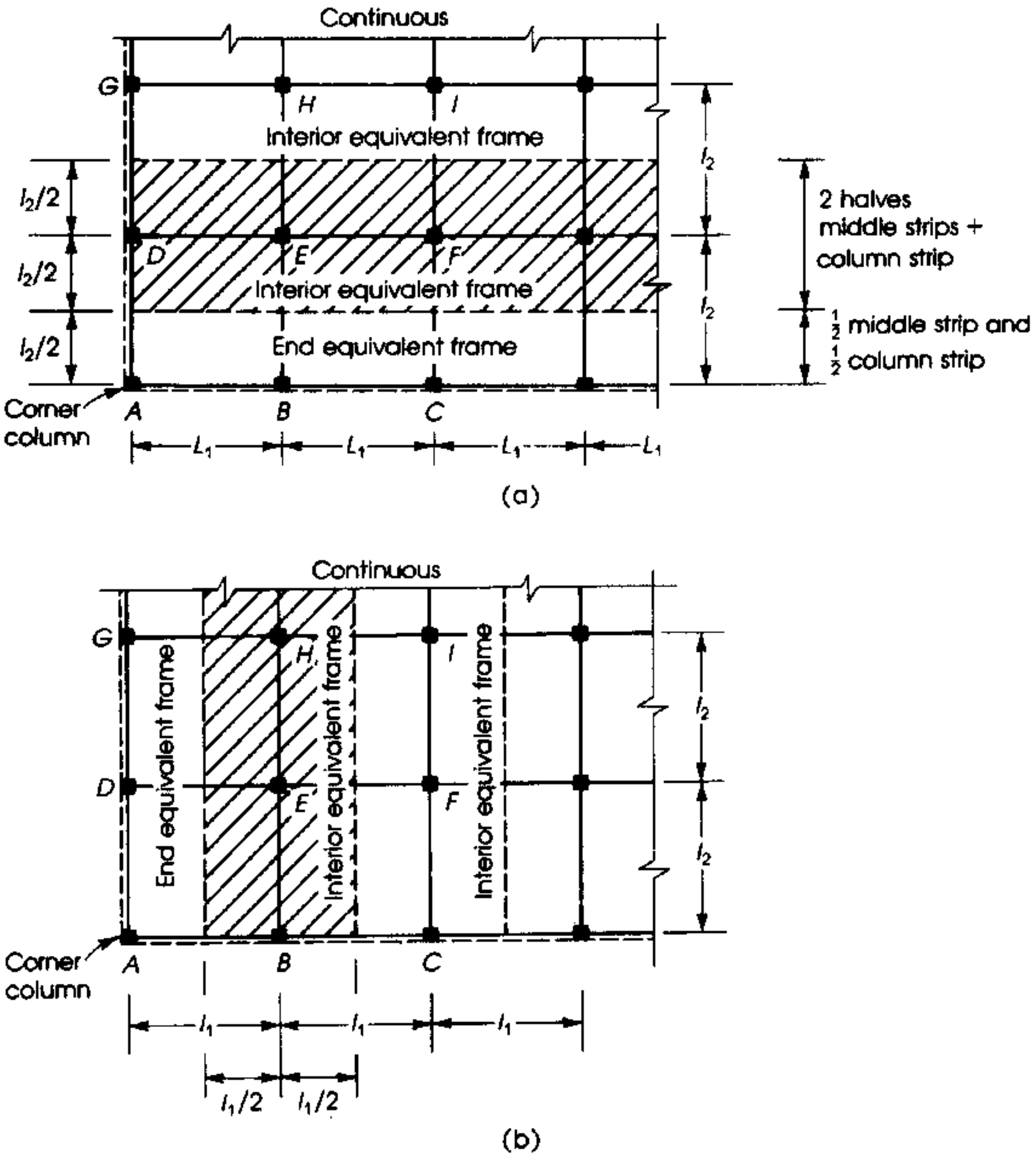
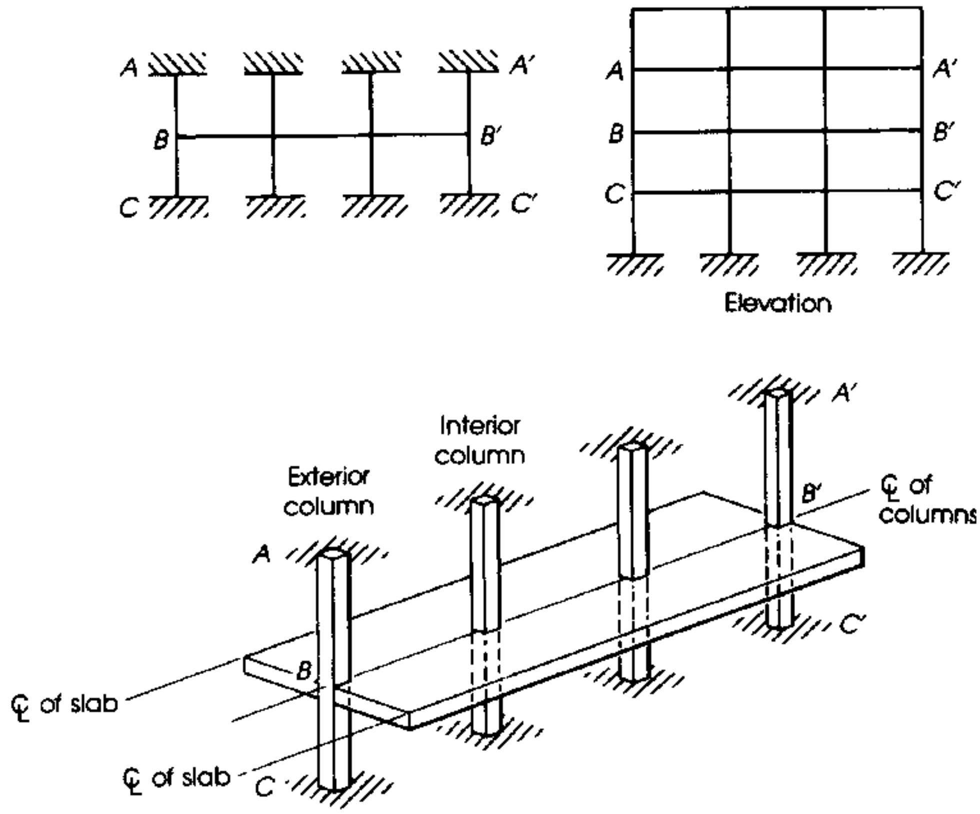


Figure 17.4 (*a*) Longitudinal and (*b*) transverse equivalent frames in plan view and (*c*) in elevation and perspective views.



(c)

Figure 17.4 (*continued*)

In the finite element method, the slab is divided into a mesh of triangles or quadrilaterals. The displacement functions of the nodes (intersecting mesh points) are usually established, and the stiffness matrices are developed for computer analysis.

2. For *plastic analysis*, three methods are available. The *yield line* method was developed by Johansen [9] to determine the limit state of the slab by considering the yield lines that occur in the slab as a collapse mechanism. The *strip* method was developed by Hillerborg [10]. The slab is divided into strips, and the load on the slab is distributed in two orthogonal directions. The strips are analyzed as simple beams. The third method is *optimal analysis*. There has been considerable research into optimal solutions. Rozvany and others [11] presented methods for minimizing reinforcement based on plastic analysis. Optimal solutions are complex in analysis and produce complex patterns of reinforcement.
3. *Nonlinear analysis* simulates the true load deformation characteristics of a reinforced concrete slab when the finite element method takes into consideration the nonlinearity of the stress strain–relationship of the individual elements [11,12]. In this case, the solution becomes complex unless simplified empirical relationships are assumed.

The preceding methods are presented very briefly to introduce the reader to the different methods of analysis of slabs. Experimental work on slabs has not been extensive in recent years, but more research is probably needed to simplify current design procedures with adequate safety, serviceability, and economy [11].

Waffle slab with light fixtures at the centers of the squares.

17.5 COLUMN AND MIDDLE STRIPS

Figure 17.5 shows an interior panel of a two-way slab supported on columns A, B, C, and D. If the panel is loaded uniformly, the slab will deflect in both directions, with maximum deflection at the center, O. The highest points will be at the columns A, B, C, and D; thus, the part of the slab around the columns will have a convex shape. A gradual change in the shape of the slab occurs, from convexity at the columns to concavity at the center of the panel O, each radial line crossing a point of inflection. Sections at O, E, F, G, and H will have positive bending moments, whereas the periphery of the columns will have maximum negative bending moments. Considering a strip along AFB, the strip bends like a continuous beam (Fig. 17.5*b*), having negative moments at A and B and positive bending moment at F. This strip extends between the two columns A and B and continues on both sides of the panel, forming a column strip.

Similarly, a strip along EOG will have negative bending moments at E and G and a positive moment at O, forming a middle strip. A third strip along DHC will behave similarly to strip AFB. Therefore, the panel can be divided into three strips, one in the middle along EOG, referred to as the *middle strip*, and one on each side, along AFB and DHC, referred to as *column strips* (Fig. 17.5*a*). Each of the three strips behaves as a continuous beam. In a similar way, the panel is divided into three strips in the other direction, one middle strip along FOH and two column strips along AED and BGC, respectively (Fig. 17.5*e*).

Referring to Fig. 17.5*a*, it can be seen that the middle strips are supported on the column strips, which in turn transfer the loads onto the columns, A, B, C, and D in this panel. Therefore, the column strips carry more load than the middle strips. Consequently, the positive bending moment in each column strip (at E, F, G, and H) is greater than the positive bending moment at O in the middle strip. Also, the negative moments at the columns A, B, C, and D in the column strips are greater than the negative moments at E, F, G, and H in the middle strips. The portions of the design moments assigned to each critical section of the column and middle strips are discussed in Section 17.8.

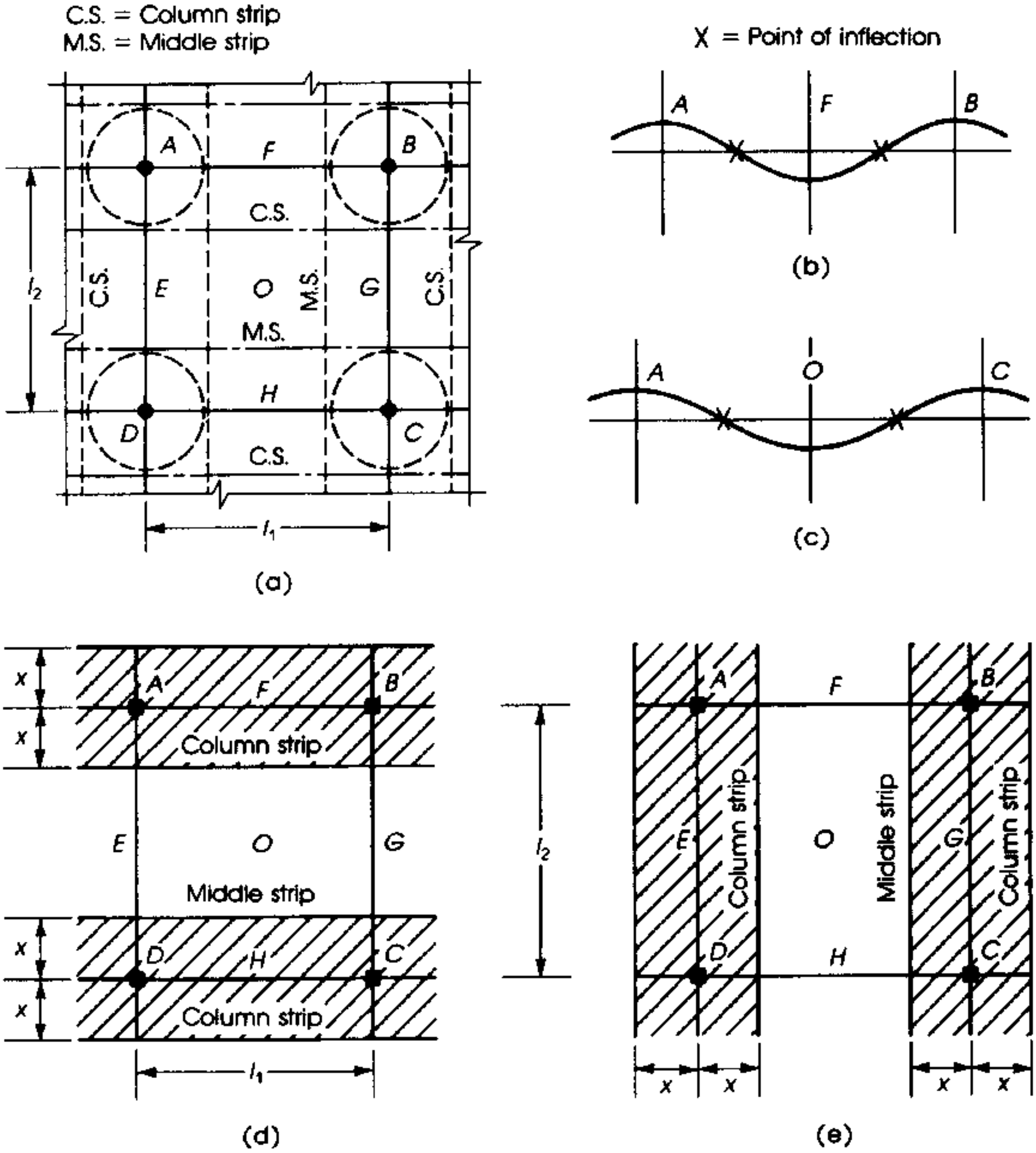


Figure 17.5 Column and middle strips; $x = 0.25l_1$ or $0.25l_2$, whichever is smaller.

The extent of each of the column and middle strips in a panel is defined by the ACI Code, Section 13.2. The column strip is defined by a slab width on each side of the column centerline, x in Fig. 17.5, equal to one-fourth the smaller of the panel dimensions l_1 and l_2, including beams if they are present, where

l_1 = span length, center to center of supports, in the direction moments are being determined

l_2 = span length, center to center of supports, in the direction perpendicular to l_1

The portion of the panel between two column strips defines the middle strip.

17.6 MINIMUM SLAB THICKNESS TO CONTROL DEFLECTION

The ACI Code, Section 9.5.3, specifies a minimum slab thickness in two-way slabs to control deflection. The magnitude of a slab's deflection depends on many variables, including the flexural stiffness of the slab, which in turn is a function of the slab thickness, h. By increasing the slab thickness, the flexural stiffness of the slab is increased, and consequently the slab deflection is reduced [13]. Because the calculation of deflections in two-way slabs is complicated and to avoid excessive deflections, the ACI Code limits the thickness of these slabs by adopting the following three empirical limitations, which are based on experimental research. If these limitations are not met, it will be necessary to compute deflections.

1. For $0.2 \leq \alpha_{fm} \leq 2$,

$$h = \frac{l_n\left(0.8 + \dfrac{f_y}{200{,}000}\right)}{36 + 5\beta(\alpha_{\text{fm}} - 0.2)} \quad (f_y \text{ in psi}) \qquad h = \frac{l_n\left(0.8 + \dfrac{f_y}{1400}\right)}{36 + 5\beta(\alpha_{\text{fm}} - 0.2)} \quad (f_y \text{ in MPa}) \tag{17.1}$$

   but not less than 5 in.

2. For $\alpha_{fm} > 2.0$,

$$h = \frac{l_n\left(0.8 + \dfrac{f_y}{200{,}000}\right)}{36 + 9\beta} \quad (f_y \text{ in psi}) \qquad h = \frac{l_n\left(0.8 + \dfrac{f_y}{1400}\right)}{36 + 9\beta} \quad (f_y \text{ in MPa}) \tag{17.2}$$

   but not less than 3.5 in.

3. For $\alpha_{fm} < 0.2$,

$$h = \text{minimum slab thickness without interior beams (Table 17.1)} \tag{17.3}$$

where

l_n = clear span in the long direction measured face to face of columns (or face to face of beams for slabs with beams)

β = the ratio of the long to the short clear spans

Table 17.1 Minimum Thickness of Slabs Without Interior Beams

	Without Drop Panels[b]			With Drop Panels[b]		
	Exterior Panels			Exterior Panels		
Yield Stress f_y psi (1)[a]	Without Edge Beams	With Edge Beams	Interior Panels	Without Edge Beams	With Edge Beams[c]	Interior Panels
40,000	$\frac{l_n}{33}$	$\frac{l_n}{36}$	$\frac{l_n}{36}$	$\frac{l_n}{36}$	$\frac{l_n}{40}$	$\frac{l_n}{40}$
60,000	$\frac{l_n}{30}$	$\frac{l_n}{33}$	$\frac{l_n}{33}$	$\frac{l_n}{30}$	$\frac{l_n}{36}$	$\frac{l_n}{36}$

[a]For values of reinforcement, yield stress between 40,000 and 60,000 psi minimum thickness shall be obtained by linear interpolation.

[b]Drop panel is defined in ACI Sections 13.3.7.1 and 13.3.7.2.

[c]Slabs with beams between columns along exterior edges. The value of α_f for the edge beam shall be not less than 0.8.

α_{fm} = the average value of α for all beams on the sides of a panel

α_f = the ratio of flexural stiffness of a beam section $E_{cb}I_b$ to the flexural stiffness of the slab $E_{cs}I_s$, bounded laterally by the centerlines of the panels on each side of the beam

$$\alpha_f = \frac{E_{cb}I_b}{E_{cs}I_s} \tag{17.4}$$

where E_{cb} and E_{cs} are the moduli of elasticity of concrete in the beam and the slab, respectively, and

I_b = the gross moment of inertia of the beam section about the centroidal axis (the beam section includes a slab length on each side of the beam equal to the projection of the beam above or below the slab, whichever is greater, but not more than four times the slab thickness)

I_s = the moment of inertia of the gross section of the slab

However, the thickness of any slab shall not be less than the following:

1. For slabs with $\alpha_{fm} \leq 2.0$ then thickness ≥ 5.0 in. (125 mm)
2. For slabs with $\alpha_{fm} > 2.0$ then thickness ≥ 3.5 in. (90 mm)

If no beams are used, as in the case of flat plates, then $\alpha_f = 0$ and $\alpha_{fm} = 0$. The ACI Code equations for calculating slab thickness, h, take into account the effect of the span length, the panel shape, the steel reinforcement yield stress, f_y, and the flexural stiffness of beams. When very stiff beams are used, Eq. 17.1 may give a small slab thickness, and Eq. 17.2 may control. For flat plates and flat slabs, when no interior beams are used, the minimum slab thickness may be determined directly from Table 9.5c of the ACI Code, which is shown here as Table 17.1.

Other ACI Code limitations are summarized as follows:

1. For panels with discontinuous edges, end beams with a minimum α equal to 0.8 must be used; otherwise, the minimum slab thickness calculated by Eqs. 17.1 and 17.2 must be increased by at least 10% (ACI Code, Section 9.5.3).
2. When drop panels are used without beams, the minimum slab thickness may be reduced by 10%. The drop panels should extend in each direction from the centerline of support a distance not less than one-sixth of the span length in that direction between center to center of supports and also project below the slab at least $h/4$. This reduction is included in Table 17.1.
3. Regardless of the values obtained by Eqs. 17.1 and 17.2, the thickness of two-way slabs shall not be less than the following: (1) for slabs without beams or drop panels, 5 in. (125 mm); (2) for slabs without beams but with drop panels, 4 in. (100 mm); (3) for slabs with beams on all four sides with $\alpha_{fm} \geq 2.0$, $3\frac{1}{2}$in. (90 mm), and for $\alpha_{fm} < 2.0$, 5 in. (125 mm) (ACI Code, Section 9.5.3.).

The following steps summarize these calculations:

1. For slabs without interior beams (flat plates and flat slabs),
   a. Calculate the minimum slab thickness directly from Table 17.1. However, Eqs. 17.1 and 17.2 may be used, and Eq. 17.1 normally controls. Minimum slab thickness shall be greater than or equal to 5 in. (125 mm) for slabs without drop panels and greater than or equal to 4 in. (100 mm) for slabs with drop panels.

b. At discontinuous edges, an edge beam with $\alpha_f \geq 0.8$ should be used. Otherwise, the minimum slab thickness calculated by Eqs. 17.1 and 17.2 should be increased by 10%. This increase of 10% has already been included in the second columns of Table 17.1.

c. If drop panels are used in flat slabs, the minimum slab thickness may be reduced by 10% on the condition that the drop panel extends in each direction from the centerline of the support a distance not less than one-sixth of the span and projects below the slab at least $h/4$. This reduction is included in the factors of Table 17.1.

2. For slabs with beams on all sides ($\alpha_{fm} > 0$),

a. Calculate α_{fm} and then calculate the minimum slab thickness from Eqs. 17.1 and 17.2. In most cases, Eq. 17.2 controls.

b. The slab thickness should be greater than or equal to 5 in. for slabs with $\alpha_{fm} < 2.0$ and should be greater than or equal to 3.5 in. for slabs with $\alpha_{fm} \geq 2.0$.

3. For all slabs: A slab thickness less than the minimum thickness given in steps 1 and 2 may be used if shown by computation that deflection will not exceed the ACI Code, Table 9.5*b* limitations explained earlier in Chapter 6.

Example 17.1

A flat-plate floor system with panels 24 by 20 ft is supported on 20-in. square columns. Using the ACI Code equations, determine the minimum slab thickness required for the interior and corner panels shown in Fig. 17.6. Edge beams are not used. Use $f'_c = 4$ ksi and $f'_c = 60$ ksi.

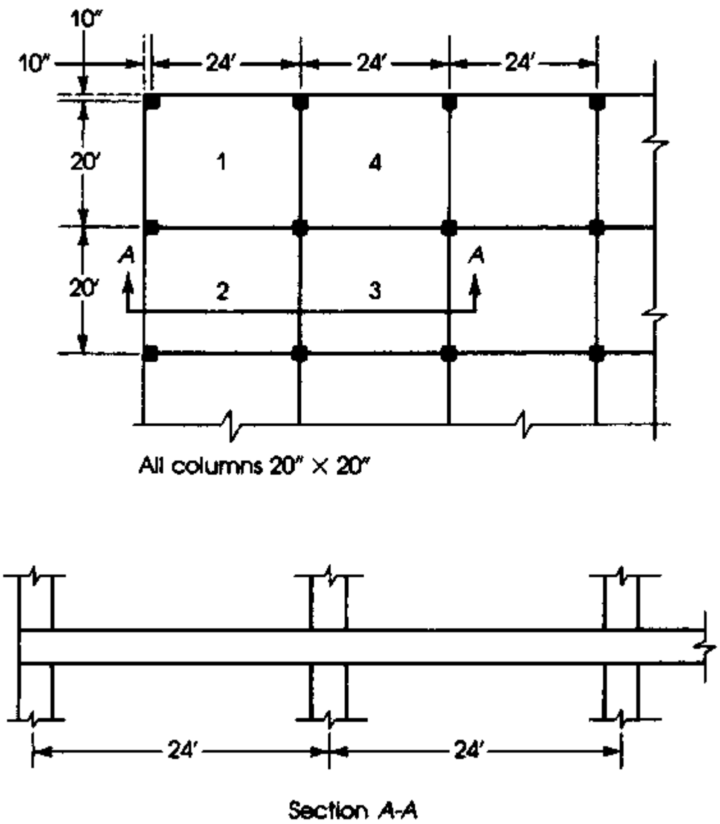


Figure 17.6 Example 17.1.

Solution

1. For corner panel no. 1, the minimum thickness is $l_n/30$ ($f_y = 60$ ksi, and no edge beams are used; see Table 17.1).

$$l_{n_1} = 24 - \frac{20}{12} = 22.33 \text{ ft} \quad \text{(long direction)}$$

$$h_{\min} = \frac{22.33 \times 12}{30} = 8.93 \text{ in.}, \quad \text{say, } 9.0 \text{ in.}$$

Alternatively, Eqs. 17.1 and 17.2 can be used to calculate the minimum thickness with $\alpha_f = \alpha_{fm} = 0$.

2. For the interior panel no. 3 and $f_y = 60$ ksi, the minimum slab thickness is $l_n/33 = (22.33 \times 12)/33 = 8.12$ in., say, 8.5 in. Alternatively, Eqs. 17.1 and 17.2 can be used. If a uniform slab thickness is used for all panels, then $h = 9.0$ in. will be adopted.

Example 17.2

The floor system shown in Fig. 17.7 consists of solid slabs and beams in two directions supported on 20-in. square columns. Using the ACI Code equations, determine the minimum slab thickness required for an interior panel. Use $f'_c = 3$ ksi and $f_y = 60$ ksi.

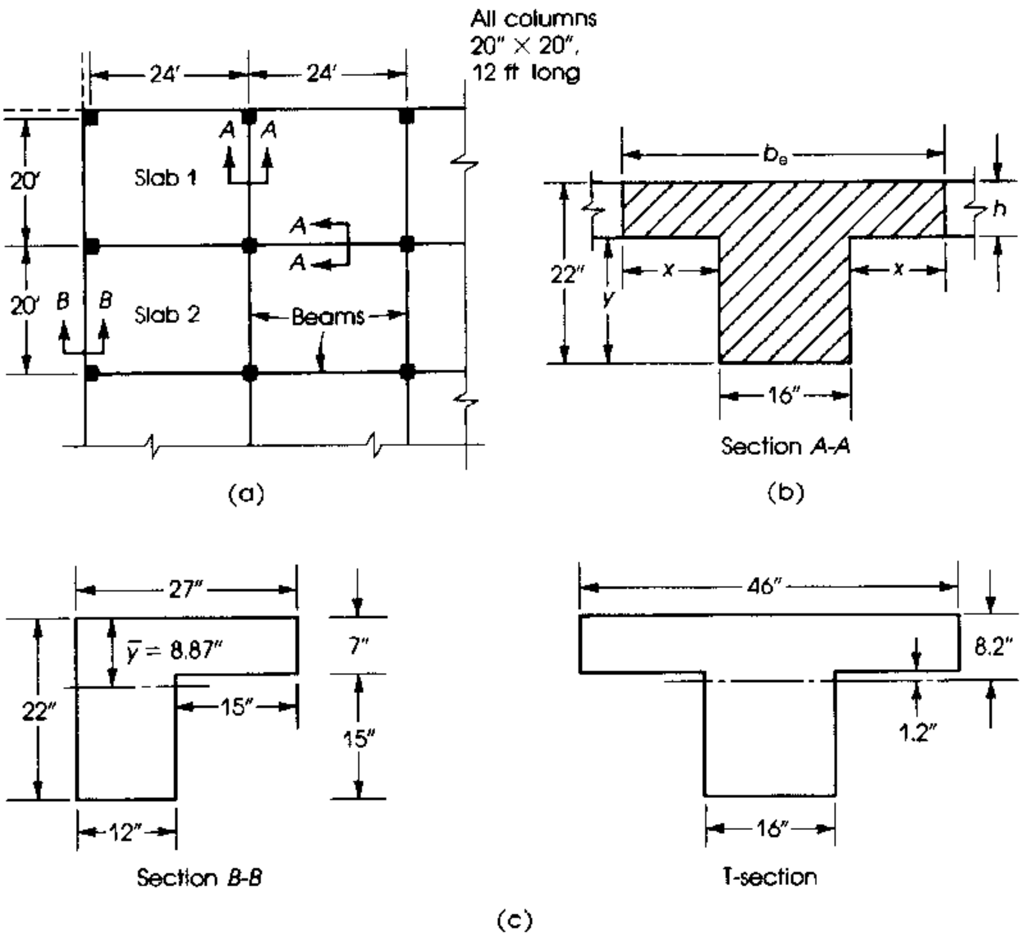


Figure 17.7 Example 17.2.

Solution

1. To use Eq. 17.1, α_m should be calculated first. Therefore, it is required to determine I_b, I_s, and α_f for the beams and slabs in the long and short directions.
2. The gross moment of inertia of the beam, I_b, is calculated for the section shown in Fig. 17.7*b*, which is made up of the beam and the extension of the slab on each side of the beam $x = y$ but not more than four times the slab thickness. Assume $h = 7$ in., to be checked later; then x $x = y = 22 - 7 = 15$ in. $< 4 \times 7 = 28$ in. Therefore, $b_e = 16 + 2 \times 15 = 46$ in., and the T-section is shown in Fig. 17.7*c*. Determine the centroid of the section by taking moments about the top of the flange:

$$\text{Area of flange} = 7 \times 46 = 322 \text{ in.}^2$$

$$\text{Area of web} = 16 \times 15 = 240 \text{ in.}^2$$

$$\text{Total area} = 562 \text{ in.}^2$$

$$(322 \times 3.5) + 240 \times (7 + 7.5) = 562y$$

$$y = 8.20 \text{ in.}$$

$$I_b = \left[\frac{46}{12}(7)^3 + 322 \times (4.7)^2\right]$$

$$+ \left[\frac{16(15)^3}{12} + 240(7.5 - 1.2)^2\right] = 22{,}453 \text{ in.}^4$$

3. The moment of inertia of the slab in the long direction is $I_s = (bh^3)/12$, where $b = 20$ ft and $h = 7$ in.

$$I_l = \frac{(20 \times 12)(7)^3}{12} = 6860 \text{ in.}^4$$

$$\alpha_{f1}(\text{in the long direction}) = \frac{EI_b}{EI_s} = \frac{22{,}453}{6860} = 3.27$$

4. The moment of inertia of the slab in the short direction is $I_s = (bh^3)/12$ where $b = 24$ ft and $h = 7$ in.

$$I_s = \frac{(24 \times 12)(7)^3}{12} = 8232 \text{ in.}^4$$

$$\alpha_s = \frac{EI_b}{EI_s} = \frac{22{,}453}{8232} = 2.27$$

5. α_{fm} is the average of α_{f1} and α_s:

$$\alpha_{\text{fm}} = \frac{3.27 + 2.72}{2} = 3.0$$

6.

$$\beta = \frac{(24 - \frac{20}{12})}{(20 - \frac{20}{12})} = \frac{22.33}{18.33} = 1.22$$

7. Determine h_{min} using Eq. 17.1 ($l_n = 22.33$ ft):

$$h_{\text{min}} = \frac{(22.33 \times 12)(0.8 + 0.005 \times 60)}{36 + (5 \times 1.22)[3.0 - 0.2]} = 5.57 \text{ in.}$$

However, this value must not be less than h given by Eq. 17.2 ($\alpha_{fm} > 2.0$):

$$h = \frac{294.8}{36 + 9(1.22)} = 6.27 \text{ in.}$$

Also, $h_{min} = 3.5$ in. Therefore, $h = 6.27$ in. controls. A slab thickness of 6.5 in. or 7.0 in. may be adopted. Note that in most practical cases, Eq. 17.2 controls.

17.7 SHEAR STRENGTH OF SLABS

In a two-way floor system, the slab must have adequate thickness to resist both bending moments and shear forces at the critical sections. To investigate the shear capacity of two-way slabs, the following cases should be considered.

17.7.1 Two-Way Slabs Supported on Beams

In two-way slabs supported on beams, the critical sections are at a distance d from the face of the supporting beams, and the shear capacity of each section is $\phi V_c = \phi(2\lambda\sqrt{f'_c}bd)$. When the supporting beams are stiff and are capable of transmitting floor loads to the columns, they are assumed to carry loads acting on floor areas bounded by 45° lines drawn from the corners, as shown in Fig. 17.8. The loads on the trapezoidal areas will be carried by the long beams AB and CD, whereas the loads on the triangular areas will be carried by the short beams AC and BD. The shear per unit width of slab is highest between E and F in both directions, and $V_u = w_u(l_2/2)$, where w_u is the uniform factored load per unit area.

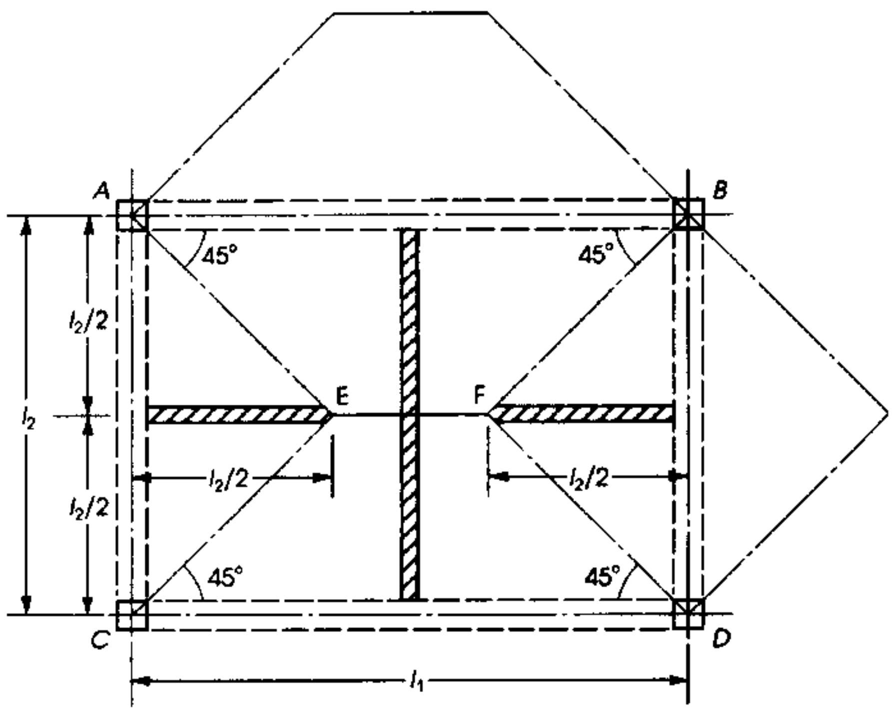


Figure 17.8 Areas supported by beams in two-way slab floor system.

If no shear reinforcement is provided, the shearing force at a distance d from the face of the beam, V_{ud}, must be equal to

$$V_{ud} \le \phi V_c \le \phi(2\lambda\sqrt{f'_c}bd),$$

where

$$V_{ud} = w_u\left(\frac{l_2}{2} - d\right).$$

17.7.2 Two-Way Slabs Without Beams

In flat plates and flat slabs, beams are not provided, and the slabs are directly supported by columns. In such slabs, two types of shear stresses must be investigated; the first is one-way shear, or beam shear. The critical sections are taken at a distance d from the face of the column, and the slab is considered as a wide beam spanning between supports, as in the case of one-way beams. The shear capacity of the concrete section is $\phi V_c = \phi(2\lambda\sqrt{f'_c}bd)$. The second type of shear to be studied is two-way, or punching, shear, as was previously discussed in the design of footings. Shear failure occurs along a truncated cone or pyramid around the column. The critical section is located at a distance $d/2$ from the face of the column, column capital, or drop panel (Fig. 17.9*a*). If shear reinforcement is not provided, the shear strength of concrete is the smaller of Eq. 17.5 and 17.6:

$$\phi V_c = \left(2 + \frac{4}{\beta}\right)\lambda\sqrt{f'_c}b_o d \le 4\phi\sqrt{f'_c}b_o d \tag{17.5}$$

where

b_o = perimeter of the critical section

β = ratio of the long side of column (or loaded area) to the short side

$$\phi V_c = \phi\left(\frac{\alpha_s d}{b_o} + 2\right)\lambda\sqrt{f'_c}b_o d \tag{17.6}$$

where α_s is 40 for interior columns, 30 for edge columns, and 20 for corner columns. When shear reinforcement is provided, the shear strength should not exceed

$$\phi V_n \le \phi(6\sqrt{f'_c}b_o d) \tag{17.7}$$

17.7.3 Shear Reinforcement in Two-Way Slabs Without Beams

In flat-slab and flat-plate floor systems, the thickness of the slab selected may not be adequate to resist the applied shear stresses. In this case, either the slab thickness must be increased or shear reinforcement must be provided. The ACI Code allows the use of shear reinforcement by shearheads and anchored bars or wires.

Shearheads consist of steel I-shapes or channel shapes welded into four cross-arms and placed in the slabs above the column (Fig. 17.9*c*, *d*). Shearhead designs do not apply to exterior columns, where large torsional and bending moments must be transferred between slab and column. The ACI Code, Section 11.11.4.8, indicates that on the critical section the nominal shear strength, V_n, should not exceed $4\sqrt{f'_c}b_o d$, but if shearhead reinforcement is provided, V_n

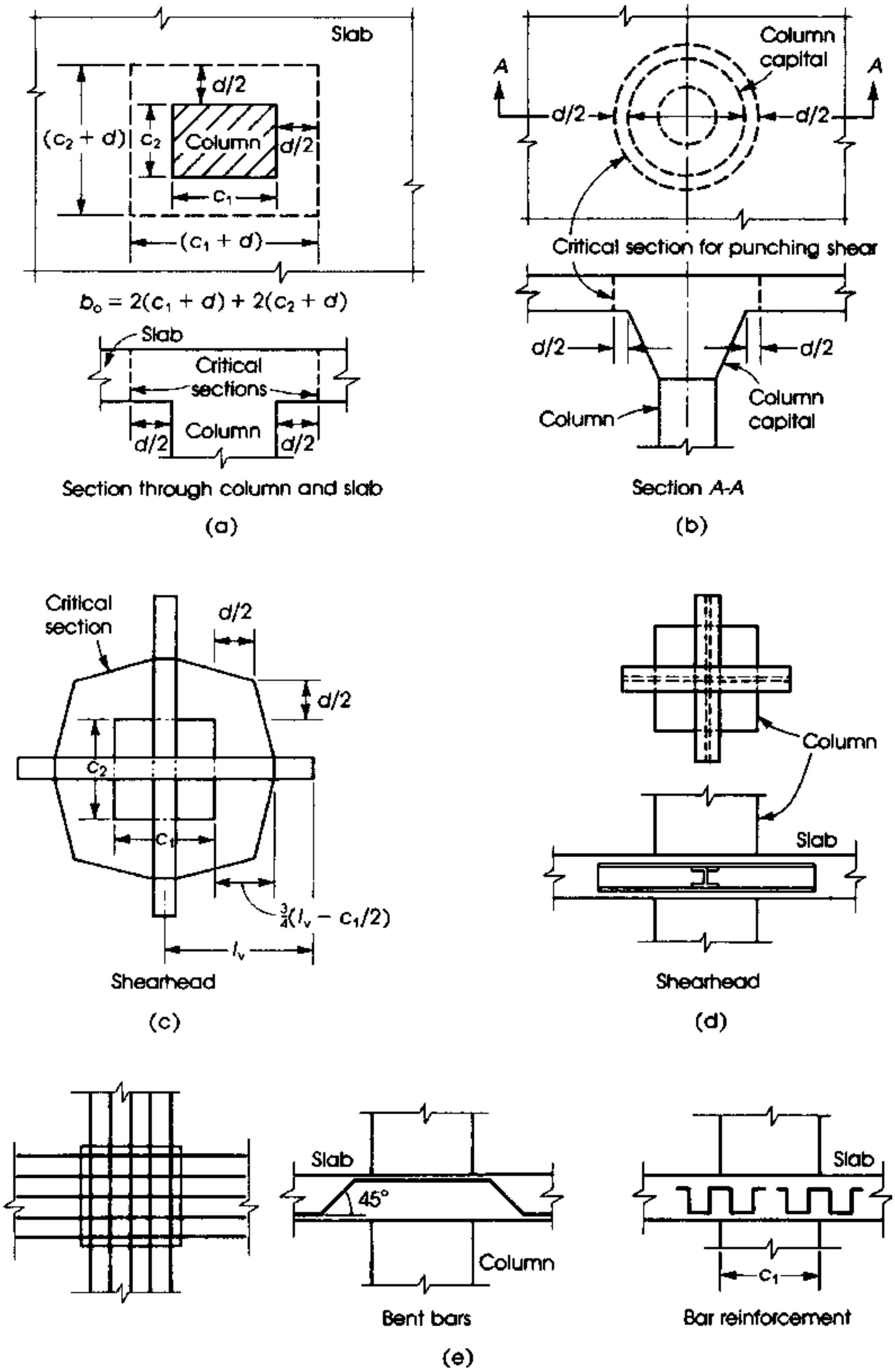
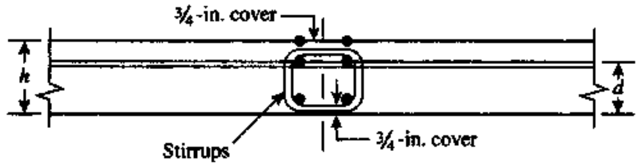
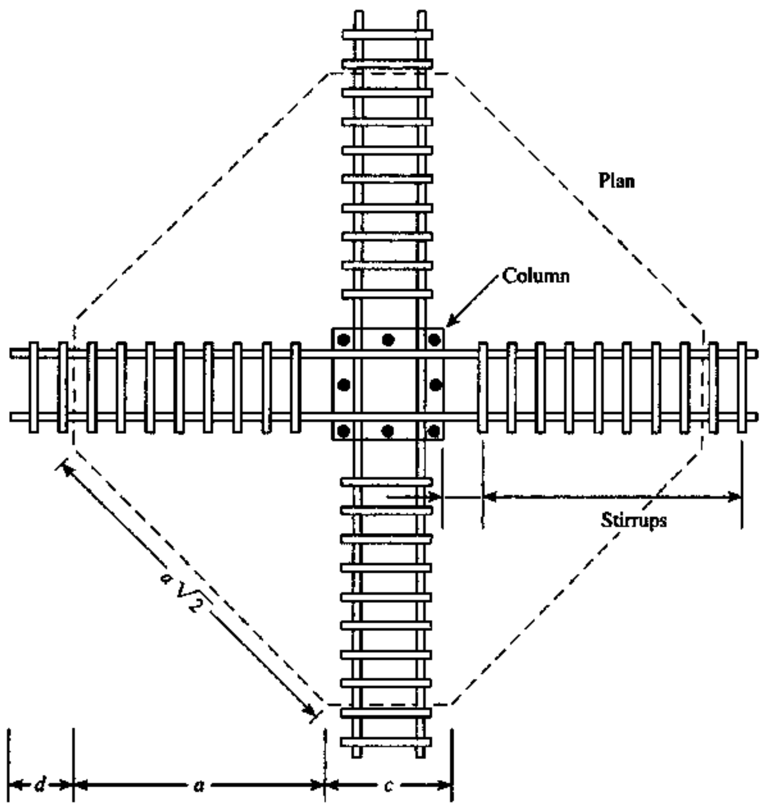


Figure 17.9 Critical section for punching shear in (*a*) flat plates and (*b*) flat slabs, reinforcement by (*c*, *d*) shearheads and (*e*) anchored bars, (*f*) conventional stirrup cages, and (*g*) studded steel strips used as shear reinforcement.





(f)

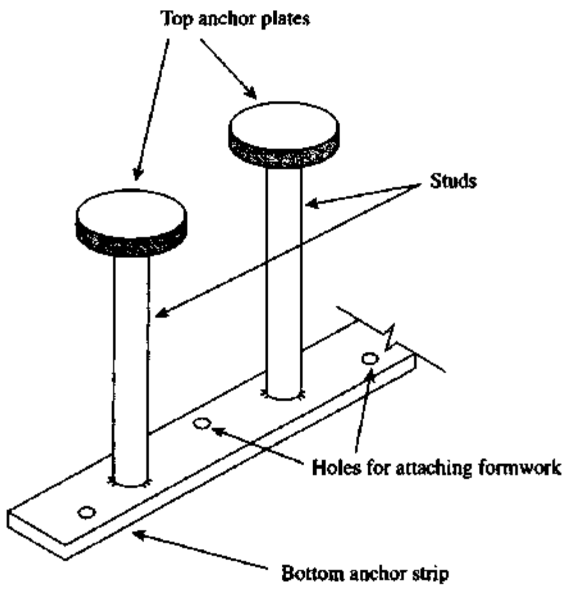


Figure 17.9 (*continued*)

should not exceed $7\sqrt{f_c'}b_od$. To determine the size of the shearhead, the ACI Code, Section 11.11.4.8, gives the following limitations:

1. The ratio α_v between the stiffness of shearhead arm, E_sI, and that of the surrounding composite cracked section of width, $c_2 + d$, must not be less than 0.15.
2. The compression flange of the steel shape must be located within $0.13d$ of the compression surface of the slab.
3. The depth of the steel shape must not exceed 70 times the web thickness.
4. The plastic moment capacity, M_p, of each arm of the shearhead is computed by

$$\phi M_p = \frac{V_u}{2n}\left[h_v + \alpha_v\left(l_v - \frac{c_1}{2}\right)\right] \quad \text{(ACI Code, Eq. 11.37)} \qquad (17.8)$$

where

$\phi = 0.9$
V_u = factored shear force around the periphery of the column face
n = number of arms
h_v = depth of the shearhead
l_v = length of the shearhead measured from the centerline of the column.

5. The critical slab section for shear must cross each shearhead arm at a distance equal to (3/4) $(l_v - c_1/2)$ from the column face to the end of the shearhead arm, as shown in Fig. 17.9*c*. The critical section must have a minimum perimeter, b_o, but it should not be closer than $d/2$ from the face of the column.
6. The shearhead is considered to contribute a moment resistance, M_v, to each slab column strip as follows:

$$M_v = \frac{\phi}{2n}\alpha_v V_u\left(l_v - \frac{c_1}{2}\right) \quad \text{(ACI Code, Eq. 11.38)} \qquad (17.9)$$

but it should not be more than the smallest of 30% of the factored moment required in the column strip or the change in the column strip moment over the length l_v or M_p given in Eq. 17.8.

The use of anchored bent bars or wires is permitted by the ACI Code, Section 11.11.3. The bars are placed on top of the column, and the possible arrangements are shown in Fig. 17.9*e*. When bars or wires are used as shear reinforcement, the nominal shear strength is

$$V_n = V_c + V_s = (2\lambda\sqrt{f_c'})b_od + \frac{A_v f_y d}{s} \qquad (17.10)$$

where A_v is the total stirrup bar area and b_o is the length of the critical section of two-way shear at a distance $d/2$ from the face of the column. The nominal shear strength, V_n, should not exceed $6\sqrt{f_c'}b_od$.

The use of shear reinforcement in flat plates reduces the slab thickness and still maintains the flat ceiling to reduce the cost of formwork. Typical stirrup cages for shear reinforcement are shown in Fig. 17.9*f*. Another type of shear reinforcement consists of studded steel strips (Fig. 17.9*g*). The steel strip is positioned with bar chairs and fastened to the formwork, replacing the stirrup gages. The yield strength of the stud material is specified between 40 and 60 ksi to achieve complete anchorage at ultimate load.

17.8 ANALYSIS OF TWO-WAY SLABS BY THE DIRECT DESIGN METHOD

The direct design method is an approximate method established by the ACI Code to determine the design moments in uniformly loaded two-way slabs. To use this method, some limitations must be met, as indicated by the ACI Code, Section 13.6.1.

17.8.1 Limitations

1. There must be a minimum of three continuous spans in each direction.
2. The panels must be square or rectangular; the ratio of the longer to the shorter span within a panel must not exceed 2.0.
3. Adjacent spans in each direction must not differ by more than one-third of the longer span.
4. Columns must not be offset by a maximum of 10% of the span length, in the direction of offset, from either axis between centerlines of successive columns.
5. All loads must be uniform, and the ratio of the unfactored live to unfactored dead load must not exceed 2.0.
6. If beams are present along all sides, the ratio of the relative stiffness of beams in two perpendicular directions, $\alpha_{f_1} l_l^2 / \alpha_{f_2} l_1^2$ must not be less than 0.2 nor greater than 5.0.

17.8.2 Total Factored Static Moment

If a simply supported beam carries a uniformly distributed load w K/ft, then the maximum positive bending moment occurs at midspan and equals $M_o = wl_1^2/8$, where l_1 is the span length. If the beam is fixed at both ends or continuous with equal negative moments at both ends, then the total moment $M_o = M_p$(positive moment at midspan) + M_n(negative moment at support) = $wl_1^2/8$ (Fig. 17.10). Now if the beam AB carries the load W from a slab that has a width l_2 perpendicular to l_1, then $W = w_u l_2$, and the total moment is $M_o = \frac{(w_u l_2) l_1^2}{8}$, where w_u = load intensity in K/ft^2. In this expression, the actual moment occurs when l_1 equals the clear span between supports A and B. If the clear span is denoted by l_n, then

$$M_o = (w_u l_2)\frac{l_n^2}{8} \quad \text{(ACI Code, Eq. 13.4)} \qquad (17.11)$$

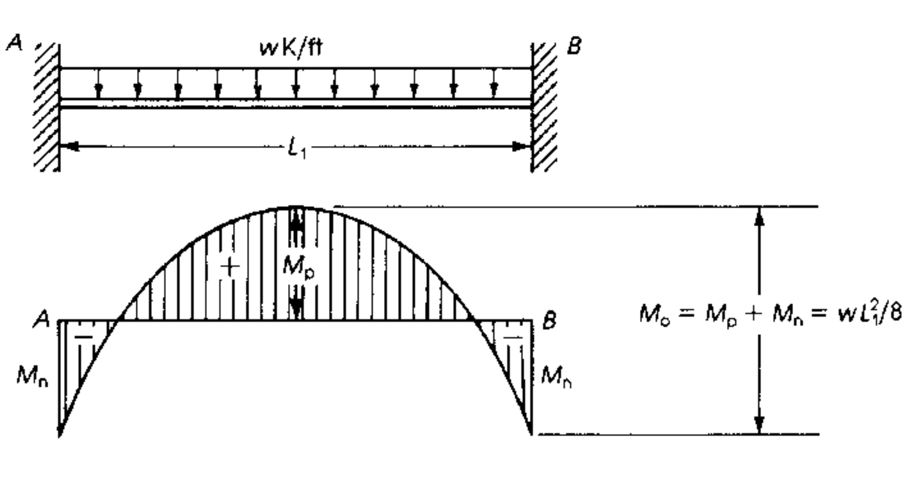


Figure 17.10 Bending moment in a fixed-end beam.

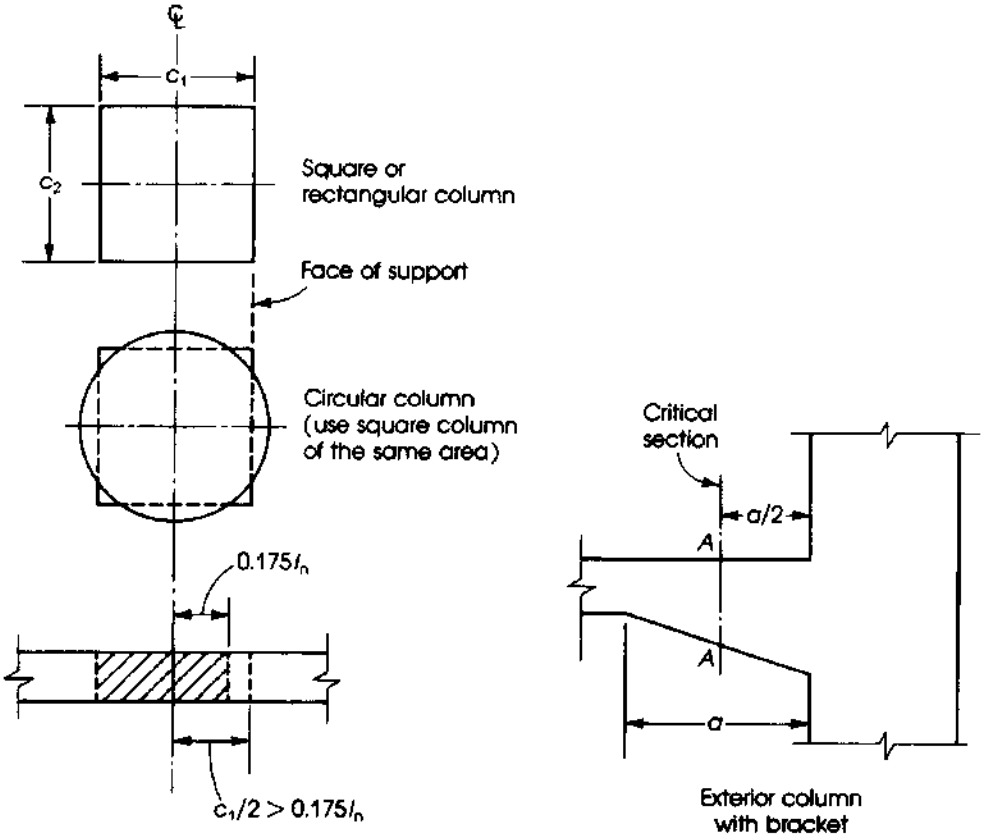


Figure 17.11 Critical sections for negative design moments. *A-A*, section for negative moment at exterior support with bracket.

The clear span, l_n, is measured face to face of supports in the direction in which moments are considered, but not less than 0.65 times the span length from center to center of supports. The face of the support where the negative moments should be calculated is illustrated in Fig. 17.11. The length l_2 is measured in a direction perpendicular to l_n and equals the direction between center to center of supports (width of slab). The total moment M_o calculated in the long direction will be referred to here as M_{ol} and that in the short direction, as M_{os}.

Once the total moment, M_o, is calculated in one direction, it is divided into a positive moment, M_p, and a negative moment, M_n, such that $M_o = M_p + M_n$ (Fig. 17.10). Then each moment, M_p and M_n, is distributed across the width of the slab between the column and middle strips, as is explained shortly.

17.8.3 Longitudinal Distribution of Moments in Slabs

In a typical *interior panel*, the total static moment, M_o, is divided into two moments, the positive moment, M_p, at midspan, equal to $0.35M_o$, and the negative moment, M_n, at each support, equal to $0.65M_o$, as shown in Fig. 17.12. These values of moment are based on the assumption that the interior panel is continuous in both directions, with approximately equal spans and loads, so that the interior joints have no significant rotation. Moreover, the moment values are approximately the same as those in a fixed-end beam subjected to uniform loading, where the negative moment at the support is twice the positive moment at midspan. In Fig. 17.12, if $l_1 > l_2$, then the distribution of moments in the long and short directions is as follows:

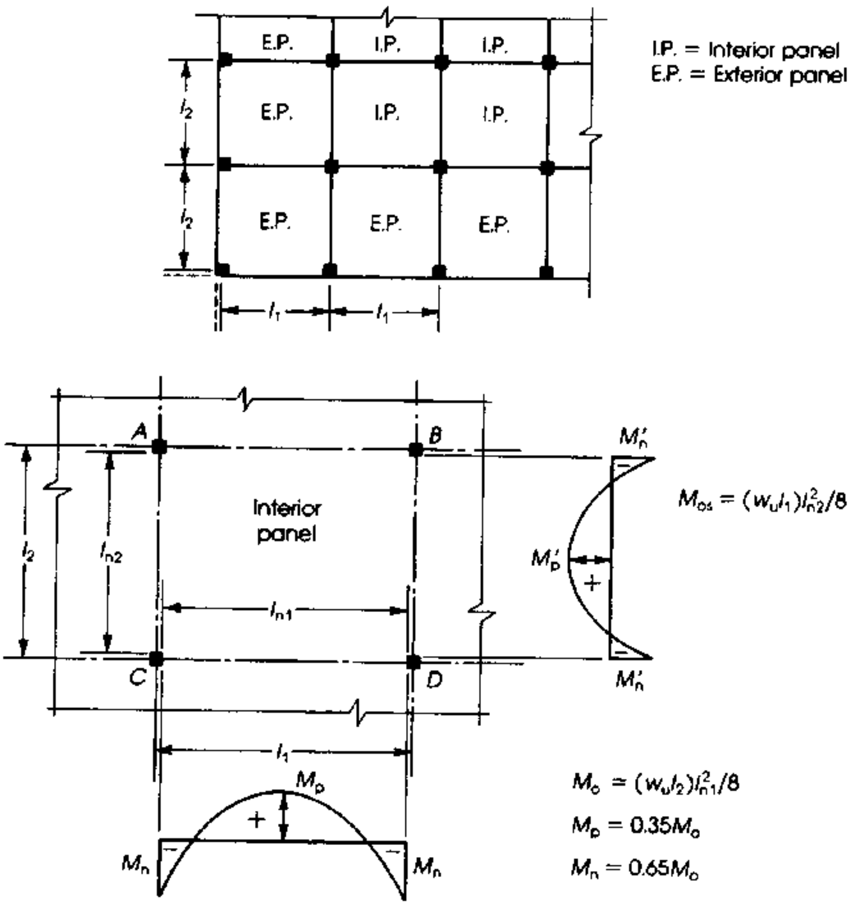


Figure 17.12 Distribution of moments in an interior panel.

$$M_{ol} = (w_u l_2)\frac{l_{n1}^2}{8} \quad M_{pl} = 0.35M_{ol} \quad M_{nl} = 0.65M_{ol}$$

$$M_{os} = (w_u l_1)\frac{l_{n2}^2}{8} \quad M_{ps} = 0.35M_{os} \quad M_{ns} = 0.65M_{os}$$

If the magnitudes of the negative moments on opposite sides of an interior support are different because of unequal span lengths, the ACI Code specifies that the larger moment should be considered to calculate the required reinforcement.

In an *exterior panel*, the slab load is applied to the exterior column from one side only, causing an unbalanced moment and a rotation at the exterior joint. Consequently, there will be an increase in the positive moment at midspan and in the negative moment at the first interior support. The magnitude of the rotation of the exterior joint determines the increase in the moments at midspan and at the interior support. For example, if the exterior edge is a simple support, as in the case of a slab resting on a wall (Fig. 17.13), the slab moment at the face of the wall there is 0, the positive moment at midspan can be taken as $M_p = 0.63M_o$, and the negative moment at the interior support is $M_n = 0.75\ M_o$. These values satisfy the static equilibrium equation

$$M_o = M_p + \frac{1}{2}M_n = 0.63M_o + \frac{1}{2}(0.75M_o)$$

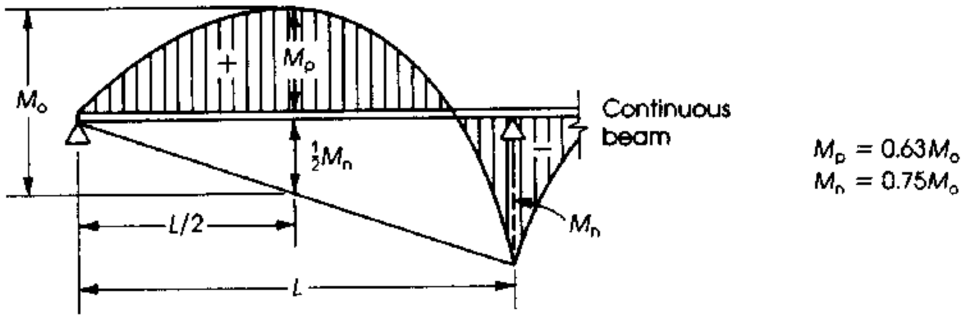


Figure 17.13 Exterior panel.

In a slab-column floor system, there is some restraint at the exterior joint provided by the flexural stiffness of the slab and by the flexural stiffness of the exterior columns.

According to Section 13.6.3 of the ACI Code, the total static moment M_o in an end span is distributed in different ratios according to Table 17.2 and Fig. 17.14. The moment coefficients in column 1 for an unrestrained edge are based on the assumption that the ratio of the flexural stiffness of columns to the combined flexural stiffness of slabs and beams at a joint, α_{ec} is equal to 0. The coefficients of column 2 are based on the assumption that the ratio α_{ec} is equal to infinity. The moment coefficients in columns 3, 4, and 5 have been established by analyzing the slab systems with different geometries and support conditions.

17.8.4 Transverse Distribution of Moments

The longitudinal moment values mentioned in the previous section are for the entire width of the equivalent building frame. This frame width is the sum of the widths of two half-column strips and two half-middle strips of two adjacent panels, as shown in Fig. 17.15. The transverse distribution of the longitudinal moments to the middle and column strips is a function of the ratios l_2/l_1,

$$\alpha_f = \frac{E_{cb}I_b}{E_{cs}I_s} = \frac{\text{beam stiffness}}{\text{slab stiffness}} \tag{17.12}$$

$$\beta_t = \frac{E_{cb}C}{2E_{cs}I_s} = \frac{\text{torsional rigidity of edge beam section}}{\text{flexural rigidity of a slab of width equal to beam span length}} \tag{17.13}$$

Table 17.2 Distribution of Moments in an End Panel

	Exterior Edge		Slab with Beams Between All Supports	Slab Without Beams Between Interior Supports	
	Unrestrained (1)	Fully Restrained (2)	(3)	With Edge Beam (4)	Without Edge Beam (5)
Exterior negative factored moment	0	0.65	0.16	0.30	0.26
Positive factored moment	0.63	0.35	0.57	0.50	0.52
Interior negative factored moment	0.75	0.65	0.70	0.70	0.70

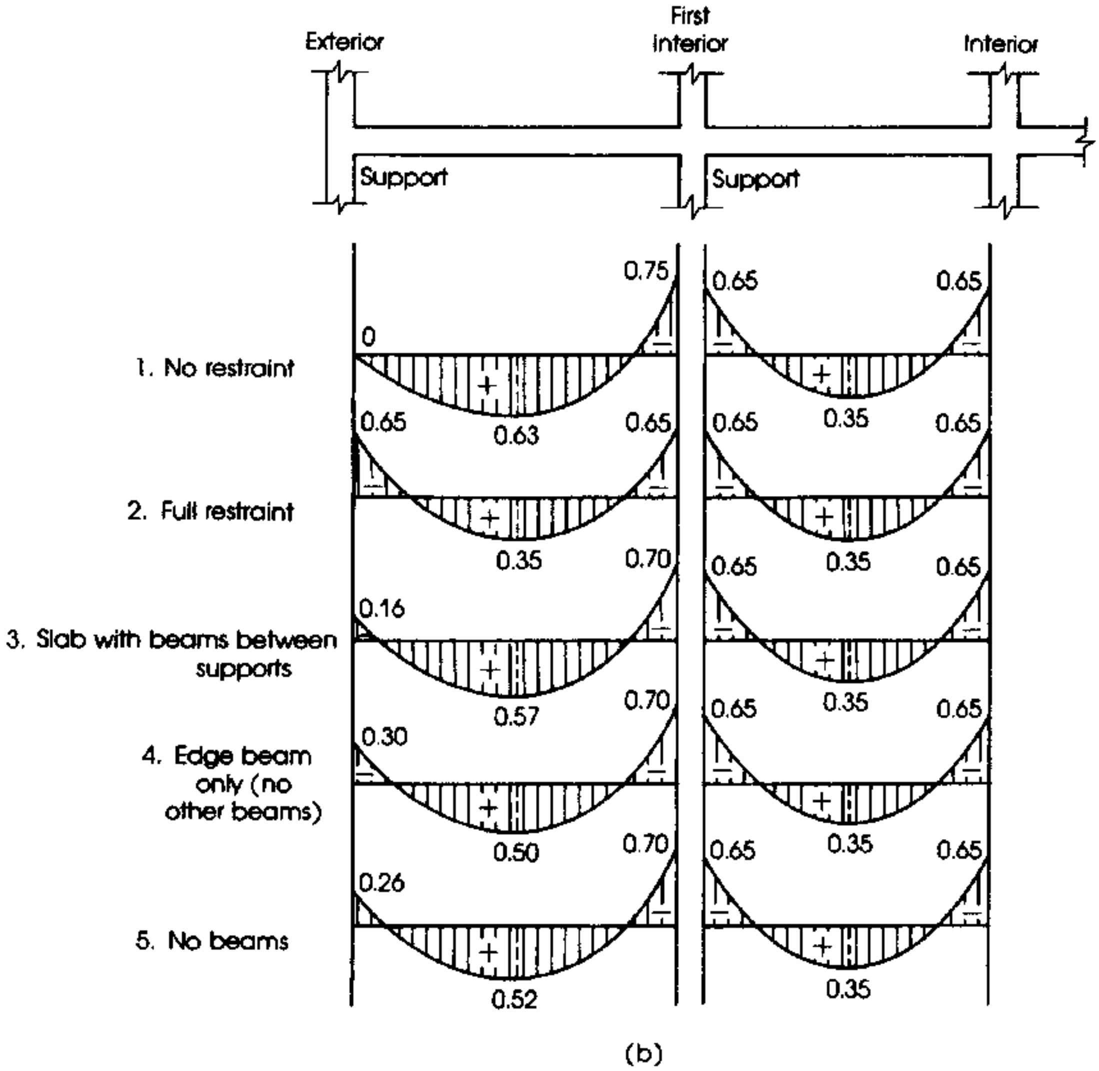


Figure 17.14 Distribution of total static moment into negative and positive span moments.

where

$$C = \text{torsional constant} = \sum \left(1 - \frac{0.63x}{y}\right)\left(\frac{x^3 y}{3}\right) \tag{17.14}$$

where x and y are the shorter and longer dimension of each rectangular component of the section. The percentages of each design moment to be distributed to column and middle strips for interior and exterior panels are given in Tables 17.3 through Table 17.6. In a typical *interior* panel, the portion of the design moment that is not assigned to the column strip (Table 17.3) must be resisted by the corresponding half-middle strips. Linear interpolation of values of l_2/l_1 between 0.5 and 2.0 and of $\alpha_{f_1} l_2/l_1$ between 0 and 1 is permitted by the ACI Code. From Table 17.3 it can be seen that when no beams are used, as in the case of flat plates or flat slabs, $\alpha_{f_1} = 0$. The final percentage of moments in the column and middle strips as a function of M_o are given in Table 17.4.

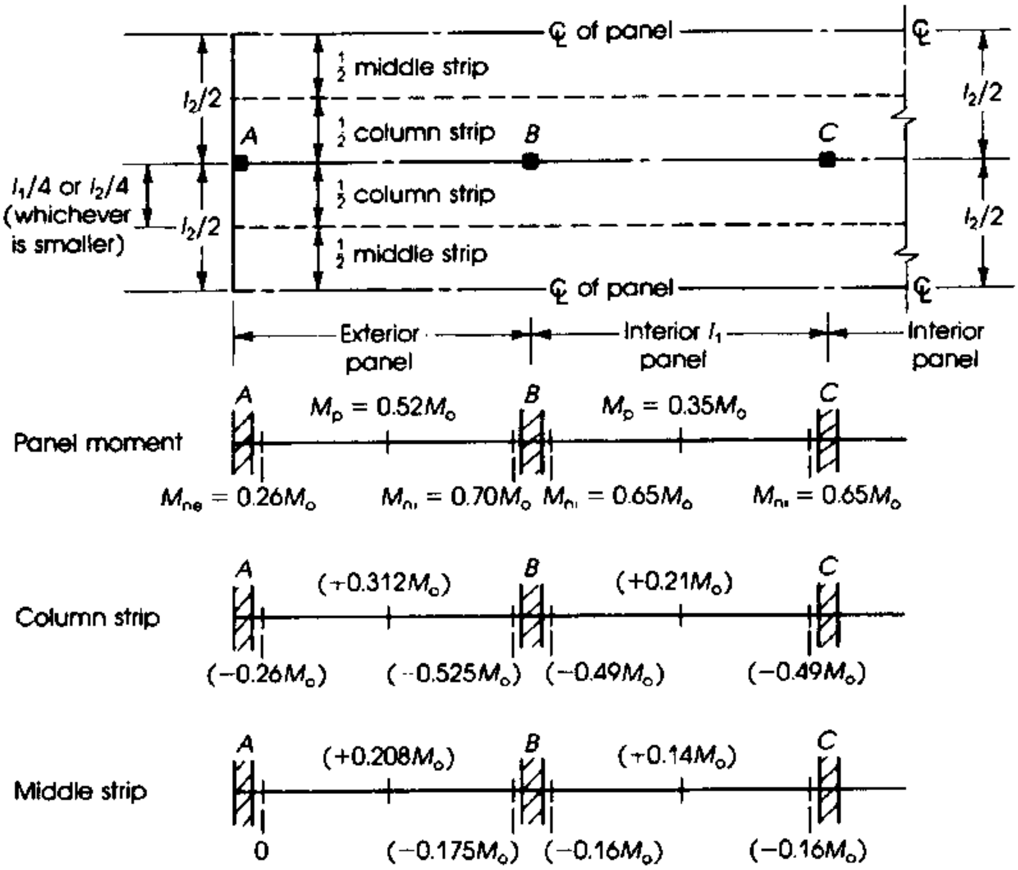


Figure 17.15 Width of the equivalent rigid frame (equal spans in this figure) and distribution of moments in flat plates, flat slabs, and waffle slabs with no beams.

Table 17.3 Percentage of Longitudinal Moment in Column Strips, Interior Panels (ACI Code, Section 13.6.4)

		Aspect Ratio, l_2/l_1		
	$\alpha_{f_1} l_2/l_1$	0.5	1.0	2.0
Negative moment at interior support	0	75	75	75
	≥1.0	90	75	45
Positive moment near midspan	0	60	60	60
	≥1.0	90	75	45

For *exterior* panels, the portion of the design moment that is not assigned to the column strip (Table 17.5) must be resisted by the corresponding half-middle strips. Again, linear interpolation between values shown in Table 17.5 is permitted by the ACI Code, Section 13.6.4.2. When no beams are used in an exterior panel, as in the case of flat slabs or flat plates with no edge (spandrel) beam, $\alpha_{f_1} = 0, C = 0$, and $\beta_t = 0$. This means that the end column provides the restraint to the exterior end of the slab. The applicable values of Table 17.5 for this special case are shown in Table 17.6 and Fig. 17.15.

From Table 17.6 it can be seen that when no edge beam is used at the exterior end of the slab, $\beta_t = 0$ and 100% of the design moment is resisted by the column strip. The middle strip

Table 17.4 Percentage of Moments in Two-Way Interior Slabs Without Beams ($\alpha_1 = 0$)

	Total Design Moment = $M_o = (w_u l_2)\left(\frac{l_{n_1}^2}{8}\right)\frac{n!}{r!(n-r)!}$	
	Negative Moment	**Positive Moment**
Longitudinal moments in one panel	$-0.65M_o$	$\pm 0.35M_o$
Column strip	$0.75(-0.65M_o) = -0.49M_o$	$0.60(0.35M_o) = 0.21M_o$
Middle strip	$0.25(-0.65M_o) = 0.16M_o$	$0.40(0.35M_o) = 0.14M_o$

Table 17.5 Percentage of Longitudinal Moment in Column Strips, Exterior Panels (ACI Code, Section 13.6.4)

			Aspect Ratio l_2/l_1		
	$\alpha_{f_1} l_2/l_1$	β_t	0.5	1.0	2.0
Negative moment at exterior support	0	0	100	100	100
		≥ 2.5	75	75	75
	≥ 1.0	0	100	100	100
		≥ 2.5	90	75	45
Positive moment near midspan	0		60	60	60
	≥ 1.0		90	75	45
Negative moment at interior support	0		75	75	75
	≥ 1.0		90	75	45

Table 17.6 Percentage of Longitudinal Moment in Column and Middle Strips, Exterior Panels (For All Ratios of l_2/l_1), Given $\alpha_{f_1} = \beta_t = 0$

	%	Column Strip	Middle Strip	Final Moment as a Function of M_o and α_{ec} (Column Strip)
Negative moment at exterior support	100	$0.26M_o$	0	$\left[\frac{0.65}{(1 + 1/\alpha_{ec})}\right](M_o)$
Positive moment ($0.6 \times 0.52M_o$)	60	$0.312M_o$	$0.208M_o$	$\left[0.63 - \frac{0.28}{(1 + 1/\alpha_{ec})}\right](M_o)$
Negative moment at interior support ($0.75 \times 0.70M_o$)	75	$0.525M_o$	$0.175M_o$	$\left[0.75 - \frac{0.10}{(1 + 1/\alpha_{ec})}\right](M_o)$

will not resist any moment; therefore, minimum steel reinforcement must be provided. The ACI Code, Section 13.6.4.3, specifies that when the exterior support is a column or wall extending for a distance equal to or greater than three-fourths the transverse span length, l_2, used to compute M_o, the exterior negative moment is to be uniformly distributed across l_2. When beams are provided along the centerlines of columns, the ACI Code, Section 13.6.5, requires that beams must be proportioned to resist 85% of the moment in the column strip if $\alpha_{f_1}(l_2/l_1) \geq 1.0$. For values of $\alpha_{f_1}(l_2/l_1)$ between 1.0 and 0, the moment assigned to the beam is determined by linear interpolation. Beams must also be proportioned to resist additional moments caused by all loads applied directly to the beams, including the weight of the projecting beam stem. The portion of the moment that is not assigned to the beam must be resisted by the slab in the column strip.

17.8.5 ACI Provisions for Effects of Pattern Loadings

In continuous structures, the maximum and minimum bending moments at the critical sections are obtained by placing the live load in specific patterns to produce the extreme values. Placing the live load on all spans will not produce either the maximum positive or negative bending moments. The maximum and minimum moments depend mainly on the following:

1. The ratio of live to dead load. A high ratio will increase the effect of pattern loadings.
2. The ratio of column to beam stiffnesses. A low ratio will increase the effect of pattern loadings.
3. Pattern loadings. Maximum positive moments within the spans are less affected by pattern loadings.

To determine the design factored moments in continuous structures, the ACI Code, Section 13.7.6, specifies the following:

1. When the loading pattern is known, the equivalent frame shall be analyzed for that load.
2. When the live load is variable but does not exceed $\frac{3}{4}$ of the dead load, $w_L \leq 0.75w_D$, or when all the panels is almost loaded simultaneously with the live load, it is permitted to analyze the frame with full factored live load on the entire slab system.
3. For other loading conditions, it is permitted to assume that the maximum positive factored moment near a midspan occurs with 0.75 of the full factored live load on the panel and alternate panels. For the maximum negative factored moment in the slab at a support, it is permitted to assume that 0.75 of the full factored live load is applied on adjacent panels only.
4. Factored moments shall not be taken less than the moments occurring with full factored live load on all continuous panels.

17.8.6 Reinforcement Details

After all the percentages of the static moments in the column and middle strips are determined, the steel reinforcement can be calculated for the negative and positive moments in each strip, as was done for beam sections in Chapter 4:

$$M_u = \phi A_s f_y \left(d - \frac{a}{2}\right) = R_u bd^2 \tag{17.15}$$

Calculate R_u and determine the steel ratio ρ using the tables in Appendix A or use the following equation:

$$R_u = \phi \rho f_y \left(1 - \frac{\rho f_y}{1.7 f_c'}\right) \tag{17.16}$$

where $\phi = 0.9$. The steel area is $A_s = \rho bd$. When the slab thickness limitations, as discussed in Section 17.4, are met, no compression reinforcement will be required. Fig. 13.3.8 of the ACI Code indicates the minimum length of reinforcing bars and reinforcement details for slabs without beams; it is reproduced here as Fig. 17.16. The spacing of bars in the slabs must not exceed the ACI limits of maximum spacing: 18 in. (450 mm) or twice the slab thickness, whichever is smaller.

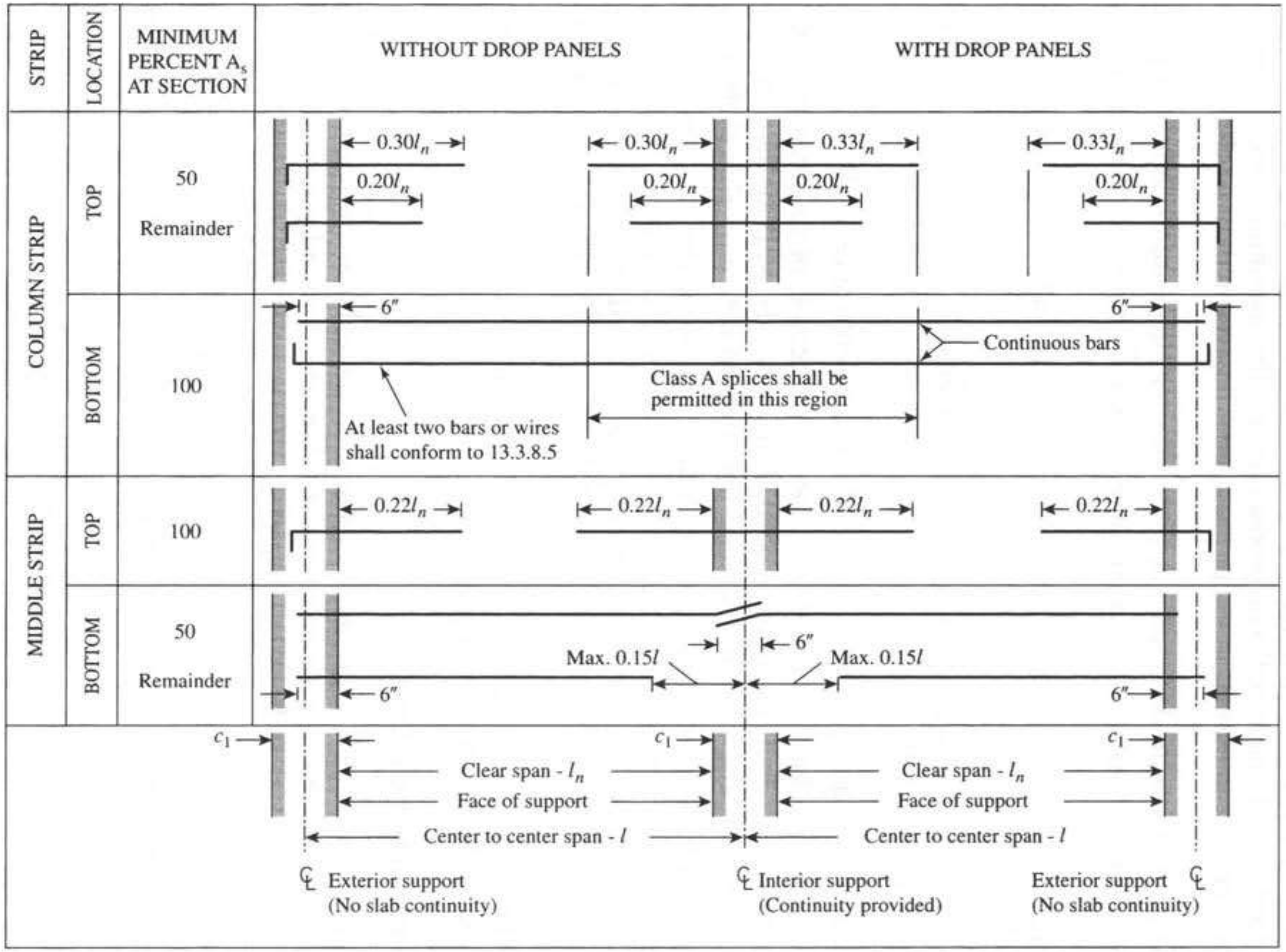


Figure 17.16 Minimum extensions for reinforcement in slabs without beams (ACI Code, Fig. 13.3.8). Courtesy of American Concrete Institute [14].

17.8.7 Modified Stiffness Method for End Spans

In this method, the stiffnesses of the slab end beam and of the exterior column are replaced by the stiffness of an equivalent column, K_{ec}. The flexural stiffness of the equivalent column, K_{ec}, can be calculated from the following expression:

$$\frac{1}{K_{ec}} = \frac{1}{\sum K_c} + \frac{1}{K_t} \quad \text{or} \quad K_{ec} = \frac{\sum K_c}{1 + \sum K_c / K_t} \tag{17.17}$$

where

K_{ec} = flexural stiffness of the equivalent column
K_c = flexural stiffness of the actual column
K_t = torsional stiffness of the edge beam

The sum of the flexural stiffness of the columns above and below the floor slab can be taken as follows:

$$\sum K_c = 4E\left(\frac{I_{c1}}{L_{c1}} + \frac{I_{c2}}{L_{c2}}\right) \tag{17.18}$$

where I_{c1} and L_{c1} the moment of inertia and length of column above slab level and I_{c2} and L_{c2} = the moment of inertia and length of column below slab level. The torsional stiffness of the end beam, K_t, may be calculated as follows:

$$K_t = \sum \frac{9E_{cs}C}{l_2\left(1 - \frac{c_2}{l_2}\right)^3} \tag{17.19}$$

where

c_2 = size of the rectangular or equivalent rectangular column, capital, or bracket measured on transverse spans on each side of the column
E_{cs} = modulus of elasticity of the slab concrete
C = torsion constant determined from the following expression:

$$C = \sum\left(1 - 0.63\frac{x}{y}\right)\left(\frac{x^3 y}{3}\right) \tag{17.20}$$

where x is the shorter dimension of each component rectangle and y is the longer dimension of each component rectangle. In calculating C, the component rectangles of the cross-section must be taken in such a way as to produce the largest value of C.

The preceding expressions are introduced here and will also be used in Section 17.2, "Equivalent Frame Method."

If a panel contains a beam parallel to the direction in which moments are being determined, the torsional stiffness, K_t, given in Eq. 17.19 must be replaced by a greater value, K_{ta}, computed as follows:

$$K_{ta} = K_t \times \frac{I_{sb}}{I_s}$$

where

$I_s = \frac{l_2 h^3}{12}$ = moment of inertia of a slab that has a width equal to the full width between panel centerlines (excluding that portion of the beam stem extending above or below the slab)

$I_{sb} = I_s$, including the portion of the beam stem extending above or below the slab.

Cross-sections of some attached torsional members are shown in Fig. 17.17. Once K_{ec} is calculated, the stiffness ratio, α_{ec}, is obtained as follows:

$$\alpha_{ec} = \frac{K_{ec}}{\sum(K_s + K_b)} \tag{17.21}$$

where

$K_s = \frac{4E_{cs}I_s}{l_1}$ = flexural stiffness of the slab

$K_b = \frac{4E_{cb}I_b}{l_1}$ = flexural stiffness of the beam

I_b = gross moment of inertia of the longitudinal beam section

The distribution of the total static moment, M_o, in an exterior panel is given as a function of α_{ec} as follows:

$$\text{Interior negative factored moment} = \left[0.75 - \frac{0.1}{(1 + 1/\alpha_{ec})}\right] M_o$$

$$\text{Positive factored moment} = \left[0.63 - \frac{0.28}{(1 + 1/\alpha_{ec})}\right] M_o$$

$$\text{Exterior negative factored moment} = \left[\frac{0.65}{(1 + 1/\alpha_{ec})}\right] M_o$$

These values are shown for a typical exterior panel in Fig. 17.18. These factors take into consideration the effect of the stiffness of the exterior column as well as the slab end beam giving adequate distribution of moments.

17.8.8 Summary of the Direct Design Method (DDM)

Case 1. Slabs without beams (flat slabs and flat plates).

1. Check the limitation requirements explained in Section 17.8.1. If limitations are not met, DDM cannot be used.
2. Determine the minimum slab thickness (h_{min}) to control deflection using values in Table 17.1. Exterior panels without edge beams give the highest $h_{min}(l_n/30$ for $f_y = 60$ ksi). It is a common practice to use the same slab depth for all exterior and interior panels.
3. Calculate the factored loads, $W_u = 1.2W_D + 1.6W_L$.
4. Check the slab thickness, h, as required by one-way and two-way shear. If the slab thickness, h, is not adequate, either increase h or provide shear reinforcement.
5. Calculate the total static moment, M_o, in both directions (Eq. 17.11).

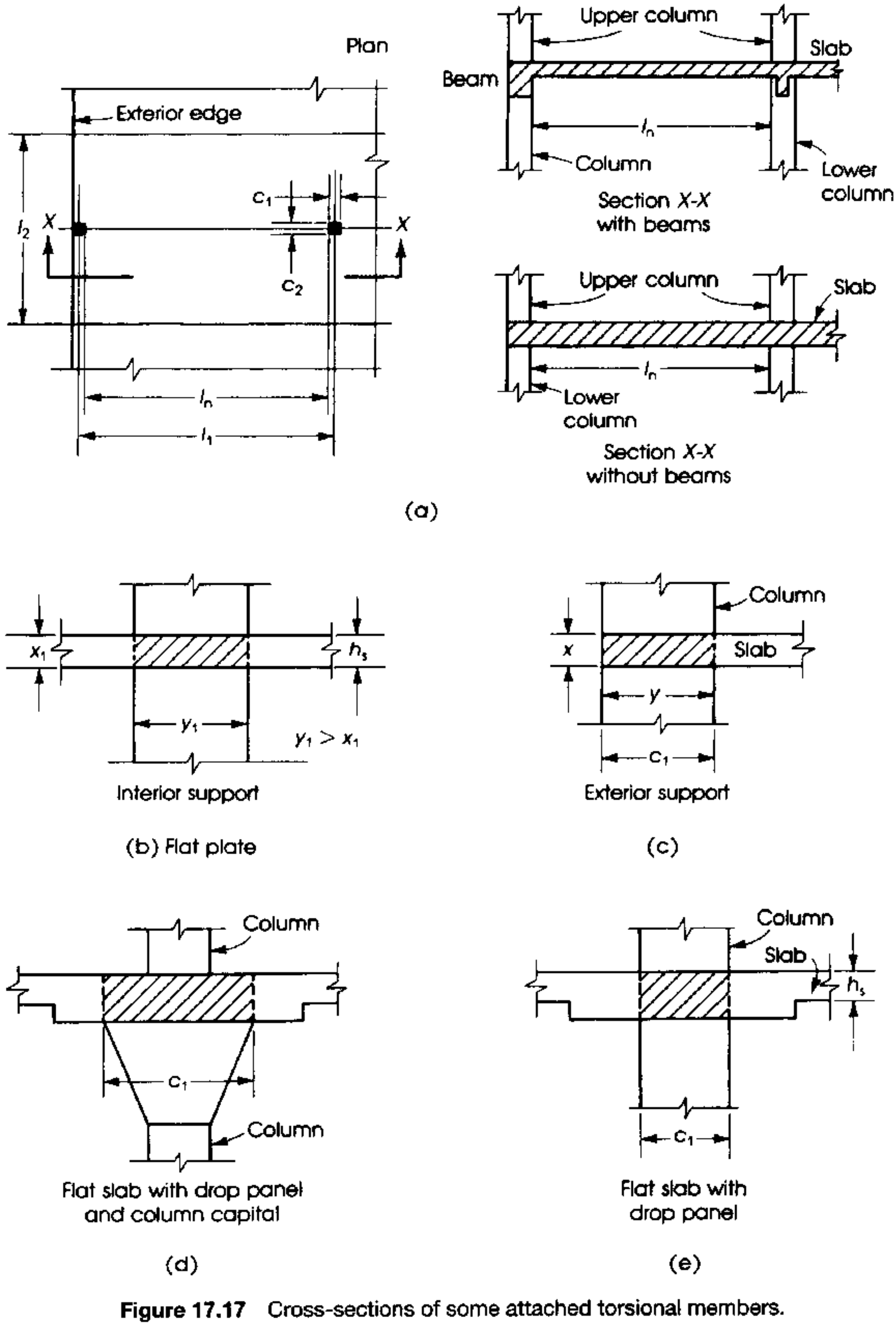


Figure 17.17 Cross-sections of some attached torsional members.

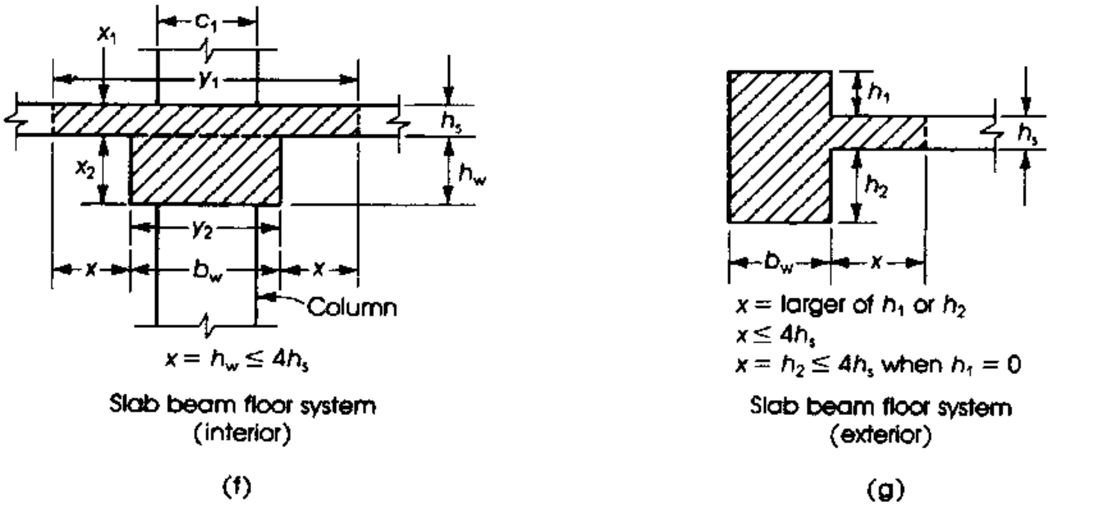


Figure 17.17 (*continued*)

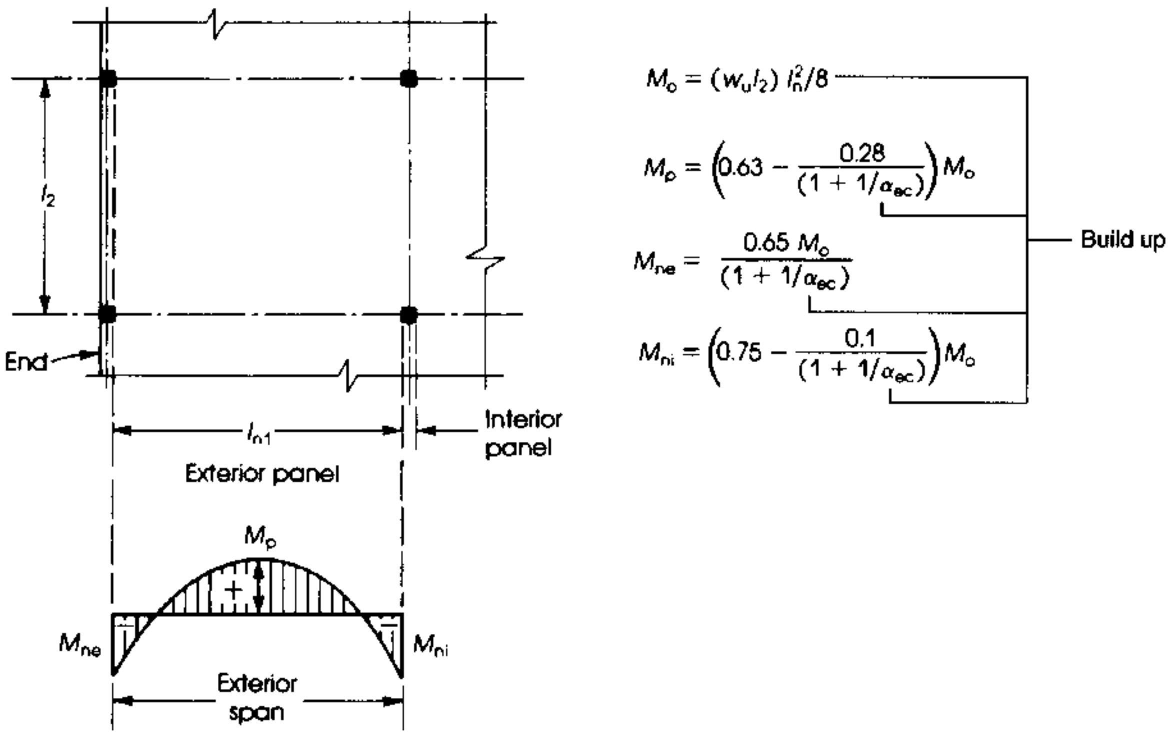


Figure 17.18 Distribution of moments in an exterior panel.

6. Determine the distribution factors for the positive and negative moments in the longitudinal and transverse directions for each column and middle strip in both interior and exterior panels as follows:
   a. For interior panels, use the moment factors given in Table 17.4 or Fig. 17.15.
   b. For exterior panels without edge beams, the panel moment factors are given in Table 17.2 or Fig. 17.14 (Case 5). For the distribution of moments in the transverse direction, use

Table 17.6 or Fig. 17.15 for column-strip ratios. The middle strip will resist the portion of the moment that is not assigned to the column strip.

c. For exterior panels with edge beams, the panel moment factors are given in Table 17.2 or Fig. 17.14 (Case 4). For the distribution of moments in the transverse direction, use Table 17.5 for the column strip. The middle strip will resist the balance of the panel moment.

7. Determine the steel reinforcement for all critical sections of the column and middle strips and extend the bars throughout the slab according to Fig. 17.16.
8. Compute the unbalanced moment and check if transfer of unbalanced moment by flexure is adequate. If not, determine the additional reinforcement required in the critical width. (Refer to Section 17.10.)
9. Check if transfer of the unbalanced moment by shear is adequate. If not, increase h or provide shear reinforcement. (Refer to Section 17.10.)

Case 2. Slabs with interior and exterior beams.

1. Check the limitation requirements as explained in Section 17.8.1.
2. Determine the minimum slab thickness ($h_{\min}$) to control deflection using Eqs. 17.1 through 17.3. In most cases, Eq. 17.2 controls. Equation 17.1 should be calculated first, as shown in Example 17.1.
3. Calculate the factored load, W_u.
4. Check the slab thickness, h, according to one-way and two-way shear requirements. In general, shear is not critical for slabs supported on beams.
5. Calculate the total static moment, M_o in both directions (Eq. 17.17).
6. Determine the distribution factors for the positive and negative moments in the longitudinal and transverse directions for each column and middle strips in both interior and exterior panels as follows:
   a. For interior panels, use moment factors in Fig. 17.14 (Case 3) or Fig. 17.12. For the distribution of moments in the transverse direction, use Table 17.3 for column strips. The middle strips will resist the portion of the moments not assigned to the column strips. Calculate α_1 from Eq. 17.12.
   b. For exterior panels, use moment factors in Table 17.2 or Fig. 17.14 (Case 3). For the distribution of moments in the transverse direction, use Table 17.5 for the column strip. The middle strip will resist the balance of the panel moment.
   c. In both cases (a) and (b), the beams must resist 85% of the moment in the column strip when $\alpha_{f_1}(l_2/l_1) \geq 1.0$, whereas the ratio varies between 85% and 0% when $\alpha_{f_1}(l_2/l_1)$ varies between 1.0 and 0.
7. Determine the steel reinforcement for all critical sections in the column strip, beam, and middle strip; then extend the bars throughout the slab according to Fig. 17.16.
8. Compute the unbalanced moment and then check the transfer of moment by flexure and shear. (Refer to Section 17.10.)

Example 17.3

Using the direct design method, design the typical *interior flat-plate* panel shown in Figs. 17.6 and 17.19. The floor system consists of four panels in each direction with a panel size of 24 by 20 ft. All panels are supported by 20- by 20-in. columns, 12 ft long. The slab carries a uniform service live

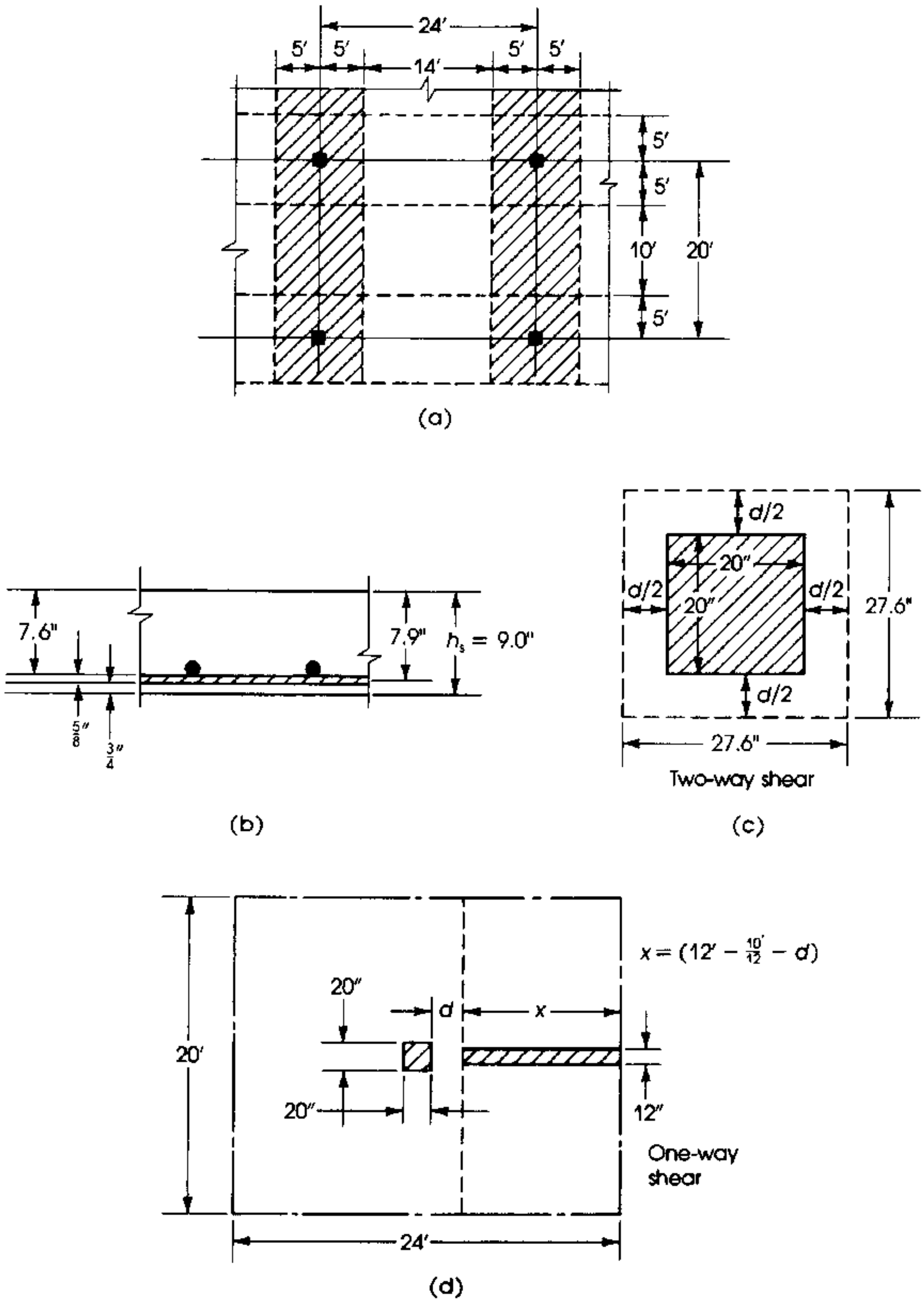


Figure 17.19 Example 17.3: interior flat plate.

load of 100 psf and a service dead load that consists of 24 psf of floor finish in addition to the slab self-weight. Use normal-weight concrete with $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Determine the minimum slab thickness using Table 17.1 for flat plates. From Example 17.1, a 9-in. slab thickness is adopted.
2. Calculate the factored loads:

$$w_D = 24 + \text{weight of slab} = 24 + \frac{9.0}{12} \times 150 = 136.5 \text{ psf}$$

$$w_u = 1.2 \times (136.5) + 1.6 \times (100) = 323 \quad \text{say, 330 psf}$$

3. Check one- and two-way shears:
   a. Check punching shear at a distance $d/2$ from the face of the column (two-way action): Assuming $\frac{3}{4}$ in. concrete cover and no. 5 bars, then the average d is $9.0 - 0.75 - \frac{5}{8} = 7.6$ in. and $b_o = 4(20 + 7.16) = 110$ in. (See Fig. 17.19c).

$$V_u = \left[l_1 l_2 - \left(\frac{27.6}{12} \times \frac{27.6}{12}\right)\right] \times w_u = (24 \times 20 - 5.3) \times 0.330 = 156.7 \text{ K}$$

$$\phi V_c = \phi(4\sqrt{f'_c})b_o d = \frac{0.75 \times 4}{1000} \times \sqrt{4000} \times 110 \times 7.6 = 158.6 \text{ K}$$

   which is greater than V_u.

   b. Check beam shear at a distance d from the face of the column; average d is 7.6 in. Consider a 1-ft strip (Fig. 17.19d), with the length of the strip being

$$x = 12 - \frac{10}{12} - \frac{7.6}{12} = 10.5 \text{ ft}$$

$$V_u = w_u(1 \times 10.5) = 0.330 \times 1 \times 10.5 = 3.47 \text{ K}$$

$$\phi V_c = \phi(2\lambda\sqrt{f'_c})bd = \frac{0.75 \times 2 \times 1}{1000} \times \sqrt{4000} \times (12 \times 7.6) = 8.7 \text{ K}$$

   which is greater than $V_u = 3.47$ K. In normal loadings, one-way shear does not control.

4. Calculate the total static moments in the long and short directions. In the long direction,

$$M_{ol} = \frac{w_u l_2 l_{n1}^2}{8} = \frac{0.33}{8} \times 20(22.33)^2 = 411.4 \text{ K·ft}$$

In the short direction,

$$M_{os} = \frac{w_u l_1 l_{n2}^2}{8} = \frac{0.33}{8} \times 24 \times (18.33)^2 = 333 \text{ K·ft}$$

Because $l_2 < l_1$, the width of half a column strip in the long direction is $0.25 \times 20 = 5$ ft, and the width of the middle strip is $20 - 2 \times 5 = 10$ ft. The width of half the column strip in the short direction is 5 ft, and the width of the middle strip is $24 - 2 \times 5 = 14$ ft. To calculate the effective depth, d, in each direction, assume that steel bars in the short direction are placed on top of the bars in the long direction. Therefore, d(long direction) $= 9.0 - 0.75 - \frac{5}{16} = 7.9$ in. and d (short direction) $= 9.0 - 0.75 - \frac{5}{8} - \frac{5}{16} - = 7.3$ in. For practical applications, an average $d = 9 - 1.5 = 7.15$ in. can be used for both directions.

The design procedure can be conveniently arranged in a table form, as in Tables 17.7 and 17.8.

Table 17.7 Design of Interior Flat-Plate Panel (Long Direction)

	$M_o = 411.4$ K·ft $M_n = 0.65M_o = -267.4$ K·ft $M_p = +0.35M_o = +144$ K·ft			
	Column Strip		**Middle Strip**	
Long Direction	**Negative**	**Positive**	**Negative**	**Positive**
Moment distribution (%)	75	60	25	40
M_u (K·ft)	$0.75M_n = -201.6$	$0.6M_p = \pm 86.4$	$0.25M_n = -66.8$	$0.4M_p = \pm 57.16$
Width of strip b (in.)	120	120	120	120
Effective depth d (in.)	7.9	7.9	7.9	7.9
$R_u = \frac{M_u}{bd^2}$ (psi)	323	128	107	93
Steel ratio ρ (%)	0.633	0.262	0.2	0.175
$A_s = \rho bd$ (in.^{2})	6.00	2.48	1.92	1.66
Min. $A_s = 0.0018bh_s$ (in.^{2})	1.94	1.94	1.94	1.94
Bars selected (Straight)	20 no. 5	10 no. 5	10 no. 4	10 no. 4
Spacing $\leq 2h_s = 18$ in.	6 in.	12	12	12

Table 17.8 Design of Interior Flat-Plate Panel (Short Direction)

	$M_o = 333$ K·ft $M_n = 0.65M_o = -216.5$ K·ft $M_p = +0.35M_o = +116.5$ K·ft			
	Column Strip		**Middle Strip**	
Short Direction	**Negative**	**Positive**	**Negative**	**Positive**
Moment distribution (%)	75	60	25	40
M_u (K·ft)	$0.75M_n = -162.4$	$0.6M_p = \pm 69.9$	$0.25M_n = -54.1$	$0.4M_p = \pm 46.16$
Width of strip b (in.)	120	120	168	168
Effective depth d (in.)	7.3	7.3	7.3	7.3
$R_u = \frac{M_u}{bd^2}$ (psi)	305	131	73	62
Steel ratio ρ (%)	0.60	0.25	0.14	0.12
$A_s = \rho bd$ (in.^{2})	5.23	2.18	1.72	1.46
Min. $A_s = 0.0018bh_s$ (in.^{2})	1.94	1.94	2.72	2.72
Bars selected (Straight)	18 no. 5	10 no. 5	14 no. 4	14 no. 4
Spacing $\leq 2h_s = 18$ in.	6.7	12	12	12

The details for the bars selected for this interior slab are shown in Fig. 17.20 using the staight bar system. Minimum lengths of the bars must meet those shown in Fig. 17.16.

Straight bars and $f_y = 60$ ksi steel bars are more often preferred by contractors.

$$\text{Maximum spacing} = \frac{\text{width of panel}}{\text{no. of bars}} = \frac{168}{14} = 12 \text{ in.}$$

occurs at the middle strip in the short direction; this spacing of 12 in. is adequate, because it is less than $2h_s = 18$ in. and less than 18 in. specified by the ACI Code. Note that all steel ratios are less than $\rho_{max} = 0.018$. Thus $\phi = 0.9$.

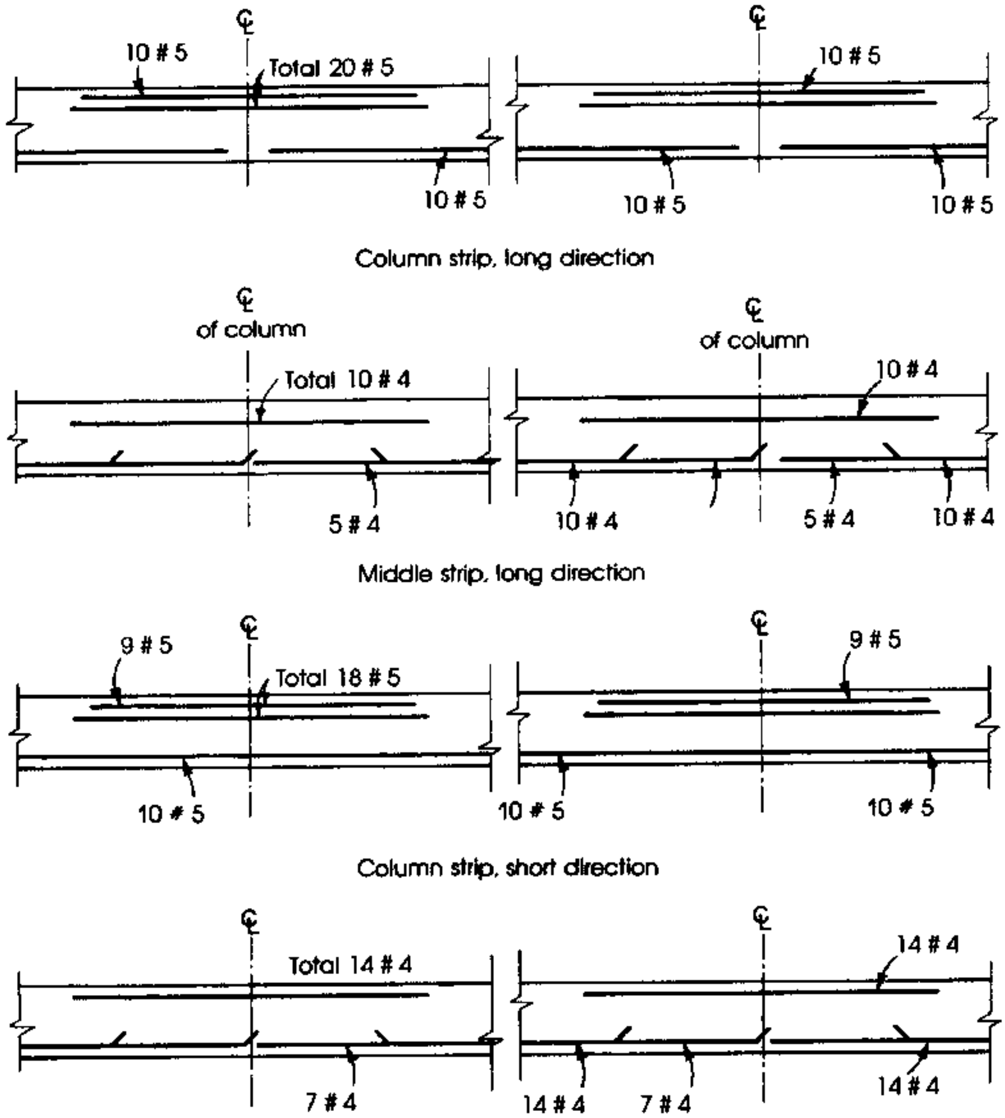


Figure 17.20 Example 17.3: reinforcement details. For bar length, refer to Fig. 17.16.

Example 17.4

Using the direct design method, design an *exterior flat-plate* panel that has the same dimensions, loads, and concrete and steel strengths given in Example 17.3. No beams are used along the edges (Fig. 17.21).

Solution

1. Determine the minimum slab thickness using Table 17.1 for flat plates. From Example17.1, a 9.0-in. slab thickness is adopted.
2. Calculate factored loads: $W_u = 330$ psf. (See Example 17.3.)
3. Check one- and two-way shear (refer to Example 17.3 and Fig. 17.19).
   a. Check punching shear at an interior column, $V_u = 156.7 < \phi V_c = 158.6$ K.
   b. Check one-way shear: $V_u = 3.47$ K $< \phi\ V_c = 8.7$ K.

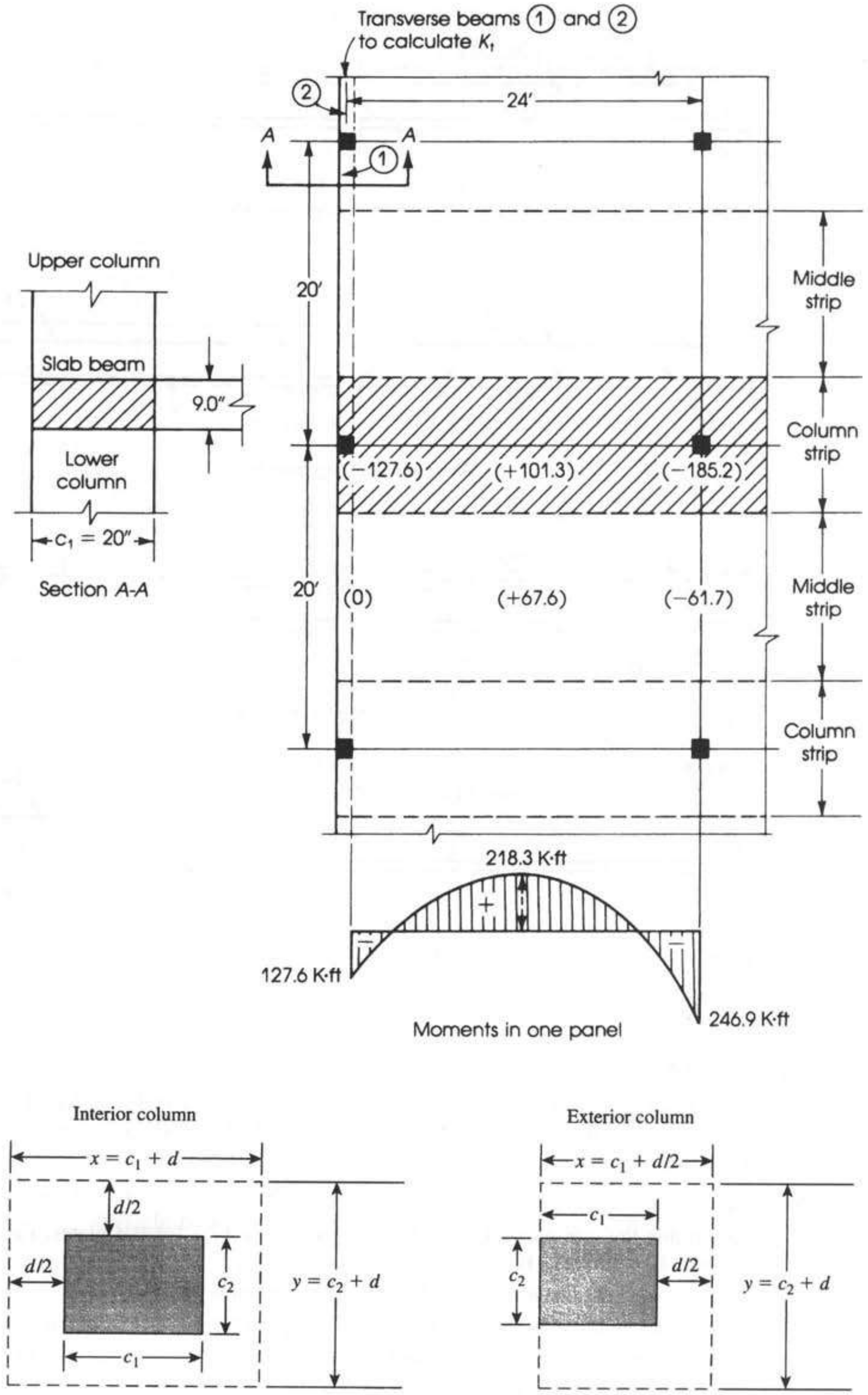


Figure 17.21 Example 17.4: distribution of bending moments.

c. Check punching shear at the exterior column: $d = 7.6$ in.

$$x = 20 + \frac{d}{2} = 20 + \frac{7.6}{2} = 23.8 \text{ in.} = 1.98 \text{ ft}$$

$$y = 20 + d = 20 + 7.6 = 27.6 \text{ in.} = 2.30 \text{ ft}$$

$$b_o = 2x + y = 75.2 \text{ in.}$$

$$V_u = \left[20\left(12 + \frac{10}{12}\right) - 1.98(2.30)\right]0.33 = 83.2 \text{ K}$$

$$\phi V_c = \phi 4\sqrt{f'_c}b_o d = 108.4 \text{ K} > V_u$$

d. Check punching shear at a corner column: $d = 7.6$ in.

$$x = y = 20 + \frac{d}{2} = 23.8 \text{ in.} = 1.98 \text{ ft}$$

$$b_o = x + y = 47.6 \text{ in.}$$

$$V_u = \left[\left(10 + \frac{10}{12}\right)\left(12 + \frac{10}{12}\right) - (1.98)(1.98)\right]0.33 = 44.6 \text{ K}$$

$$\phi V_c = \phi 4\sqrt{f'_c}b_o d = 68.6 \text{ K} > V_u$$

4. Calculate the total static moments. From Example 17.3,

$$M_{ol} \text{ (long direction)} = 411.4 \text{ K·ft} \qquad d = 7.9 \text{ in.}$$

$$M_{os} \text{ (short direction)} = 333 \text{ K·ft} \qquad d = 7.3 \text{ in.}$$

The width of the column strip is 120 in., and the width of the middle strip is 168 in.

5. Calculate the design moments in the long direction: $l_1 = 24$ ft. (Refer to Table 17.5 or Fig. 17.15). The distribution of the total moment, M_{ol}, in the column and middle strips is computed as follows:

a. Column strip:

$$\text{Interior negative moment} = -0.525M_o = -0.525(411.4) = -216 \text{ K·ft}$$

$$\text{Positive moment within span} = 0.312M_o = 0.312(411.4) = +128.4 \text{ K·ft}$$

$$\text{Exterior negative moment} = -0.26M_o = -0.26(411.14) = -107 \text{ K·ft}$$

b. Middle strip:

$$\text{Interior negative moment} = -0.175M_o = -0.175 \times 411.4 = -72 \text{ K·ft}$$

$$\text{Positive moment within span} = 0.208M_o = 0.208 \times 411.4 = +85.6 \text{ K·ft}$$

$$\text{Exterior negative moment} = 0$$

6. Calculate the design moments in the short direction: $l_s = 20$ ft. It will be treated as an interior panel because it is continuous on both sides. Referring to Table 17.4 or Fig. 17.15, the distribution of the total moment, M_{os}, in the column and middle strips is computed as follows:

a. Column strip:

$$\text{Negative moment} = 0.49M_o = -0.49(333) = -163.2 \text{ K·ft}$$

$$\text{Positive moment} = +0.21M_o = +0.21(333) = +70.0 \text{ K·ft}$$

Table 17.9 Design of Exterior Flat-Plate Panel for Example 17.4 ($d = 7.9$ in.)

Long Direction	Column Strip Exterior	Column Strip Positive	Column Strip Interior	Middle Strip Exterior	Middle Strip Positive	Middle Strip Interior
M_u (K·ft)	−107.06	±128.4	−216.0	0	±85.6	−72.0
b (in.)	120	120	120	120	120	120
$R_u = \frac{M_u}{bd^2}$ (psi)	172	206	346	0	138	116
Steel ratio ρ (%)	0.33	0.4	0.682	0	0.262	0.22
$A_s = \rho bd$	3.11	3.75	6.47	0	2.48	2.10
Min. $A_s = 0.0018bh_s$	1.94	1.94	1.94	1.94	1.94	1.94
Bars selected (Straight)	12 no. 5	12 no. 5	22 no. 5	10 no. 4	14 no. 4	14 no. 4
Spacing # 18 in.	10	10	5.5	12	8.5	8.5

Short Direction	Column Strip		Middle Strip	
M_u (K·ft)	−163.2	±70.0	−53.3	±46.6
Width of strip b (in.)	120	120	168	168
d(in.)	7.3	7.3	7.3	7.3
$R_u = \frac{M_u}{bd^2}$ (psi)	306	131	71	63
Steel ratio ρ (%)	0.6	0.25	0.133	0.12
$A_s = \rho bd$ (in.^{2})	5.26	2.20	1.63	1.47
Min. $A_s = 0.0018bh_s$	1.94	1.94	2.72	2.72
Bars selected (Straight)	18 no. 5	8 no. 5	14 no. 4	14 no. 4
Spacing 18 in.	6.67	15	12	12

b. Middle strip:

$$\text{Negative moment} = -0.16M_o = -0.16(333) = -53.3 \text{ K·ft}$$

$$\text{Positive moment} = +0.14M_o = +0.14(333) = +46.6 \text{ K·ft}$$

The design procedure can be conveniently arranged in Table 17.9. The details for bars selected are shown in Fig. 17.22 using the straight-bar system in the long direction. Details of reinforcement in the short direction will be similar to Fig. 17.20 using the bars chosen in Table 17.9.

Note that all steel ratios are less than $\rho_{max} = 0.018$. Thus $\phi = 0.9$.

Example 17.5

Repeat Example 17.4 using the modified stiffness method. (Similar calculations are needed for the equivalent frame method, Section 17.12.)

Solution

1. Steps 1 through 4 will be the same as in Example 17.4.
2. Calculate the equivalent column stiffness, K_{ec}:

$$\frac{1}{K_{ec}} = \frac{1}{\sum K_c} + \frac{1}{K_t}$$

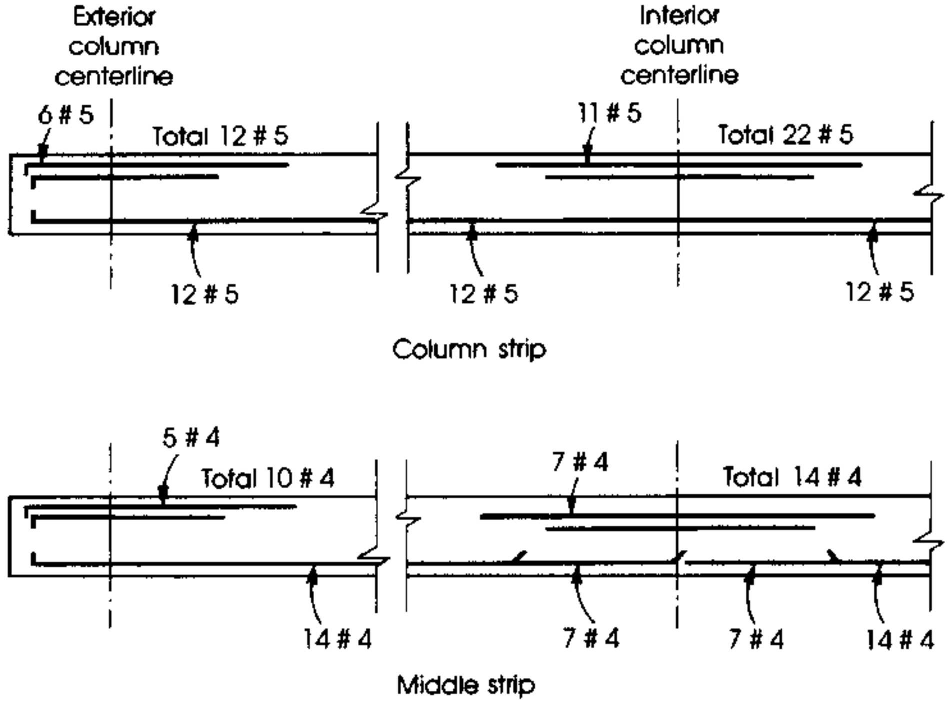


Figure 17.22 Example 17.4: reinforcement details (longitudinal direction). For bar lengths, refer to Fig. 17.16.

It can be assumed that the part of the slab strip between exterior columns acts as a beam resisting torsion. The section of the slab-beam is 20 in. (width of the column) × 9.0 in. (thickness of the slab), as shown in Fig. 17.21.

a. Determine the torsional stiffness, K_t, from Eq. 17.20:

$$C = \left(1 - 0.63\frac{x}{y}\right)\frac{x^3 y}{3} \qquad x = 9 \text{ in} \qquad y = 20 \text{ in.}$$

$$C = \left(1 - 0.63 \times \frac{9}{20}\right)\frac{9^3 \times 20}{3} = 3482 \text{ in.}^4$$

$$K_t = \frac{9E_c C}{l_2\left(l - \frac{c_2}{l_2}\right)^3} = \frac{9E_c \times 3482}{(20 \times 12)\left(1 - \frac{20}{20 \times 12}\right)^3} = 170E_c$$

For the two adjacent slabs (on both sides of the column) acting as transverse beams,

$$K_t = 2 \times 170E_c = 340E_c$$

b. Calculate the column stiffness, K_c; the column height $L_c = 12$ ft:

$$K_c = \frac{4E_c I_c}{L_c} = \frac{4E_c}{(12 \times 12)} \times \frac{(20)^4}{12} = 370.4E_c$$

For two columns above and below the floor slab,

$$K_c = 2 \times 370.14E_c = 740.18E_c$$

c. Calculate K_{ec}:

$$\frac{1}{K_{ec}} = \frac{1}{740.8E_c} + \frac{1}{340E_c}$$

To simplify the calculations, multiply by $1000E_c$:

$$\frac{1000E_c}{K_{ec}} = \frac{1000}{740.8} + \frac{1000}{340} = 4.29 \qquad K_{ec} = 233E_c$$

3. Calculate slab stiffness and the ratio α_{ec}:

$$K_s = \frac{4E_c I_s}{l_1} \qquad h_s = 9 \text{ in.} \qquad l_2 = 20 \text{ ft} \qquad I_s = \frac{l_2 h_s^3}{12}$$

$$K_s = \frac{4E_c}{(24 \times 12)} \times \frac{(20 \times 12)(9.0)^3}{12} = 202.5E_c$$

$$\alpha_{ec} = \frac{K_{ec}}{\sum (K_s + K_b)} \tag{17.21}$$

$K_b = 0$ (no beams are provided)

thus

$$\alpha_{ec} = \frac{233E_c}{202.5E_c} = 1.15$$

Let

$$Q = \left(1 + \frac{1}{\alpha_{ec}}\right) = 1 + \frac{1}{1.15} = 1.87$$

4. Calculate the design moments in the long direction: $l_1 = 24$ ft. The distribution of moments in one panel is shown in Fig. 17.18. The interior negative moment is

$$M_{nt} = \left[0.75 - \frac{0.10}{Q}\right] M_{ol} = \left(0.75 - \frac{0.10}{1.87}\right)(411.4) = -286.6 \text{ K·ft}$$

The positive moment is

$$M_p = \left[0.63 - \frac{0.28}{Q}\right] M_{ol}$$
$$= \left(0.63 - \frac{0.28}{1.87}\right)(411.4) = 197.6 \text{ K·ft}$$

The exterior negative moment is

$$M_{ne} = \frac{0.65}{Q}(M_{ol}) = \frac{0.65}{1.87}(411.4) = -143.0 \text{ K·ft}$$

5. Calculate the distribution of panel moments in the transverse direction to column and middle strips. The moments M_{ni}, M_p, and M_{ne} are distributed as follows (refer to Table 17.6):

a. The interior moment (M_{nl}) = −286.6 K·ft is distributed 75% for the column strip and 25% for the middle strip.

Column strip = 0.75(−286.6) = −215 K·ft

Middle strip = 0.25(−286.6) = −71.6 K·ft

b. The positive moment, $M_p = 197.6$ K·ft, is distributed 60% for the column strip and 40% for the middle strip.

$$\text{Column strip} = 0.6(197.6) = 118.5 \text{ K·ft}$$

$$\text{Middle strip} = 0.4(197.6) = 79.1 \text{ K·ft}$$

c. The exterior negative moment, $M_{\text{ne}} = -143$ K·ft, is distributed according to Table 17.5:

$$\beta_t = \frac{E_c C}{2E_c I_s} = \frac{C}{2I_s}$$

The concrete of slab and column are the same.

$$I_s = (20 \times 12)\frac{(9.0)^3}{12} = 14{,}580 \text{ in.}^4$$

$$\beta_t = \frac{3482}{2 \times 14{,}580} = 0.119$$

$$\alpha_{f_1} = \frac{E_{\text{cb}} I_b}{E_{\text{cs}} I_s} = 0 \qquad \alpha_{f_1}\frac{l_2}{l_1} = 0 \qquad \frac{l_2}{l_1} = 0.83$$

From Table 17.5 and by interpolation between $\beta_t = 0$ (percentage = 100%) and $\beta_t = 2.5$ (percentage = 100%) for $\beta_t = 0.1119$, the percentage is 99%. The exterior negative moment in the column strip is $0.99(-143.10) = -142$ K·ft and in the middle strip, it is -1.10 K·ft. It is practical to consider that the column strip carries in this case 100% of $M_{\text{ne}} = -143$ K·ft.

6. Determine the reinforcement required in the long direction in a table form similar to Example 17.4. Results will vary slightly from those of Table 17.9.
7. Comparison of results between Examples 17.4 and 17.5 shows that the exterior moment in the column strip (− 143 K·ft) is greater than that calculated in Example 17.4 (− 107 K·ft) by about 34%, whereas the positive moment (± 118.15) is reduced by about 8% (relative to ± 128.14). Other values are almost compatible.

Example 17.6

Design an interior panel of the two-way slab floor system shown in Fig. 17.7. The floor consists of six panels in each direction, with a panel size of 24 by 20 ft. All panels are supported on 20- by 20-in. columns, 12 ft long. The slabs are supported by beams along the column lines with the cross-sections shown in the figure. The service live load is to be taken as 100 psf, and the service dead load consists of 22 psf of floor finish in addition to the slab weight. Use normal-weight concrete with $f'_c = 3$ ksi, $f_y = 60$ ksi, and the direct design method.

Solution

1. The limitations required by the ACI Code are met. Determine the minimum slab thickness using Eqs. 17.1 and 17.2. The slab thickness has been already calculated in Example 17.3, and a 7.0-in. slab can be adopted. Generally, the slab thickness on a floor system is controlled by a corner panel, as the calculations of h_{min} for an exterior panel give greater slab thickness than for an interior panel.
2. Calculate factored loads:

$$w_D = 22 + \text{weight of slab} = 22 + \frac{7}{12} \times 150 = 109.5 \text{ psf}$$

$$w_u = 1.2(109.5) + 1.6(100) = 292 \text{ psf}$$

3. The shear stresses in the slab are not critical. The critical section is at a distance d from the face of the beam. For a 1-ft width:

$$V_u = w_u\left(10 - \frac{1}{2}\text{beam width} - d\right)$$

$$= 0.292\left(10 - \frac{16}{2 \times 12} - \frac{6}{12}\right) = 2.58 \text{ K}$$

$$\phi V_c = \phi(2\lambda\sqrt{f_c'})bd = \frac{0.75 \times 2 \times 1 \times \sqrt{3000} \times 12 \times 6}{1000} = 6.3 \text{ K} > V_u$$

4. Calculate the total static moments in the long and short directions:

$$M_{\text{ol}} = \frac{w_u}{8} l_2 (l_{n1})^2 = \frac{0.292}{8}(20)(22.33)^2 = 364.0 \text{ K·ft}$$

$$M_{\text{os}} = \frac{w_u}{8} l_1 (l_{n2})^2 = \frac{0.292}{8}(24)(18.33)^2 = 294.3 \text{ K·ft}$$

5. Calculate the design moments in the long direction: $l_1 = 24$ ft.

a. Distribution of moments in one panel:

$$\text{Negative moment } (M_n) = 0.65 M_{\text{ol}} = 0.65 \times 364 = -236.6 \text{ K·ft}$$

$$\text{Positive moment } (M_p) = 0.35 M_{\text{ol}} = 0.35 \times 364 = 127.4 \text{ K·ft}$$

b. Distributions of panel moments in the transverse direction to the beam, column, and middle strips are as follows:

$$\frac{l_2}{l_1} = \frac{20}{24} = 0.83 \quad \alpha_{f_1} = \alpha_s = \frac{EI_b}{EI_s} = 3.27 \quad \text{(from Example 17.2)}$$

$$\alpha_{f_1}\frac{l_2}{l_1} = 3.27 \times 0.83 = 2.71 > 1.0$$

c. Distribute the negative moment, M_n. The portion of the interior negative moment to be resisted by the column strip is obtained from Table 17.3 by interpolation and is equal to 80% (for $l_2/l_1 = 0.183$ and $\alpha_{f_1}(l_2/l_1) > 1.0$).

$$\text{Column strip} = 0.18 M_n = 0.18 \times 236.16 = -189.13 \text{ K·ft}$$

$$\text{Middle strip} = 0.12 M_n = 0.12 \times 236.16 = -47.13 \text{ K·ft}$$

Because $\alpha_{f_1}(l_2/l_1) > 1.0$, the ACI Code, Section 13.6.5, indicates that 85% of the moment in the column strip is assigned to the beam and the balance of 15% is assigned to the slab in the column strip.

$$\text{Beam} = 0.85 \times 189.3 = -160.9 \text{ K·ft}$$

$$\text{Column strip} = 0.15 \times 189.3 = -28.4 \text{ K·ft}$$

$$\text{Middle strip} = -47.3 \text{ K·ft}$$

d. Distribute the positive moment, M_p. The portion of the interior positive moment to be resisted by the column strip is obtained from Table 17.3 by interpolation and is equal to 80% (for $l_2/l_1 = 0.83$ and $\alpha_{f_1}(l_1/l_2) > 1.0$).

$$\text{Column strip} = 0.8 M_p = 0.8 \times 127.4 = +101.9 \text{ K·ft}$$

$$\text{Middle strip} = 0.2 M_p = 0.2 \times 127.4 = +25.5 \text{ K·ft}$$

Since $\alpha_{f1}(l_2/l_1) > 1.0$, 85% of the moment in the column strip is assigned to the beam and the balance of 15% is assigned to the slab in the column strip:

$$\text{Beam} = 0.85 \times 101.9 = +86.6 \text{ K·ft}$$

$$\text{Column strip} = 0.15 \times 101.9 = +15.3 \text{ K·ft}$$

$$\text{Middle strip} = +25.5 \text{ K·ft}$$

Moment details are shown in Fig. 17.23.

6. Calculate the design moment in the short direction: span = 20 ft. The procedure is similar to step 5.

$$\text{Negative moment } (M_n) = 0.65M_{os} = 0.65 \times 294.3 = -191.3 \text{ K·ft}$$

$$\text{Positive moment } (M_p) = 0.35M_{os} = 0.35 \times 294.3 = +103.0 \text{ K·ft}$$

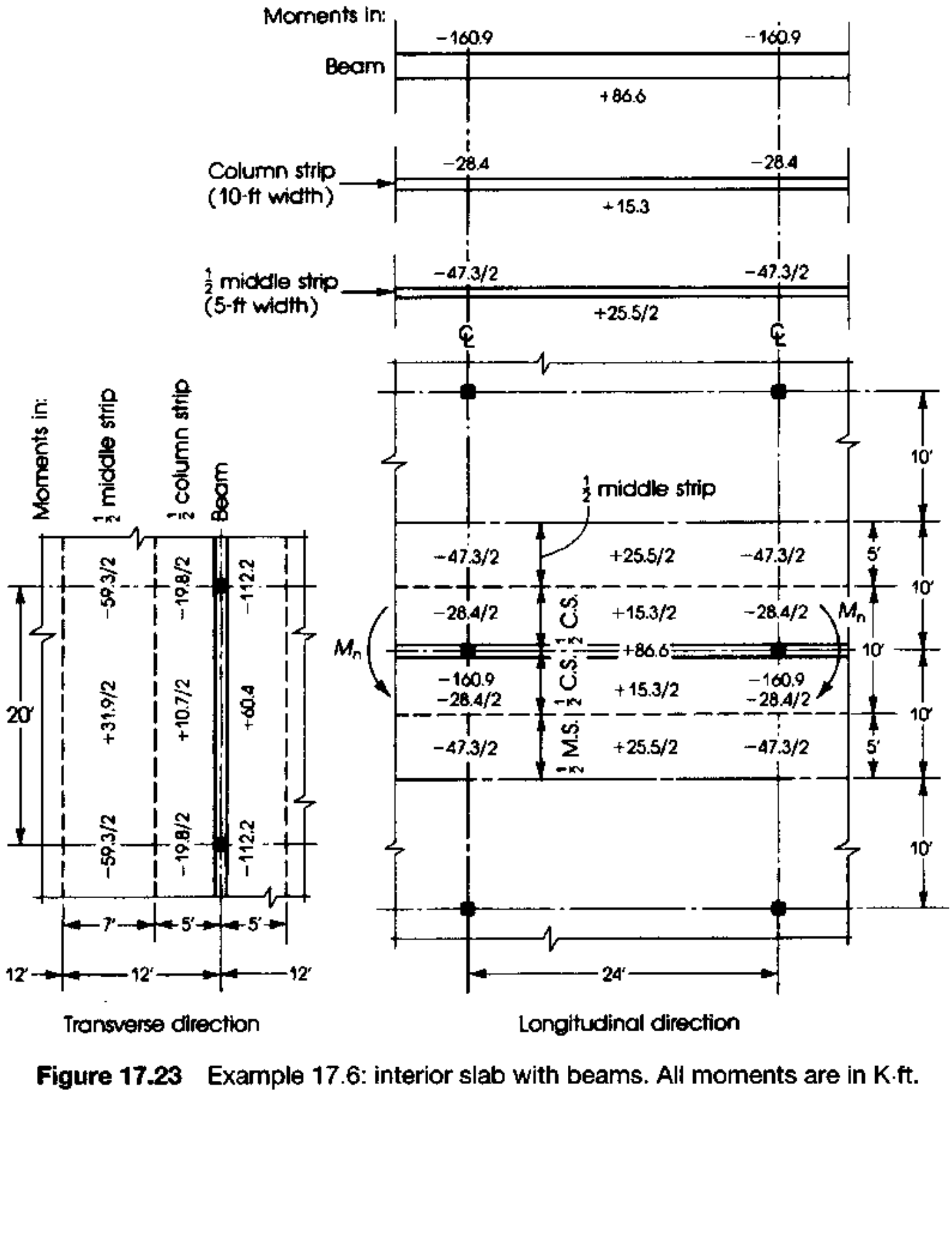


Figure 17.23 Example 17.6: interior slab with beams. All moments are in K·ft.

Distribution of M_n and M_p to beam, column, and middle strips:

$$\frac{l_2}{l_1} = \frac{24}{20} = 1.2 \quad \alpha_{f_1} = \alpha_s = \frac{EI_b}{EI_s} = 2.72 \quad \text{(from Example 17.2)}$$

$$\alpha_{f_1}\frac{l_2}{l_1} = 2.72 \times 1.2 = 3.26 > 1.0$$

The percentages of the column strip negative and positive moments are obtained from Table 17.3 by interpolation. (For $l_2/l_1 = 1.12$ and $\alpha_{f_1}(l_2/l_1) > 1.0$, the percentage is 69%.)

$$\text{Column strip negative moment} = 0.69M_n = 0.69 \times 191.3 = -132 \text{ K·ft}$$

$$\text{Middle strip negative moment} = 0.31M_n = 0.31 \times 191.3 = -59.3 \text{ K·ft}$$

Since $\alpha_{f_1}(l_2/l_1) > 1.0$, 85% of − 132 K·ft is assigned to the beam. Therefore,

$$\text{Beam negative moment} = 0.85 \times 132 = -112.2 \text{ K·ft}$$

$$\text{Column strip negative moment} = 0.15 \times 132 = -19.8 \text{ K·ft}$$

$$\text{Beam positive moment} = (0.85)(0.69 \times 103.0) = +60.4 \text{ K·ft}$$

$$\text{Column strip positive moment} = (0.15)(0.69 \times 103.0) = +10.7 \text{ K·ft}$$

$$\text{Middle strip positive moment} = (1 - 0.169)(103.10) = \pm 31.19 \text{ K·ft}$$

7. The steel reinforcement required and number of bars are shown in Table 17.10. Note all steel ratios are less than $\rho_{max} = 0.0135$. Thus, $\phi = 0.9$.

Example 17.7

Using the direct design method, determine the negative and positive moments required for the design of the exterior panel (no. 2) of the two-way slab system with beams shown in Fig. 17.7. Use the loads and the data given in Example 17.6.

Solution

1. Limitations required by the ACI Code are satisfied in this problem. Determine the minimum slab thickness, h_s, using Eqs. 17.1 and 17.2 and the following steps: Assume $h_s = 7.10$ in. The sections of the interior and exterior beams are shown in Fig. 17.7. Note that the extension of the slab on each side of the beam $x = y = 15$ in.
2. **a.** The moments of inertia for the *interior* beams and slabs were calculated earlier in Example 17.2:

$$I_b(\text{in both directions}) = 22{,}453 \text{ in.}^4$$

$$I_s(\text{in the long direction}) = 6860 \text{ in.}^4$$

$$I_s(\text{in the short direction}) = 8232 \text{ in.}^4$$

b. Calculate I_b and I_s for the *edge* beam and end slab.

$$I_b(\text{edge beam}) = \left[\frac{27}{12}(7)^3 + (27 \times 7)(5.37)^2\right] + \left[\frac{12}{12}(15)^3 + (12 \times 15)(5.63)^2\right]$$

$$= 15{,}302 \text{ in.}^4$$

Calculate I_s for the end strip parallel to the edge beam, which has a width $= \frac{24}{2}$ ft $+ \frac{1}{2}$ column width $= 12 + \frac{10}{12} = 12.83$ ft.

$$I_s(\text{end slab}) = \frac{(12.83 \times 2)}{12}(7)^3 = 4401 \text{ in.}^4$$

Table 17.10 Design of an Interior Two-Way Slab with Beams

	Long Direction			
	Column Strip		Middle Strip	
M_u (K·ft)	−28.14	±15.13	−47.13	±25.15
Width of strip (in.)	120	120	120	120
Effective depth (in.)	6.0	6.0	6.0	6.0
$R_u = \frac{M_u}{bd^2}$ (psi)	79	43	132	71
Steel ratio ρ	0.0016	Low	0.0026	0.0015
$A_s = \rho bd$ (in.^{2})	1.15	Low	1.87	1.08
Min. $A_s = 0.10018bh_s$ (in.^{2})	1.52	1.52	1.52	1.52
Selected bars	8 no. 4	8 no. 4	10 no. 4	8 no. 4

	Short Direction			
	Column Strip		Middle Strip	
M_u (K·ft)	−19.18	±10.17	−59.13	±31.19
Width of strip (in.)	120	120	168	168
Effective depth (in.)	5.5	5.5	5.5	5.5
$R_u = \frac{M_u}{bd^2}$ (psi)	65	35	196	105
Steel ratio ρ	Low	Low	0.0039	0.002
$A_s = \rho bd$ (in.^{2})	Low	Low	3.60	1.85
Min. $A_s = 0.0018bh_s$ (in.^{2})	1.52	1.52	2.10	2.10
Selected bars	8 no. 4	8 no. 4	18 no. 4	10 no. 4

3. a. Calculate $\alpha_f (\alpha_f = EI_b/EI_s)$:

$$\alpha_l(\text{long direction}) = \frac{22{,}453}{6860} = 3.27$$

$$\alpha_s(\text{short direction}) = \frac{22{,}453}{8232} = 2.72$$

$$\alpha(\text{edge beam}) = \frac{15{,}302}{4401} = 3.48$$

$$\text{Average}\alpha = \alpha_{fm} = \frac{3.27 + 2.72 \times 2 + 3.48}{4} = 3.05$$

b. β = ration of long to short clear span.

$$\frac{22.33}{18.33} = 1.22$$

c. Calculate h_s:

$$\text{Min. } h_s = \frac{(22.33 \times 12)(0.8 + 0.005 \times 60)}{36 + (5 \times 1.22)[3.05 - 0.2]} = 5.52 \text{ in.}$$

but this value must not be less than

$$\text{Min. } h_s = \frac{294.756}{36 + 9(1.22)} = 6.30 \text{ in.} \quad \text{(controls)}$$

Use $h_s = 7$ in. > 3.15 in. (minimum code limitations).

4. Calculate factored loads:

$$w_u = 292 \text{ psf} \quad \text{(from Example 17.6)}$$

5. Calculate total static moments:

$$M_{ol} = 364.0 \text{ K·ft} \qquad M_{os} = 294.3 \text{ K·ft} \qquad \text{(from previous example)}$$

6. Calculate the design moments in the short direction (span = 20 ft): Because the slab is continuous in this direction, the moments are the same as those calculated in Example 17.6 and shown in Fig. 17.23 for an interior panel.
7. Calculate the moments in one panel using the coefficients given in Table 17.2 or Fig. 17.14 (Case 3):

$$\text{Interior negative moment } (M_{nt}) = 0.7M_o = 0.7 \times 364 = -254.8 \text{ K·ft}$$

$$\text{Positive moment within span } (M_p) = 0.57M_o = 0.57 \times 364 = +207.5 \text{ K·ft}$$

$$\text{Exterior negtive moment } (M_{ne}) = 0.16M_o = 0.16 \times 364 = -58.2 \text{ K·ft}$$

Note: If the modified stiffness method is used, then $C = 9528$, $K_t = 1520E_c$, $K_c = 370E_c$, $K_b = 312E_c$, $K_s = 95E_c$, $K_{ec} = 498E_c$, and $\alpha_{ec} = 1.22$. The interior negative moment becomes −253.13 K·ft (same as before). The positive moment becomes −173.19 K·ft (16% decrease) and the exterior moment becomes −128.16 K·ft (220% increase).

8. Distribute the panel moments to beam, column, and middle strips:

$$\frac{l_2}{l_1} = \frac{20}{24} = 0.83 \quad \alpha_{f_1} = \alpha_s = 3.27$$

$$\alpha_{f_1}\frac{l_2}{l_1} = 3.27 \times 0.83 = 2.71 > 1.0$$

Calculate C:

$$C = \sum\left(1 - 0.63\frac{x}{y}\right)\frac{x^3 y}{3}$$

Divide the section of the edge beam into two rectangles in such a way as to obtain maximum C. Use for a beam section 12 by 22 in., $x_1 = 12$ in., $y_1 = 22$ in., and a slab section 7 by 15 in., $x_2 = 7$ in., and $y_2 = 15$ in.

$$C = \left(1 - 0.63 \times \frac{12}{22}\right)\left(\frac{12^3 \times 22}{3}\right) + \left(1 - 0.63 \times \frac{7}{15}\right)\left(\frac{7^3 \times 15}{3}\right)$$

$$= 9528 \text{ in.}^4$$

$$\beta_t = \frac{E_{ch}C}{2E_{cs}I_s} = \frac{9528}{2 \times 6860} = 0.69$$

a. Distribute the interior negative moment, M_{ni}: Referring to Table 17.5 and by interpolation, the percentage of moment assigned to the column strip (for $l_2/l_1 = 0.83$ and $\alpha_{f_1} l_2/l_1 > 1.0$ is 80%.

$$\text{Column strip} = 0.8 \times 254.8 = -203.8 \text{ K/ft}$$

$$\text{Middle strip} = 0.2 \times 254.8 = -51.0 \text{ K·ft}$$

Because $\alpha_{f1}\, l_2/l_1 > 1.0$, 85% of the moment in the column strip is assigned to the beam. Therefore,

$$\text{Beam} = 0.85 \times 203.8 = -173.3 \text{ K·ft}$$

$$\text{Column strip} = 0.15 \times 203.18 = -30.6 \text{ K·ft}$$

$$\text{Middle strip} = -51.0 \text{ K·ft}$$

b. Distribute the positive moment, M_p: Referring to Table 17.5 and by interpolation, the percentage of moment assigned to the column strip is 80% (85% of this value is assigned to the beam). Therefore,

$$\text{Beam} = (0.85)(0.8 \times 207.5) = +141.1 \text{ K·ft}$$

$$\text{Column strip} = (0.15)(0.8 \times 207.5) = 24.9 \text{ K·ft}$$

$$\text{Middle strip} = 0.2 \times 207.5 = +41.5 \text{ K·ft}$$

c. Distribute the exterior negative moment, M_{ne}: Referring to Table 17.5 and by interpolation, the percentage of moment assigned to the column strip (for $l_2/l_1 = 0.83$, $\alpha_{f1}\, l_2/l_1 > 1.0$, and $\beta_t = 0.69$) is 94%, and 85% of the moment is assigned to the beam. Therefore,

$$\text{Beam} = (0.85)(0.94 \times 58.2) = -46.5 \text{ K·ft}$$

$$\text{Column strip} = (0.15)(0.94 \times 58.2) = -8.2 \text{ K·ft}$$

$$\text{Middle strip} = 0.06 \times 58.2 = -3.5 \text{ K·ft}$$

17.9 DESIGN MOMENTS IN COLUMNS

When the analysis of the equivalent frames is carried out by the direct design method, the moments in columns due to the unbalanced loads on adjacent panels are obtained from the following equation, which is specified by the ACI Code, Section 13.6.9:

$$M_u = 0.07[(w_d + 0.5w_l)l_2 l_n^2 - w_d' l_2' (l_n')^2] \tag{17.22a}$$

If the modified stiffness method using K_{ec} and α_{ec} is used, then the moment M_u is computed as follows:

$$M_u = \frac{0.08[(w_d + 0.5w_l)l_2 l_n^2 - w_d' l_2' (l_n')^2]}{\left(1 + \dfrac{1}{\alpha_{ec}}\right)} \tag{17.22b}$$

where

w_d and w_l = factored dead and live loads on the longer span
w_d' = factored dead load on the shorter span
l_n and l_n' = length of the longer and shorter spans, respectively

$$\alpha_{ec} = \frac{K_{ec}}{\sum (K_s + K_b)} \tag{17.21}$$

The moment in Eq. 17.22 should be distributed between the columns above and below the slab at the joint in proportion to their flexural stiffnesses (Fig. 17.24). For equal spans $l_2 = l_2'$ and $l_n = l_n'$,

$$M_u = 0.07(0.5w_l l_2 l_n^2) \tag{17.23a}$$

$$M_u = \frac{0.08(0.5w_l l_2 l_n^2)}{\left(1 + \dfrac{1}{\alpha_{ec}}\right)} \tag{17.23b}$$

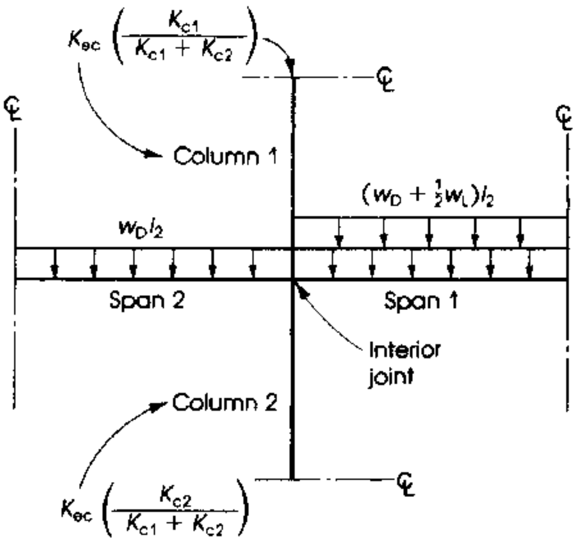


Figure 17.24 Interior column loading.

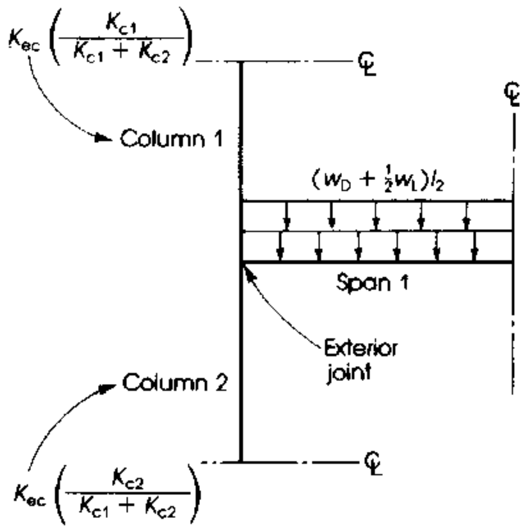


Figure 17.25 Exterior column loading.

The development of these equations is based on the assumption that half the live load acts on the longer span, whereas the dead load acts on both spans. Equation 17.22 can also be applied to an exterior column by assuming the shorter span length is 0 (Fig. 17.25).

17.10 TRANSFER OF UNBALANCED MOMENTS TO COLUMNS

17.10.1 Transfer of Moments

In the analysis of an equivalent frame in a building, moments develop at the slab-column joints due to lateral loads, such as wind, earthquakes, or unbalanced gravity loads, causing unequal moments in the slab on opposite sides of columns. A fraction of the unbalanced moment in the slabs must be transferred to the columns by flexure, and the balance must be transferred by vertical shear acting on the critical sections for punching shear. Approximately 60% of the moment transferred to both ends of the column at a joint is transferred by flexure, and the remaining 40% is transferred by eccentric shear (or torque) at the section located at $d/2$ from the face of the column [14,15]. The ACI Code, Section 13.5.3, states that the fraction of the unbalanced moment transferred by flexure M_f at a slab-column connection is determined as follows (ACI Eq. 13.1):

$$M_f = \gamma_f M_u \tag{17.24}$$

$$\gamma_f = \frac{1}{1 + \left(\frac{2}{3}\sqrt{\frac{c_1 + d}{c_2 + d}}\right)} = \frac{1}{1 + \left(\frac{2}{3}\right)\sqrt{\frac{b_1}{b_2}}} \tag{17.25}$$

and the moment transferred by shear is

$$M_v = (1 - \gamma_f)M_u = M_u - M_f \tag{17.26}$$

where c_1 and c_2 are the lengths of the two sides of a rectangular or equivalent rectangular column, $b_1 = (c_1 + d)$, and $b_2 = (c_2 + d)$. When $c_1 = c_2$, $M_f = 0.6M_u$, and $M_v = 0.4M_u$.

17.10.2 Concentration of Reinforcement Over the Column

For a direct transfer of moment to the column, it is necessary to concentrate part of the steel reinforcement in the column strip within a specified width over the column. The part of the moment transferred by flexure, M_f, is considered acting through a slab width equal to the transverse column width c_2 plus $1.15h_s$ on each side of the column or to the width $(c_2 + 3h_s)$ (ACI Code, Section 13.5.3). Reinforcement can be concentrated over the column by closer spacing of bars or the use of additional reinforcement.

17.10.3 Shear Stresses Due to M_v

The shear stresses produced by the portion of the unbalanced moment, M_v, must be combined with the shear stresses produced by the shearing force, V_u, due to vertical loads. Both shear stresses are assumed acting around a periphery plane located at a distance $d/2$ from the face of the column [16], as shown in Fig. 17.26. The equation for computing the shear stresses is

$$v_{1,2} = \frac{V_u}{A_c} \pm \frac{M_v C}{J_c} \tag{17.27}$$

where

A_c = area of critical section around the column

J_c = polar moment of inertia of the areas parallel to the applied moment in addition to that of the end area about the centroidal axis of the critical section

For an interior column,

$$A_c = 2d(x + y) \tag{17.28}$$

and

$$J_c = \frac{d}{2}\left(\frac{x^3}{3} + x^2 y\right) + \frac{xd^3}{6} \tag{17.29}$$

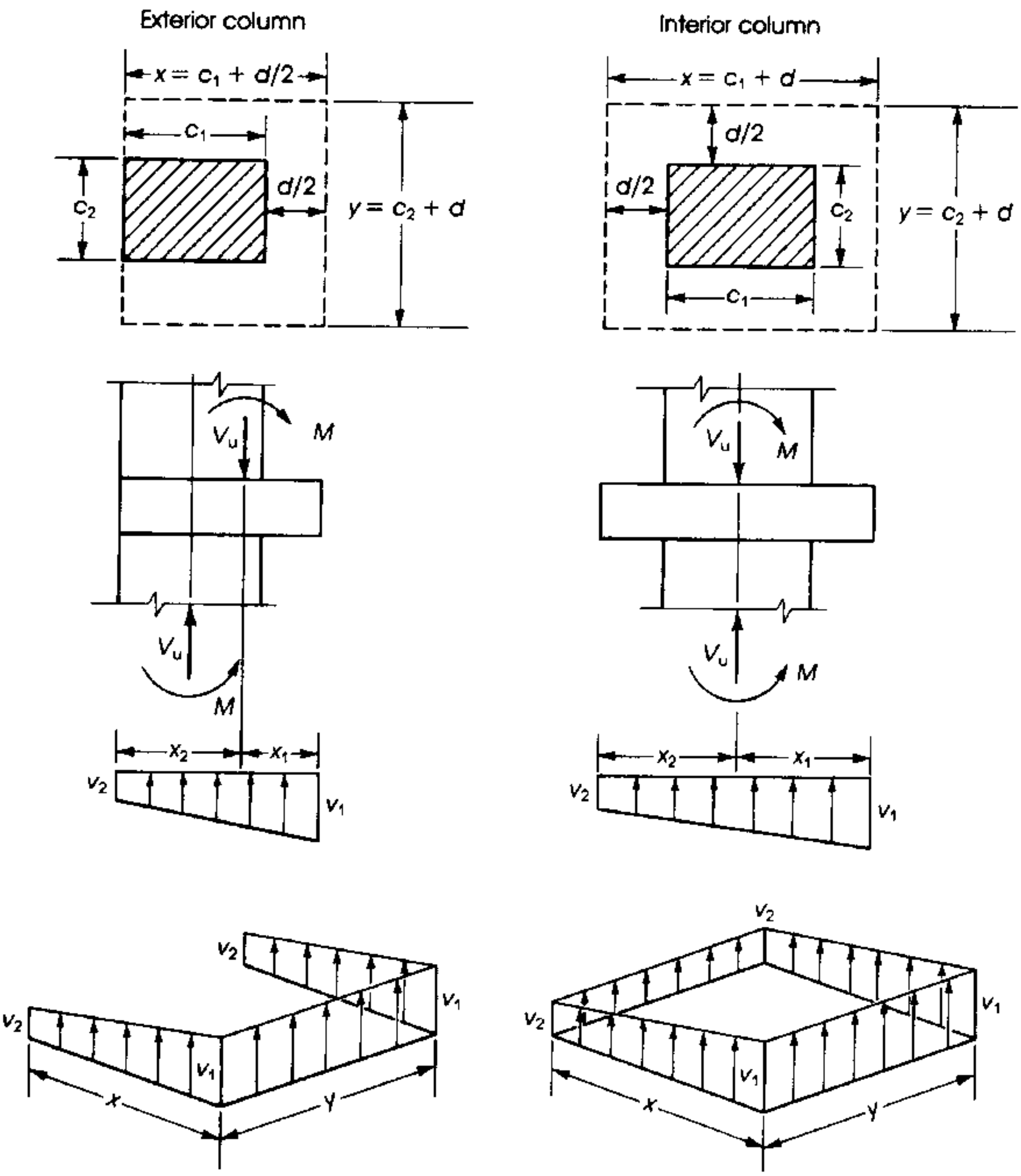


Figure 17.26 Shear stresses due to V_u and M.

For an exterior column,

$$A_c = d(2x + y) \tag{17.30}$$

and

$$J_c = \frac{2dx^3}{3} - (2x + y)dx_1^2 + \frac{xd^3}{6} \tag{17.31}$$

where x, x_1, and y are as shown in Fig. 17.26. The maximum shear stress, $v_1 = V_u/A_c + M_v C/J_c$, must be less than $\phi(4\sqrt{f_c'})$; otherwise, shear reinforcement should be provided.

Example 17.8

Determine the moments at the exterior and interior columns in the long direction of the flat plate in Example 17.4.

Solution

1. Find the exterior column moment. From Examples 17.4 and 17.5,

$$w_d = (136.5)(1.2) = 0.16 \text{ ksf}$$

$$0.5w_l = 0.5 \times (1.6 \times 100) = 80\text{psf}$$

$$l_2 = l_2' = 20 \text{ ft} \qquad l_n = l_n' = 22.33 \text{ ft} \qquad \left(1 + \frac{1}{\alpha_{ec}}\right) = 1.87$$

The unbalanced moment to be transferred to the exterior column using Eq. 17.22b is

$$M_u = \frac{0.08}{1.87}[(0.16 + 0.08)(20)(22.33)^2 - 0] = 102 \text{ K·ft}$$

If Eq. 17.22a is used, $M_u = 168$ K·ft, which is a conservative value.

2. At an interior support, the slab stiffness on both sides of the column must be used to compute α_{ec}:

$$\alpha_{ec} = \frac{K_{ec}}{\sum(K_s + K_b)} \tag{17.21}$$

From Example 17.5, $K_{ec} = 233E_c$, $K_s = 202.5E_c$, and $K_b = 0$. Therefore,

$$\alpha_{ec} = \frac{233E_c}{(2)202.5E_c} = 0.58$$

$$\left(1 + \frac{1}{\alpha_{ec}}\right) = 1 + \frac{1}{0.58} = 2.72$$

From Eq. 17.22b, the unbalanced moment at an interior support is

$$M_u = \frac{0.08}{2.72}[(0.16 + 0.08)(20)(22.33)^2 - 0.16(20)(22.33)^2] = 23 \text{ K·ft}$$

If Eq. 17.22a is used, $M_u = 42$ K·ft, which is a conservative value.

Example 17.9

For the flat plate in Example 17.4, calculate the shear stresses in the slab at the critical sections due to unbalanced moments and shearing forces at an interior and exterior column. Check the concentration of reinforcement and torsional requirements at the exterior column. Use $f_c' = 4$ ksi and $f_y = 60$ ksi.

Solution

1. The unbalanced moment at the interior support is $M_u = 20$ K·ft (Example 17.8), where $\gamma_f = 0.6$ (because $c_1 = c_2 = 20$ in.). The moment to be transferred by flexure is

$$M_f = \gamma_f M_u = 0.6 \times 23 = 13.8 \text{ K·ft}$$

The moment to be transferred by shear is

$$M_v = 23 - 13.8 = 9.2 \text{ K·ft}$$

Alternatively, moments calculated from Eq. 17.22a may be used producing higher shear stresses.

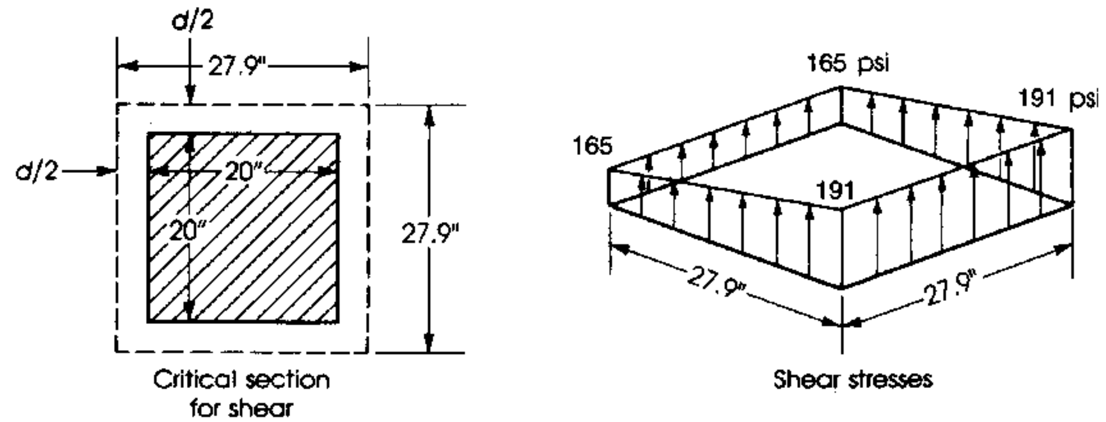


Figure 17.27 Example 17.9: shear stresses at interior column due to unbalanced moment.

Using $d = 7.9$ in. (Example 17.4),

$$V_u = 0.33\left[20 \times 24 - \left(\frac{27.9}{12}\right)^2\right] = 156.6 \text{ K}$$

From Fig. 17.27,

$$A_c = 4(27.9)(7.9) = 882 \text{ in.}$$

$$J_c = \frac{d}{2}\left(\frac{x^3}{3} + x^2 y\right) + \frac{xd^3}{6}$$

$$= \frac{7.9}{2}\left[\frac{(27.9)^3}{3} + (27.9)^2(27.9)\right] + \frac{27.9}{6}(7.9)^3 = 114{,}670 \text{ in.}^4$$

$$v_{\max} = \frac{156{,}600}{882} + \frac{9.2(12{,}000)(27.9/2)}{114{,}670}$$

$$= 178 + 13 = 191 \text{ psi}$$

$$v_{\min} = 178 - 13 = 165 \text{ psi}$$

Allowable v_c is $\phi 4\sqrt{f'_c} = 0.75 \times 4\sqrt{4000} = 190$ psi > 190 psi

2. For the exterior column, the unbalanced moment to be transferred by flexure M_f at a slabcolumn joint is equal to $\gamma_f M_u$, where $M_u = 102$ K·ft. Note that $c_1 = c_2 = 20$ in., $d = 7.9$ in. in the longitudinal direction, and $\gamma_f = 0.6$ for square columns.

$$M_f = 0.6(102) = 61.2 \text{ K·ft}$$

The moment to be transferred by shear is

$$M_v = M_u - M_f = 102 - 61.2 = 40.8 \text{ K·ft}$$

3. For transfer by shear at exterior column, the critical section is located at a distance $d/2$ from the face of the column (Fig. 17.28).

$$W_u = 330 \text{ psf}$$

$$V_u = 0.33\left(20 \times 12.83 - \frac{23.95}{12} \times \frac{27.9}{12}\right) = 83.1 \text{ K}$$

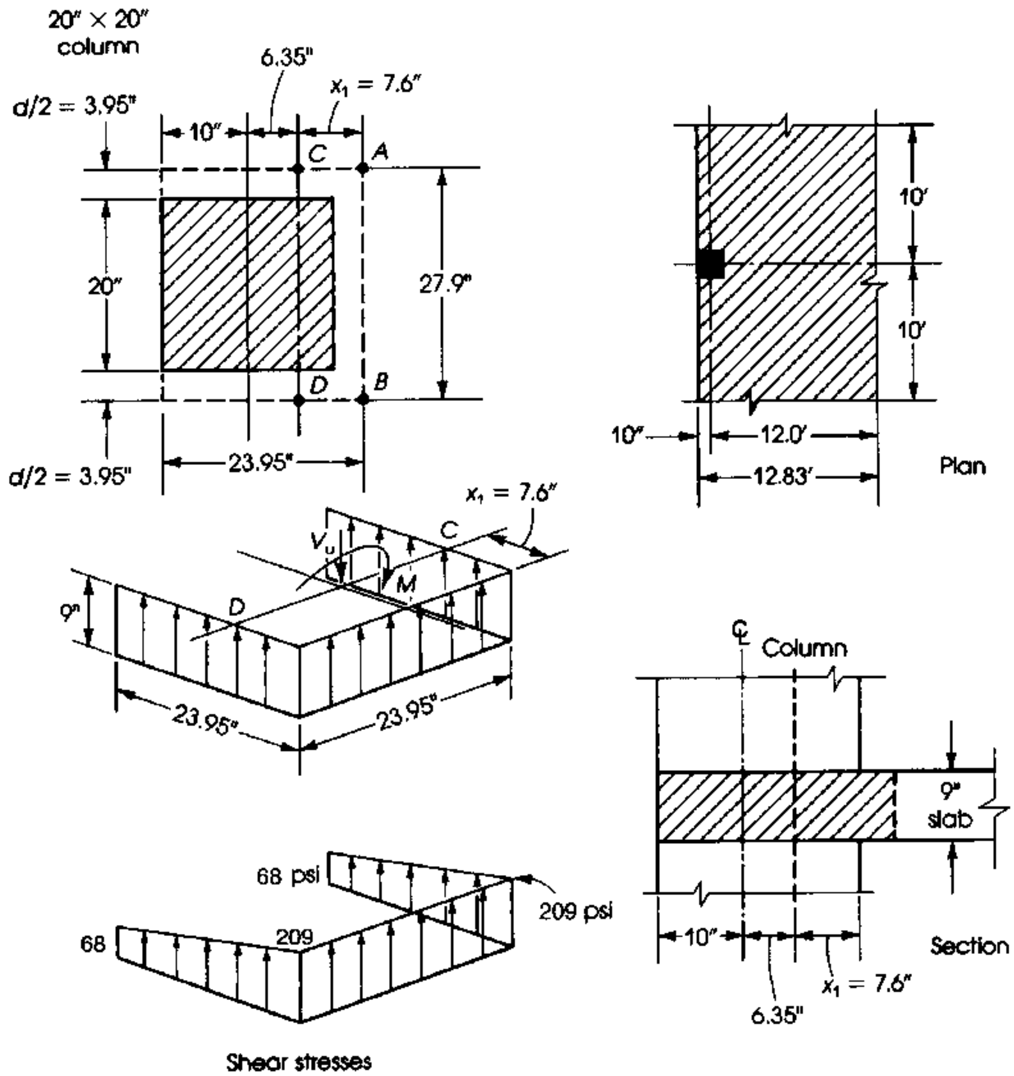


Figure 17.28 Example 17.9: shear stresses at exterior column due to unbalanced moment.

Locate the centroid of the critical section by taking moments about AB:

$$2\left(23.95 \times \frac{23.95}{2}\right) = (2 \times 23.95 + 27.9)x_l$$

Therefore, $x_l = 7.6$ in. The area of the critical section A_c is $2(23.95 \times 7.9) + (27.9 \times 7.9) = 599$ in.^{2} Calculate $J_c = I_x + I_y$ for the two equal areas (7.19×23.195) with sides parallel to the direction of moment and the area (7.19×27.19) perpendicular to the direction of moment, all about the axis through CD.

$$J_c = I_y + I_x = \sum\left(\frac{bh^3}{12} + Ax^2\right)$$

$$= 2\left[7.9\,\frac{(23.95)^3}{12} + (7.9 \times 23.95)\left(\frac{23.95}{2} - 7.6\right)^2\right]$$

$$+2\left[\frac{23.95}{12}\,(7.9)^3\right] + [(27.9 \times 7.9)(7.6)^2] = 52{,}760 \text{ in.}^4$$

or by using Eq. 17.31 for an exterior column. Calculate the maximum and minimum nominal shear stresses using Eq. 17.27:

$$v_{\max} = \frac{V_u}{A_c} + \frac{M_v c}{J_c} = \frac{83{,}100}{599} + \frac{40.2(12{,}000)(7.6)}{52{,}760} = 209 \text{ psi}$$

$$v_{\min} = 68 \text{ psi}$$

$$\text{Allowable } v_c = \phi 4\sqrt{f'_c} = 0.75 \times 4\sqrt{4000} = 190 \text{ psi.}$$

Shear stress is greater than the allowable v_c, so increase the slab thickness or use shear reinforcement.

4. Check the concentration of reinforcement at the exterior column; that is, check that the flexural capacity of the section is adequate to transfer the negative moment into the exterior column. The critical area of the slab extends $1.5h_s$ on either side of the column, giving an area $(20 + 3 \times 9) = 47$ in. wide and 9 in. deep. The total moment in the 120-in.-wide column strip is 107 K·ft, as calculated in Example 17.4 (step 5). The moment in a width, $c_2 + 3h_s = 47$ in., is equal to $107(\frac{47}{120}) = 41.9$ K·ft.

   If equal spacing in the column strip is used, then the additional reinforcement within the 47-in. width will be needed for a moment equal to $M_f - 41.9 = 66 - 41.9 = 24.1$ K·ft. The required $A_s = 0.73$ in.^{2} and four no. 4 bars ($A_s = 0.8$ in.^{2}) may be used. An alternative solution is to arrange the reinforcement within the column strip to increase the reinforcement within a width of 47 in. The amount of steel needed within this width should be enough to resist a moment of 0.6 times the negative moment in the column strip, or $0.6 \times 107 = 64.2$ K·ft.

$$A_s = \frac{M_u}{\phi f_y\left(d - \frac{a}{2}\right)} \quad \text{assume} \quad a = 1.0 \text{ in.}$$

$$A_s = \frac{64.2(12)}{0.9 \times 60(7.9 - 0.5)} = 1.93 \text{ in.}^2$$

$$\text{Check: } a = \frac{A_s f_y}{0.85 f'_c b} = \frac{1.93 \times 60}{0.85 \times 4 \times 47} = 0.73 \text{ in.}$$

Use 10 no. 4 bars within a width 47 in. divided equally at both sides from the center of the column (Fig. 17.29). Additional reinforcement of four no. 4 bars, as indicated before, provides a better solution.

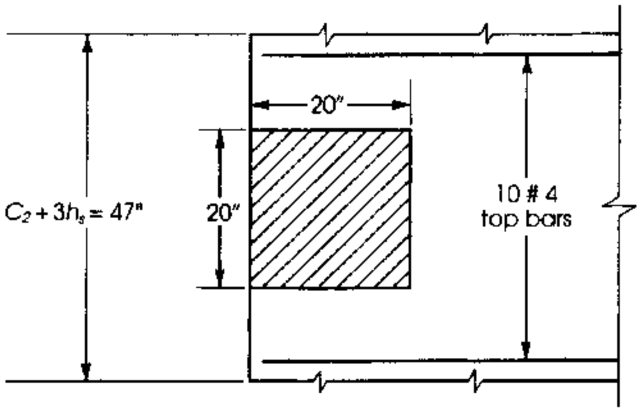


Figure 17.29 Example 17.9: concentration of reinforcement within exterior column strip.

5. Torque on slab: The torque from both sides of the exterior column is equal to 40% of the column strip moment.

$$T_u = 0.4(107) = 42.8 \text{ K·ft}$$

$$\text{Torque on each side}: \frac{42.8}{2} = 21.4 \text{ K·ft} = 257 \text{ K·in.}$$

A slab section of width equal to the column width will be assumed to resist the torsional stresses:

$$T_u = \frac{1}{3} v_{tu} \sum x^3 y$$

where $x = 9$ in. and $y = 20$ in. The critical section is at a distance d from the face of the column (Fig. 17.30). Assuming that the torque varies in a parabolic curve to the center of the slab, then the torque at a distance d is

$$T_u = 257 \left(\frac{140 - 7.9}{140}\right)^2 = 229 \text{ K·in.}$$

For torsional strength of concrete, $A_{cp} = 9 \times 20 = 180$ in., $P_{cp} = 2(9 + 20) = 58$ in. By Eq. 15.19, $\phi T_{cp} = 0.75(4)\sqrt{4000}(180)^2/58 = 106$ K·in. $T_a = 106/4 = 26.5$ K·in. $< T_u$.
Torsional reinforcement is needed. The required closed stirrups and the additional longitudinal bars are determined as explained in Chapter 15. The final section is shown in Fig. 17.30. It is advisable to provide an edge beam between the exterior columns to increase the torsional stiffness of the slab.

Example 17.10

Determine the shear reinforcement required for an interior flat plate panel considering the following: Punching shear is $V_u = 195$ K, slab thickness = 9 in., $d = 7.5$ in., $f'_c = 4$ ksi, $f_y = 60$ ksi, and column size is 20 × 20 in.

Solution

1. Determine $\phi V_c = \phi 4\sqrt{f'_c} b_o d$ for two-way shear.

$$b_o = 4(20 + d) = 4(20 + 7.5) = 110 \text{ in.}$$

$$\phi V_c = 0.75(4)\sqrt{4000}(110)(7.5) = 156.3 \text{ K}$$

Because $V_u = 195$ K $> \phi V_c$, shear reinforcement is required.

2. Maximum allowable ϕV_n using shear reinforcement is equal to $\phi 6\sqrt{f'_c} b_o d = 1.5(\phi V_c) =$ 234.5 K Because $\phi V_n > V_n$, shear reinforcement can be used.

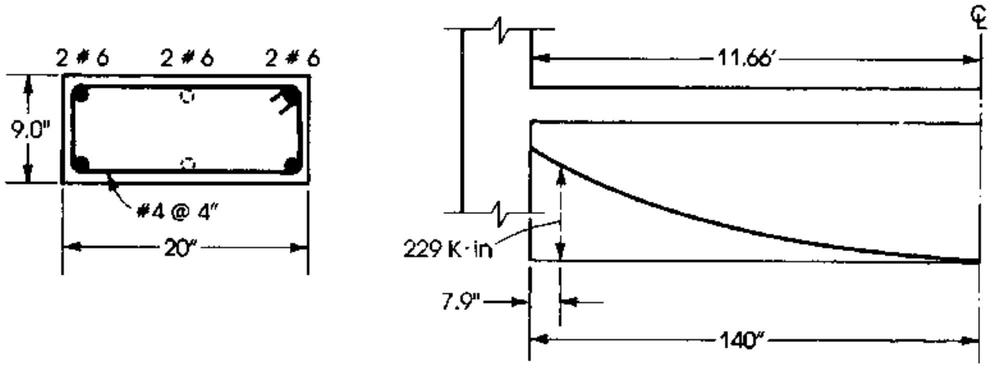


Figure 17.30 Example 17.9: reinforcement in edge of slab to resist torque.

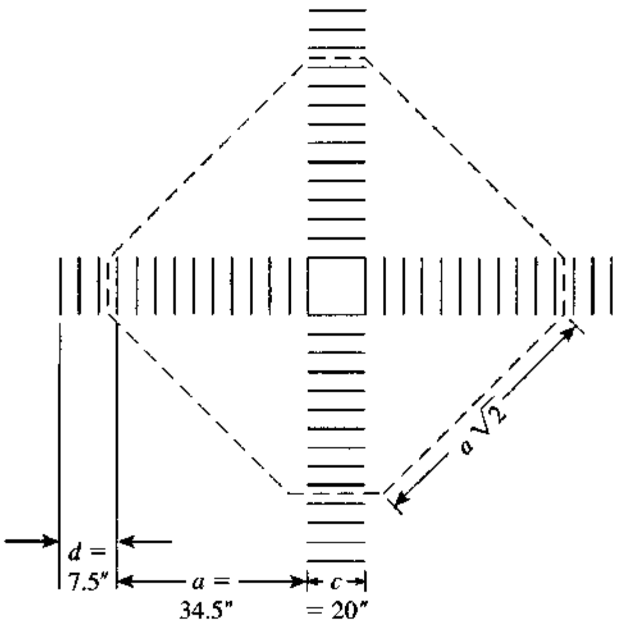


Figure 17.31 Example 17.10: Shear reinforcement no. 3 at 3.5 in.

3. Shear reinforcement may consist of reinforcing bars, structural steel sections such as I-beams, or special large-head studs welded to a steel strip. In this example, an inexpensive solution using normal shear reinforcement will be adopted. See Fig. 17.9f. Shear reinforcement must be provided on the four sides of the interior column (or three sides of an exterior column) for a distance of $d + a$. See Fig. 17.31. The distance a is determined by equating $\phi\ V_c = V_u$ at section b_o, indicated by the dashed line, and assuming $\phi V_c = \phi 2\lambda\sqrt{f_c'}b_o d$.

$$b_o = 4(c + \sqrt{2}a) = 4(20 + \sqrt{2}a)$$

$$0.75(2)\sqrt{4000}(4)(20 + \sqrt{2}a)(7.5) = 195{,}000 \text{ lb}$$

Here, $a = 34.3$ in., and $(a + d) = 34.3 + 7.5 = 41.8$ in., so use 42 in.

4. Calculate shear reinforcement:

$$\phi V_s = (V_u - \phi V_c) = 195 - 156.3 = 38.7 \qquad V_s = 51.6 \text{ K}$$

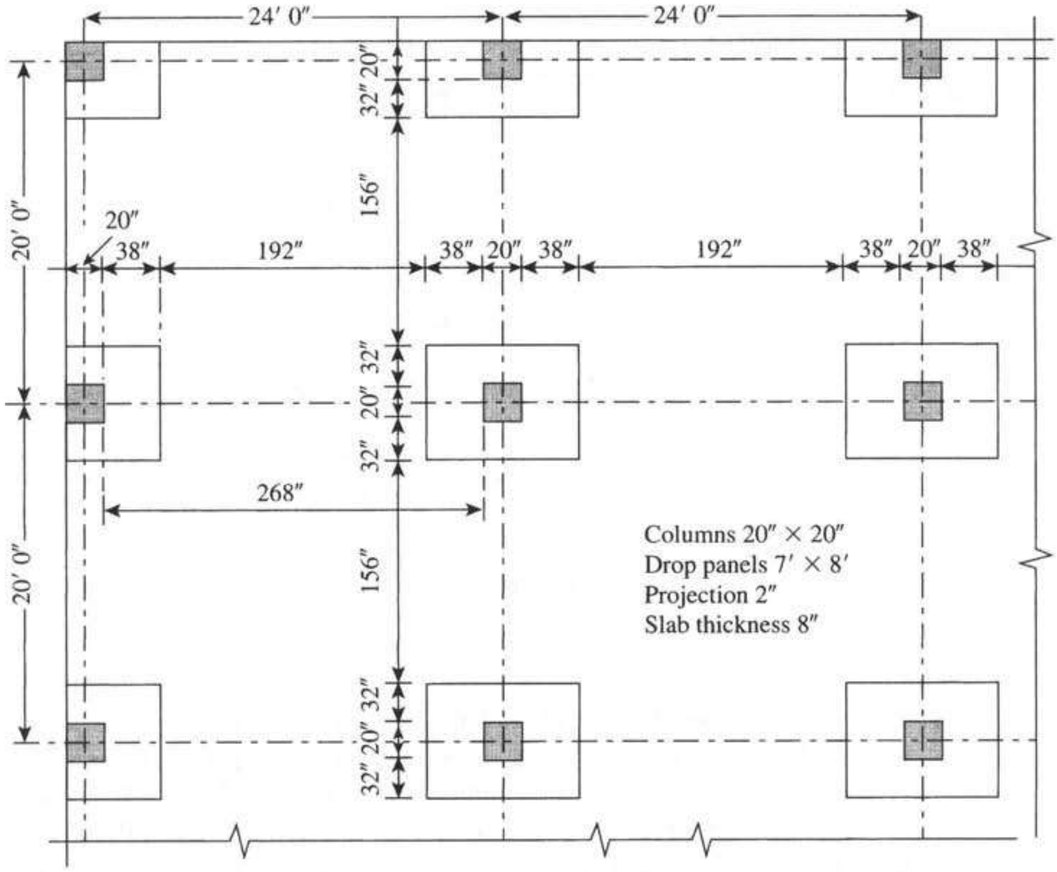
$$V_s(\text{for one face of critical section}) = \frac{V_s}{4} = \frac{51.6}{4} = 12.9 \text{ K}$$

Use no. 3 U-stirrups, $A_v = 0.22$ in.^{2} (for two legs). The spacing is $S = A_v f_y d / V_s = 0.22(60)(7.5)/12.9 = 7.7$ in. Maximum spacing is $d/2 = 7.5/2 = 3.75$ in.; let $s = 3.5$ in.

5. Distribution of stirrups: The number of stirrups per one side of column is $43/3.5 = 12.3$, or 13 stirrups. Total distance is $13(3.5) = 45.5$ in. (Fig. 17.31).

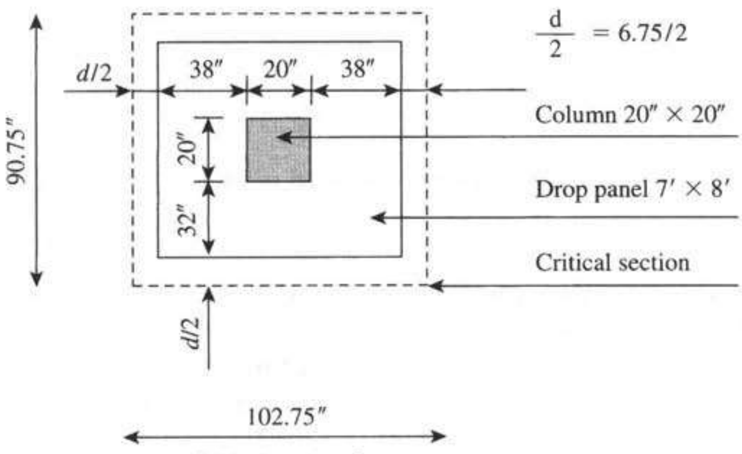
Example 17.11: Flat-Slab Floor System

Using the direct design method, design a typical 24 × 20-ft interior flat-slab panel with drop panels only (Fig. 17.32). All panels are supported by 20 × 20-in. columns, 12 ft long. The slab carries a uniform service live load of 100 psf and a service dead load of 24 psf, excluding self-weight. Use $f_c' = 4$ ksi, and $f_y = 60$ ksi. (The solution is similar to Example 17.3.)



Plan view of the flat slab

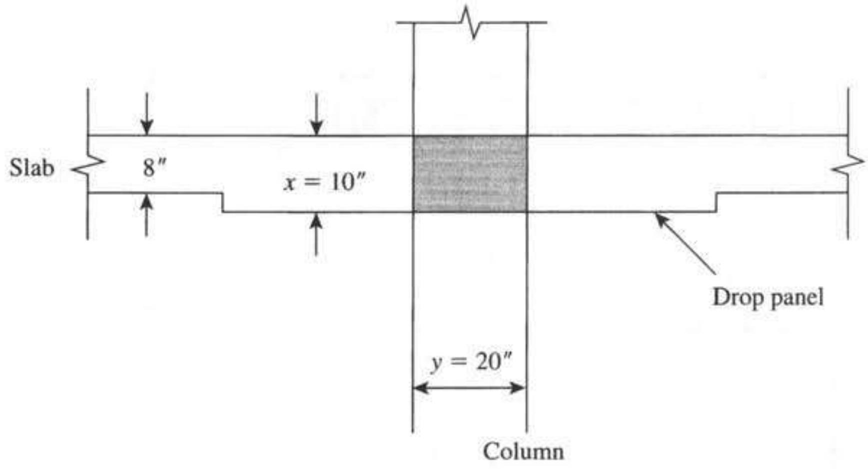
(a)



Critical section for two-way shear in slab

(b)

Figure 17.32 Example 17.11: flat slab with drop panel.



Torsional part
of the drop panel

(c)

Figure 17.32 (*continued*)

Solution

1. Determine slab and drop panel thicknesses using Table 17.1.
   a. The clear span is $24 - \frac{20}{12} = 22.33$ ft. For an exterior panel, minimum $h = l_n/33 = 8.12$ in., whereas for an interior panel, minimum $h = l_n/36 = 7.44$ in. Use a slab thickness of 8 in. The projection below the slab is $h/4 = \frac{8}{4} = 2.0$ in.; thus, the drop panel thickness is 10 in.
   b. Extend the drop panels $L/6 = \frac{24}{6} = 4$ ft in each direction from the centerline of support in the long direction and $\frac{20}{6} = 3.33$ ft, or 3.5 ft, in the short direction. Thus, the total size of one drop panel is 8×7 ft (Fig. 17.32).
2. Calculate factored loads:

$$\text{Slab load} = 24 + \frac{8(150)}{12} = 124 \text{ psf}$$

$$W_u = 1.2(124) + 1.6(80) = 277 \text{ psf}$$

$$\text{Drop panel load} = 24 + \frac{10(150)}{12} = 149 \text{ psf}$$

$$W_u = 1.2(149) + 1.6(80) = 307 \text{ psf}$$

Because the drop panel length is $L/3$ in each direction, the average W_u is $\left(\frac{2}{3}\right)(277) + \left(\frac{1}{3}\right)(307) = 287$ psf.

3. Check two-way shear (at distance $d/2$ from the face of column):
   a. In the drop panel: $d = 10 - 0.75 - 0.5 = 8.75$ in.

$$b_o = 4(20 + 8.75) = 115 \text{ in.}$$

$$V_u = 0.287\left[24 \times 20 - \left(\frac{28.75}{12}\right)^2\right] = 136.1 \text{ K}$$

$$\phi V_c = \phi 4\sqrt{f'_c}b_o d = 0.85(4)\sqrt{4000}(115)(8.75) = 214.4 \text{ K} > V_u$$

b. In the slab: $d = 8 - 0.75 - 0.5 = 6.75$ in. and b_o is measured at 6.175/2 in. (in slab) beyond the drop panel.

$$b_o = 2(8 \times 12 + 6.75) + 2(7 \times 12 + 6.75) = 387 \text{ in.}$$

$$V_u = 0.287[24 \times 20 - (102.75)(90.75)/144] = 119.2 \text{ K}$$

$$\phi V_c = 0.75(4)\sqrt{4000}(387)(6.75) = 495.6 \text{ K} > V_u$$

c. One-way shear is not critical.

4. Calculate the total static moments in the long and short directions:

$$M_{\text{ol}} = \frac{0.287(20)(22.33)^2}{8} = 357.8 \text{ K·ft}$$

$$M_{\text{os}} = \frac{0.287(24)(18.33)^2}{8} = 289.3 \text{ K·ft}$$

The width of column strip in each direction is $\frac{20}{2} = 10$ ft, whereas the width of the middle strip is 10 ft in the long direction and 14 ft in the short direction.

5. Calculations of moments and steel reinforcement are shown in Table 17.11. Use an average $d = 10 - 1.5 = 8.5$ in. in the column strip and $d = 8 - 1.5 = 6.5$ in. in the middle strip.

Bars are chosen for adequate distribution in both the column and middle strip. Reinforcement details are similar to those in flat-plate examples.

Table 17.11 Design of an Interior Flat-Slab Floor System

	Long Direction			
M_{ol} **= 358 K·ft**	**Column Strip**		**Middle Strip**	
M factor	$-0.49M_o$	$0.21M_o$	$-0.16M_o$	$0.14M_o$
M_u (K·ft)	−175.4	±75.2	−57.3	±50.1
Width of strip (in.), b	120	120	120	120
Effective depth (in.), d	8.5	6.5	6.5	6.5
$R_u = \frac{M_u}{bd^2}$ (psi)	243	178	129	119
Steel ratio ρ (%)	0.48	0.34	0.25	0.23
$A_s = \rho bd$ (in.^{2})	4.9	2.65	1.95	1.79
Min. $A_s = 0.0018bh_s$ (in.^{2})	2.16	2.16	1.73	1.73
Selected bars	16 no. 5	14 no. 4	10 no. 4	9 no. 4
	Short Direction			
M_{os} **= 289.3 K·ft**	**Column Strip**		**Middle Strip**	
M factor	$-0.49M_o$	$0.21M_o$	$-0.16M_o$	$0.14M_o$
M_u (K·ft)	−142	±60.8	−46.3	±40.5
Width of strip (in.), b	120	120	168	168
Effective depth (in.), d	8.5	6.5	6.5	6.5
$R_u = \frac{M_u}{bd^2}$ (psi)	196	144	78	68
Steel ratio ρ (%)	0.38	0.28	0.15	0.13
$A_s = \rho bd$ (in.^{2})	3.9	2.2	1.64	1.42
Min. $A_s = 0.0018bh_s$ (in.^{2})	2.16	2.16	2.42	2.42
Selected bars	13 no. 5	11 no. 4	12 no. 4	12 no. 4

17.11 WAFFLE SLABS

A two-way waffle slab system consists of concrete ribs that normally intersect at right angles. These slabs might be constructed without beams, in which case a solid column head is made over the column to prevent any punching due to shear. Wide beams can also be used on the column centerlines for uniform depth construction. Square metal or fiberglass pans are commonly used to form these joists. A thin slab of 3 to 5 in. is cast with these joists to form the waffle slab.

Each panel is divided into a column and a middle strip. The column strip includes all joists that frame into the solid head; the middle strip is located between consecutive column strips. Straight or bent bars could be used as a reinforcement in a waffle slab. The design of a two-way waffle slab is similar to that of flat slabs by considering the solid head as a drop panel. To prevent any excess in the diagonal tension in the head, a sufficient size of column must be used or a shear cap must be provided.

In the design of a waffle slab, the top slabs with each rib form a T-section, with considerable depth relative to flat plates. Consequently, long spans carrying heavy loads may be designed with great savings in concrete. Waffle slabs also provide an attractive ceiling, which is achieved by leaving the rib pattern or by integrating lighting fixtures. The standard pans that are commonly used in waffle slabs can be one of the following two types:

1. 30 × 30-in. square pans with a 3-in. top slab, from which 6-in.-wide ribs at 36 in. (3 ft) on centers are formed. These are available in standard depths of 8 to 20 in. in 2-in. increments. Refer to Example 17.12 and Fig. 17.33.
2. 19 × 19-in. square pans with a 3-in. top slab, from which 5-in.-wide ribs at 24 in. (2 ft) on centers are formed. These are available in standard depths of 4, 6, 8, 10, and 12 in. Other information about pans is shown in Table 17.12 [17].Other types, ranging from 19 × 19-in. pans to 40 × 40-in. pans, are available in the construction industry.

Example 17.12: Waffle Slab

Design a waffle floor system that consists of square panels without beams considering the following data (Fig. 17.33):

Span, center to center of columns = 33 ft

Width of rib = 6 in., spaced at 36 in. on centers

Depth of rib = 14in. and slab thickness = 3 in.

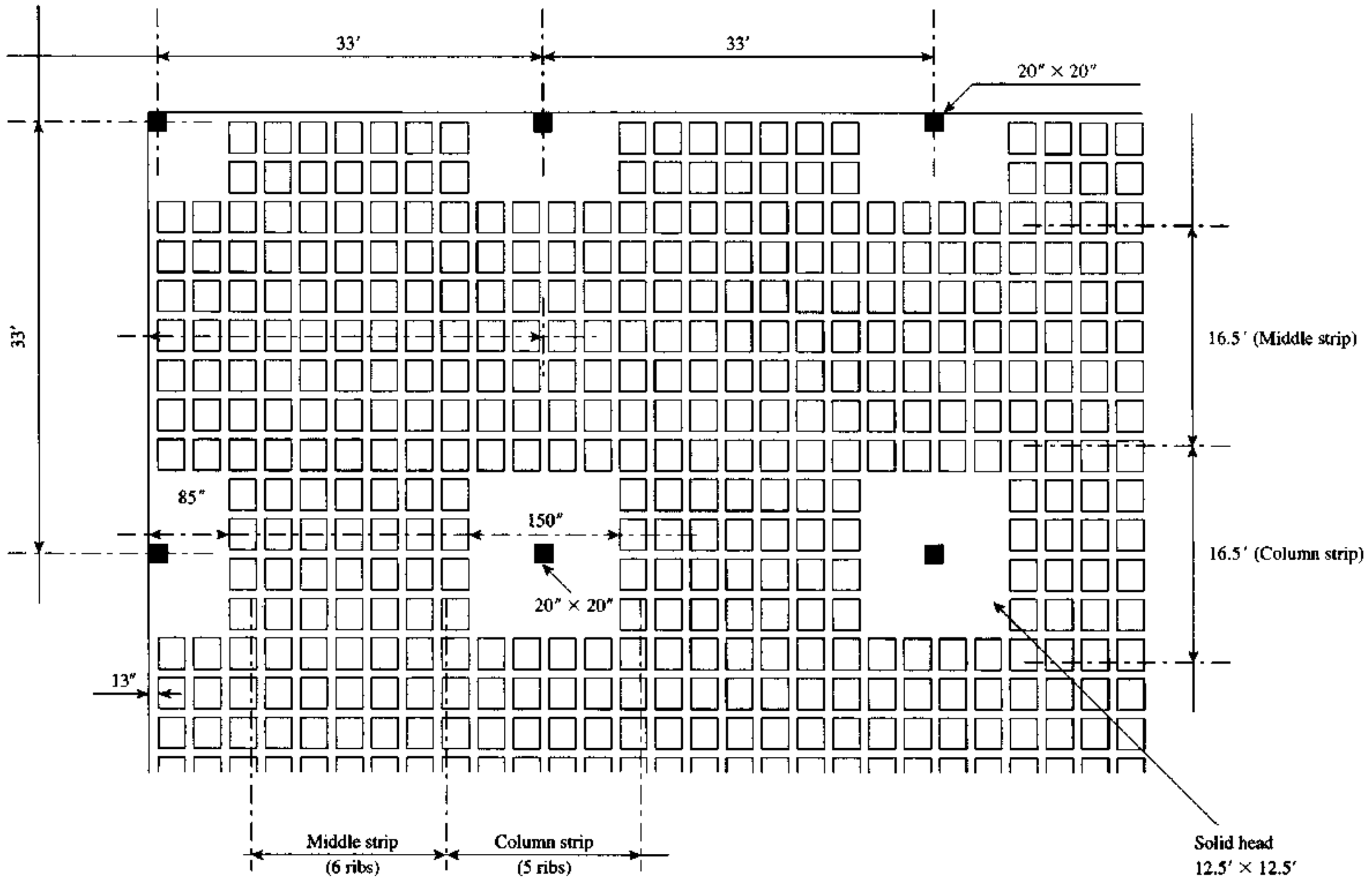
Column size = 20 × 20 in.

Dead load (excluding self-weight) = 50 psf

Live load = 100 psf $f'_c = 5$ ksi $f_y = 60$ ksi

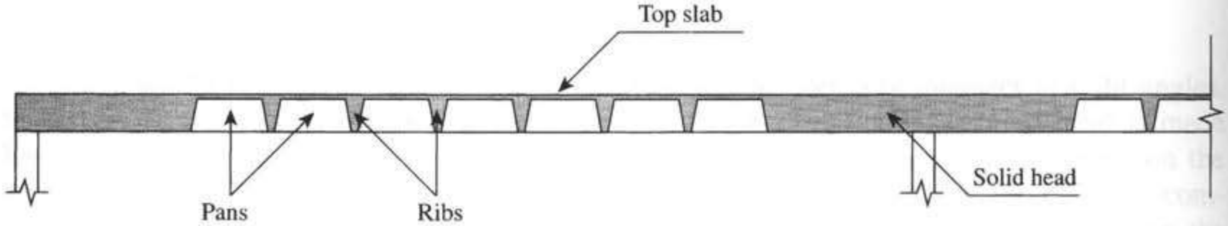
Solution

1. Determine minimum slab thickness using Table 17.1: Minimum $h = l_n/30$, $l_n = 33 - \frac{20}{12} = 31.33$ ft, $h = 31.33(12)/30 = 12.5$ in. for exterior panels, and $h = l_n/33 = 11.4$ in. for interior

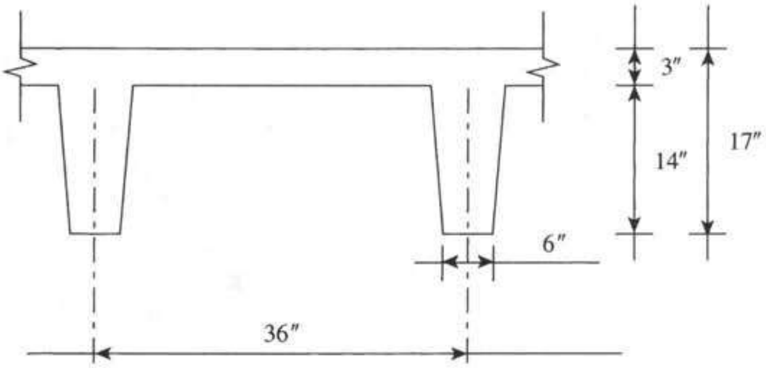


(a)

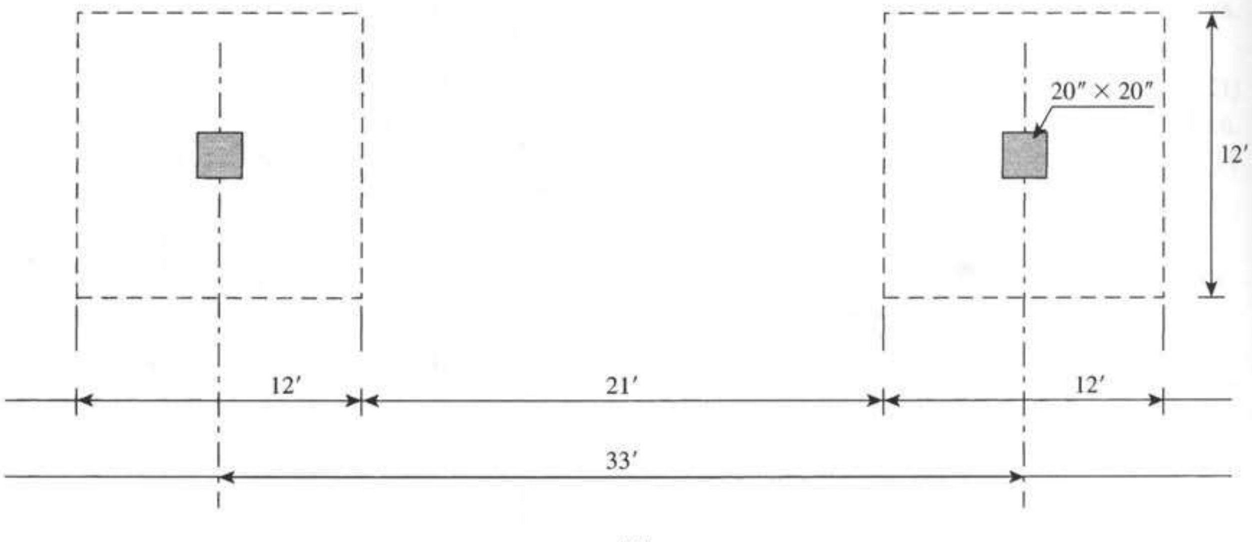
Figure 17.33 (*a*) Plan of the waffle slab, (*b*) cross section, (*c*) pan and rib dimensions, and (*d*) spacing and dimensions of solid heads (Example 17.12).



(b)



(c)



(d)

Figure 17.33 *(continued)*

panels. Equations 17.1 and 17.2 may be used. Assume the total depth is 17 in. consisting of 3-in. slab thickness and 14-in. rib depth.

2. Calculate loads on the waffle slab:

a. Factored load of solid head part = 1.2(150)(17/12) = 255 psf.

b. Voided volume of 14-in. rib = 6.54 ft^3 on 3 × 3-ft^2 area. Total weight of 9-ft^2 area is $1.2(150)(9 \times \frac{17}{12} - 6.54) = 1118$ lb. Weight per square foot is $\frac{1118}{9} = 125$ psf.

Table 17.12 Gross Section Properties [17]

Top Slab (in.)	Rib Depth (in.)	Volume (cf/pan)	Gross Area (in.^{2})	Y_{cg} (in.)	I_g (in.^{4})
For the Joists (30 x 30-in. pans)					
3	8	3.85	161.3	3.28	1393
3	10	4.78	176.3	3.95	2307
3	12	5.53	192	4.66	3541
3	14	6.54	208.3	5.42	5135
3	16	7.44	223.3	6.20	7127
3	20	9.16	261.3	7.83	12,469
4.5	8	3.85	215.3	3.77	2058
4.5	10	4.78	230.3	4.35	3227
4.5	12	5.53	246.0	4.97	4783
4.5	14	6.54	262.3	5.66	6773
4.5	16	7.44	279.3	6.36	9238
4.5	20	9.16	315.3	7.86	15,768
For the Joists (19 x 19-in. pans)					
3	6	1.09	105	2.886	598
3	8	1.41	117.4	3.564	1098
3	10	1.9	130.4	4.303	1824
3	12	2.14	144	5.083	2807
4.5	6	1.09	141	3.457	957
4.5	8	1.41	153	4.051	1618
4.5	10	1.9	166.4	4.709	2550
4.5	12	2.14	180	5.417	3794

c. Factored additional dead plus live load is 1.2(50) + 1.6(100) = 220 psf. Uniform w_u (at solid head) = 255 + 220 ≈ 500 psf. Uniform w_u (at ribbed area) = 125 + 220 = 345 psf.

d. Loads on one panel (refer to Fig. 17.34): At the solid head, $W = 0.5(12) + 0.345(21) = 13.22$ K/ft. At the ribbed area, $W = 0.345(33) = 11.39$ K/ft.

3. Calculate shear and total static moment:

$$V_u \text{ (at the face of column)} = 13.22(5.17) + \frac{(11.39)(21)}{2} = 188 \text{ K}$$

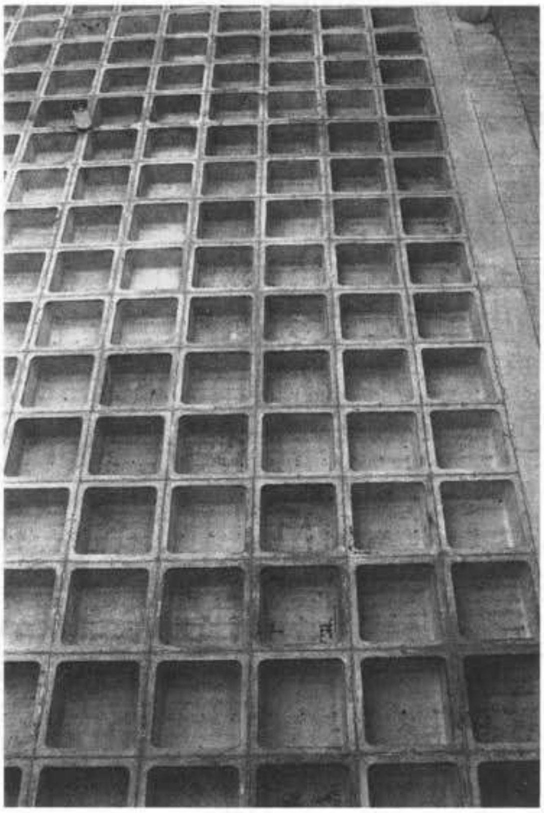
$$M_o \text{ (at midspan)} = 188(15.67) - 13.22(5.17)(13.09) - \frac{11.39(10.5)^2}{2} = 1424 \text{ K·ft}$$

4. Check punching shear (refer to Fig. 17.35):

a. In solid head at $d/2$ from column face, $h = 17$ in., $d = 17 - 1.25 = 15.75$ in., c (column) = 20 in., $b_o = 4(20 + 15.75) = 143$ in., $V_u = 11.39(21 \text{ ft}) + 13.22(12 \text{ ft}) - 0.5(37.75/12)^2 = 393.4$ K and $\phi V_c = \phi 4\sqrt{f'_c} b_o d = 0.75(4)(\sqrt{5000})(143)(15.75) = 478 \text{ K} > V_u$.

b. In the slab at distance $d/2$ from the edge of the solid head, slab thickness is 3 in.; let $d = 2.15$ in. Then

$$b_o = 4(150 + 2.5) = 610 \text{ in.}$$



Waffle slab (looking upward).

$$V_U = 11.39(21) + 13.22(12) - 0.5\left(\frac{152.5}{12}\right)^2 = 317.4 \text{ K}$$

$$\phi V_c = 0.75(4)(\sqrt{5000})(610)(2.5) = 324 \text{ K} > V_u$$

5. Design moments and reinforcement:

a. Exterior panel: $M_o = 1424$ K·ft

$$\text{Exterior negative moment} = 0.26M_o = -370 \text{ K·ft}$$

$$\text{Positive moment} = 0.52M_o = +740 \text{ K·ft}$$

$$\text{Interior negative moment} = 0.7M_o = -997 \text{ K·ft}$$

b. Interior panel: $M_o = 1424$ K·ft

$$\text{Negative moment} = 0.65(1424) = -925.6 \text{ K·ft}$$

$$\text{Positive moment} = 0.35(1424) = 498.4 \text{ K·ft}$$

Design details are shown in Table 17.13 and Fig. 17.36. Note that all steel ratios are low and $\phi = 0.9$.

6. Calculate the unbalanced moments in columns and check shear for V_u and M_v, as in Examples 17.8 and 17.9.

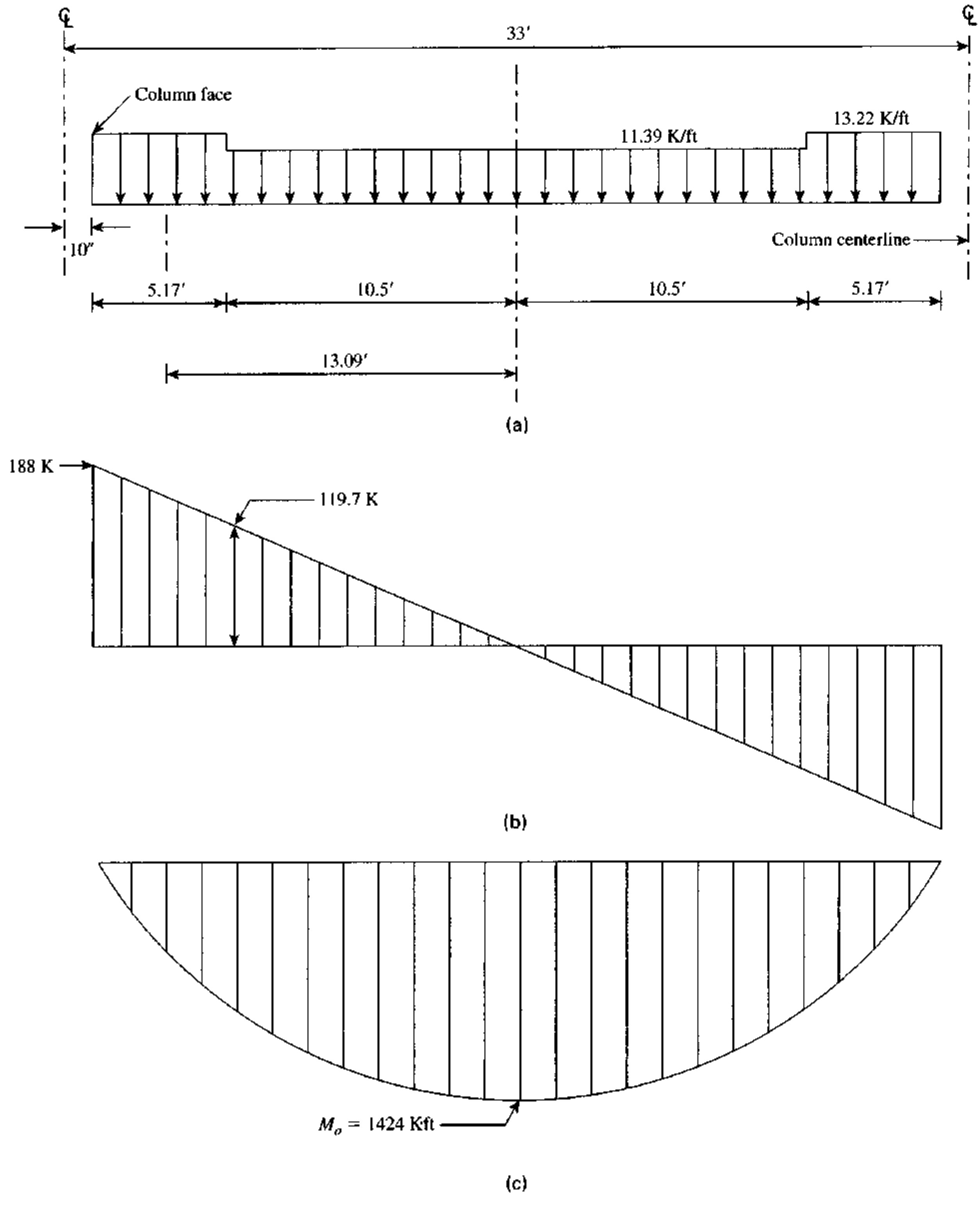
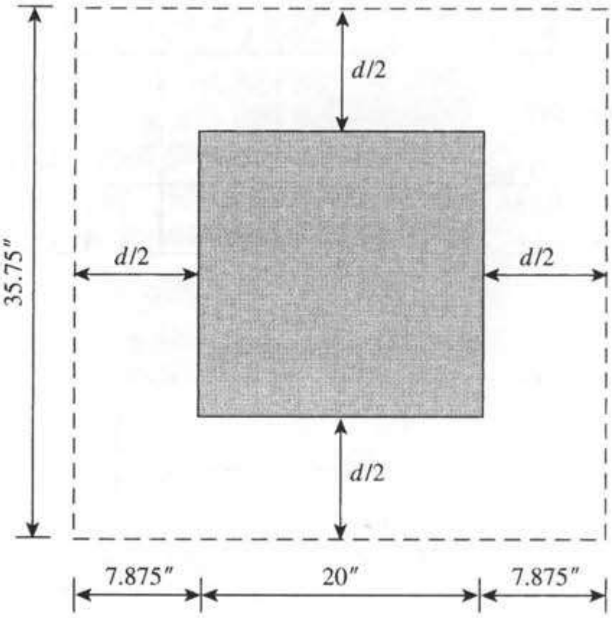
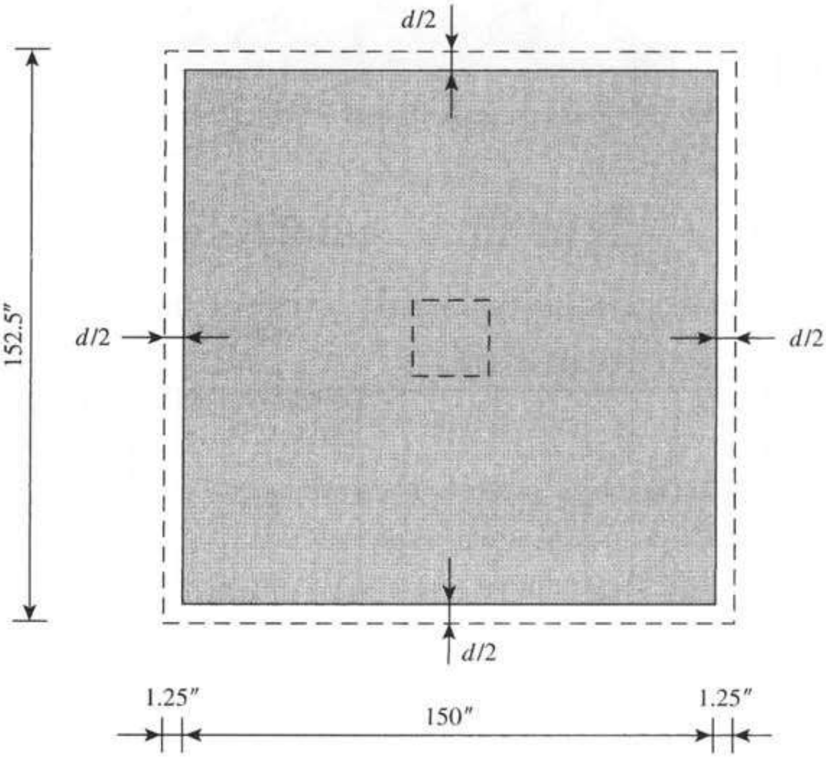


Figure 17.34 Load, shear, and moment diagrams: (*a*) load distribution on the span, (*b*) shear force diagram, and (*c*) bending moment diagram.



(a)



(b)

Figure 17.35 Punching shear locations: (*a*) punching shear in column head and (*b*) punching shear in slab.

Table 17.13 Design of an Exterior and an Interior Waffle Slab (5 Ribs in Column Strip and 6 Ribs in Middle Strips)

	Column Strip			Middle Strip	
Exterior Panel	**Exterior (−*M*)**	**±*M***	**Interior −*M***	**−*M***	**±*M***
Moment factor (%)	100	60	75	25	40
M_u (K·ft)	370	444	748	249	296
Strip width, b (in.)	150	198	150	36 (6 ribs)	198
d (in.)	15.75	15.75	15.75	15.75	15.75
$R_u = \frac{M_u}{bd^2}$ (psi)	120	108	241	334	72
Steel ratio, ρ (%)	0.226	0.204	0.465	0.657	0.135
$A_s = \rho bd$ (in.^{2})	5.33	6.36	11.0	3.73	4.2
Min. $A_s = 0.0018bh$	2.6	1.22	4.6	1.1	1.47
Bars selected	14 no. 6	2 no. 8/rib	26 no. 6	10 no. 6	2. no. 7/rib

	Column Strip			Middle Strip	
Exterior Panel	**Exterior (−*M*)**	**±*M***	**Interior −*M***	**−*M***	**±*M***
Moment factor (%)	—	60	75	25	40
M_u (K·ft)	—	299	694.2	231.4	200
Strip width, b (in.)	—	198	150	36 (6 ribs)	198
d (in.)	—	15.75	15.75	15.75	15.75
$R_u = \frac{M_u}{bd^2}$ (psi)	—	73	224	311	49
Steel ratio, ρ (%)	—	0.137	0.431	0.61	0.091
$A_s = \rho bd$ (in.^{2})	—	4.27	10.18	3.45	2.84
Min. $A_s = 0.10018bh$	—	1.22/rib	4.6	1.1	1.47
Bars selected	—	2 no. 7/rib	24 no. 6	10 no. 6	2 no. 6/rib

17.12 EQUIVALENT FRAME METHOD

When two-way floor systems do not satisfy the limitations of the direct design method, the design moments must be computed by the equivalent frame method. In the latter method, the building is divided into equivalent frames in two directions and then analyzed elastically for all conditions of loadings. The difference between the direct design and equivalent frame methods lies in the way by which the longitudinal moments along the spans of the equivalent rigid frame are determined. The design requirements can be explained as follows.

1. Description of the equivalent frame: An equivalent frame is a two-dimensional building frame obtained by cutting the three-dimensional building along lines midway between columns (Fig. 17.4). The resulting equivalent frames are considered separately in the longitudinal and transverse directions of the building. For vertical loads, each floor is analyzed separately, with the far ends of the upper and lower columns assumed to be fixed. The slab-beam may be assumed to be fixed at any support two panels away from the support considered, because the vertical loads contribute very little to the moment at that support. For lateral loads, the equivalent frame consists of all the floors and extends for the full

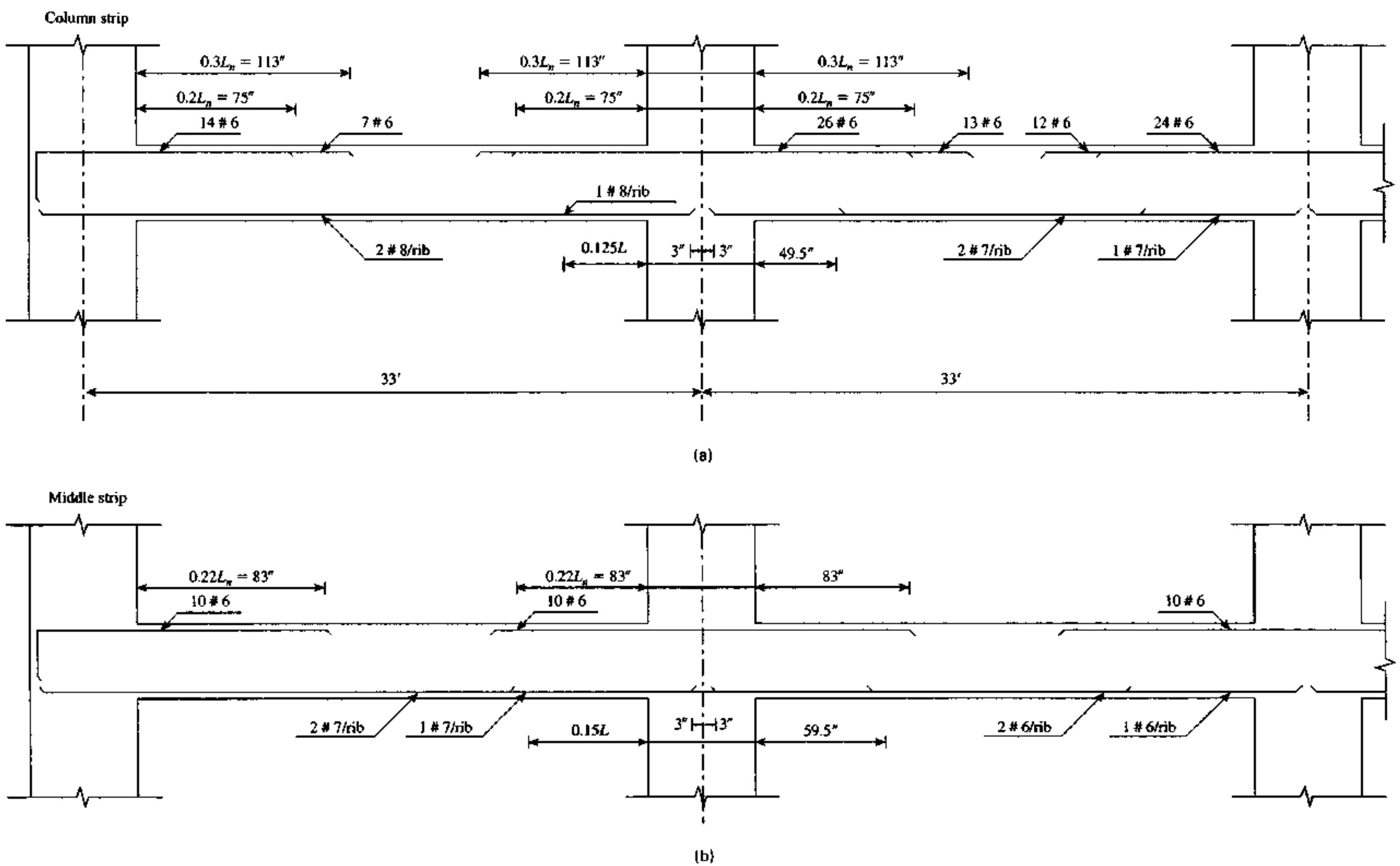


Figure 17.36 Example 17.12: reinforcement details of the waffle slab.

height of the building, because the forces at each floor are a function of the lateral forces on all floors above the considered level. Analysis of frames can also be made using computer programs.

2. Load assumptions: When the ratio of the service live load to the service dead load is less than or equal to 0.75, the structural analysis of the frame can be made with the factored dead and live loads acting on all spans instead of a pattern loading. When the ratio of the service live load to the service dead load is greater than 0.75, pattern loading must be used, considering the following conditions:
   a. Only 75% of the full-factored live load may be used for the pattern loading analysis.
   b. The maximum negative bending moment in the slab at the support is obtained by loading only the two adjacent spans.
   c. The maximum positive moment near a midspan is obtained by loading only alternate spans.
   d. The design moments must not be less than those occurring with a full-factored live load on all panels (ACI Code, Section 13.7.6).
   e. The critical negative moments are considered to be acting at the face of a rectangular column or at the face of the equivalent square column having the same area for nonrectangular sections.
3. Slab-beam moment of inertia: The ACI Code specifies that the variation in moment of inertia along the longitudinal axes of the columns and slab beams must be taken into account in the analysis of frames. The critical region is located between the centerline of the column and the face of the column, bracket, or capital. This region may be considered as a thickened section of the floor slab. To account for the large depth of the column and its reduced effective width in contact with the slab beam, the ACI Code, Section 13.7.3.3, specifies that the moment of inertia of the slab beam between the center of the column and the face of the support is to be assumed equal to that of the slab beam at the face of the column divided by the quantity $(1 - c_2/l_2)^2$, where c_2 is the column width in the transverse direction and l_2 is the width of the slab beam. The area of the gross section can be used to calculate the moment of inertia of the slab beam.
4. Column moment of inertia: The ACI Code, Section 13.7.4, states that the moment of inertia of the column is to be assumed infinite from the top of the slab to the bottom of the column capital or slab beams (Fig. 17.37).
5. Column stiffness, K_{ec}, is defined by

$$\frac{1}{K_{ec}} = \frac{1}{\sum K_c} + \frac{1}{K_t} \tag{17.17}$$

where ΣK_c is the sum of the stiffness of the upper and lower columns at their ends,

$$K_t = \sum \frac{9E_{es}C}{l_2\left(1 - \frac{c_2}{l_2}\right)} \tag{17.19}$$

$$C = \sum \left(1 - 0.63\frac{x}{y}\right)\left(\frac{x^3y}{3}\right) \tag{17.20}$$

6. Column moments: In frame analysis, moments determined for the equivalent columns at the upper end of the column below the slab and at the lower end of the column above the slab must be used in the design of a column.

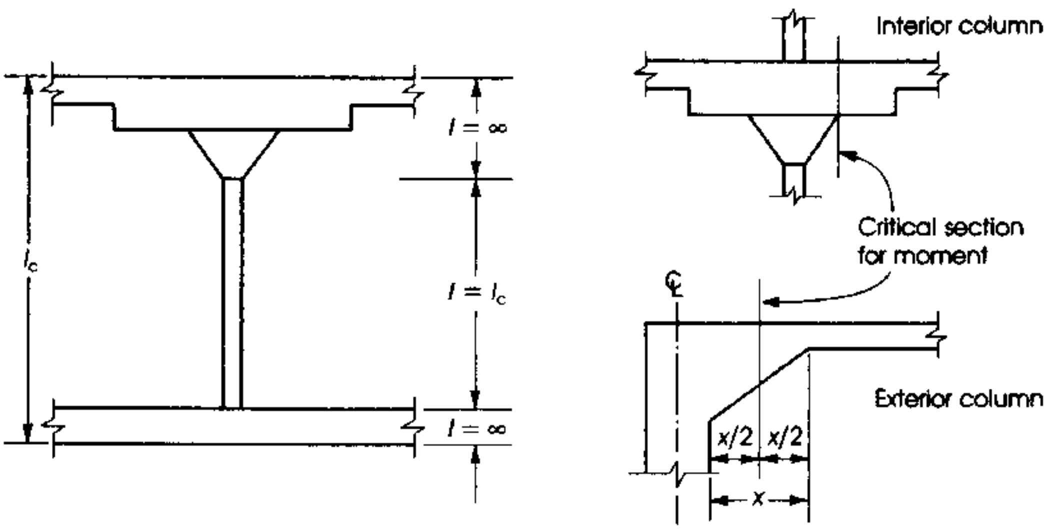


Figure 17.37 Critical sections for column moment, equivalent frame method.

7. Negative moments at the supports: The ACI Code, Section 13.7.7, states that for an interior column, the factored negative moment is to be taken at the face of the column or capital but at a distance not greater than $0.1175l_1$ from the center of the column. For an exterior column, the factored negative moment is to be taken at a section located at half the distance between the face of the column and the edge of the support. Circular section columns must be treated as square columns with the same area.
8. Sum of moments: A two-way slab floor system that satisfied the limitations of the direct design method can also be analyzed by the equivalent frame method. To ensure that both methods will produce similar results, the ACI Code, Section 13.7.7, states that the computed moments determined by the equivalent frame method may be reduced in such proportion that the numerical sum of the positive and average negative moments used in the design must not exceed the total statical moment, M_o.

Example 17.13: Waffle Slab

By the equivalent frame method, analyze a typical interior frame of the flat-plate floor system given in Example 17.3 in the longitudinal direction only. The floor system consists of four panels in each direction with a panel size of 25 by 20 ft. All panels are supported by a 20- by 20-in. columns, 12 ft long. The service live load is 80 psf and the service dead load is 124 psf (including the weight of the slab). Use $f'_c = 3$ ksi and $f_y = 60$ ksi. Edge beams are not used. Refer to Fig. 17.38.

Solution

1. A slab thickness of 8.0 in. is chosen, as explained in Example 17.3.
2. Factored load is $w_u = 1.2 \times 124 + 1.6 \times 60 = 245$ psf. The ratio of service live load to service dead load is $60/124 = 0.48 < 0.75$; therefore, the frame can be analyzed with the full factored load, w_u, acting on all spans instead of pattern loading.
3. Determine the slab stiffness, K_s:

$$K_s = k\,\frac{EI_s}{l_s}$$

where k is the stiffness factor and

$$I_s = \frac{l_2 h_s^3}{12} = \frac{(20 \times 12)}{12}(8)^3 = 10{,}240 \text{ in.}^4$$

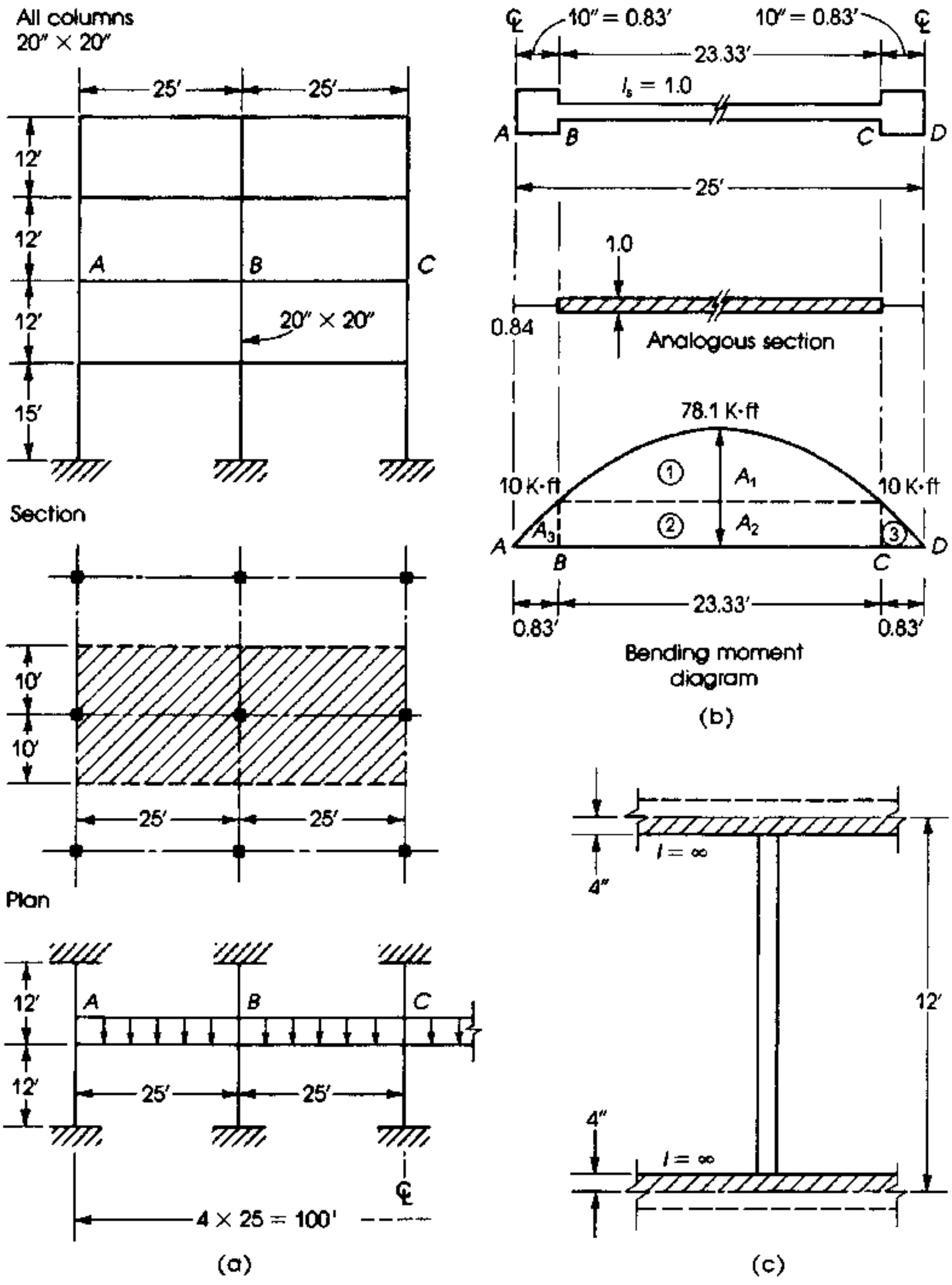


Figure 17.38 Example 17.13.

The stiffness factor can be determined by the column analogy method described in books on structural analysis. Considering the moment of inertia for the slab I_s to be 1.0 as a reference, the moment of inertia between the column centerline and the face of the column is

$$\frac{1.0}{\left(1-\frac{c_2}{l_2}\right)^2} = \frac{1.0}{\left(1-\frac{20}{(20\times 12)}\right)^2} = 1.19$$

The width of the analogous column varies with $1/I$, as shown in Fig. 17.38*b*: $(1/1.19) = 0.84$

$$\text{Slab stiffness factor } k = l_1\left(\frac{1}{A_a} + \frac{Mc}{I_a}\right)$$

where

A_a = area of the analogous column section

I_a = moment of inertia of analogous column

M = moment due to a unit load at the extreme fiber of the analogous column located at the center of the slab

$M = 1.0 \times \frac{l_1}{2}$

$A_a = 23.33 + 2 \times (0.83\text{ ft})(0.84) = 23.33 + 1.40 = 24.72$

$I_a = I$ (for slab portion of 23.33) $+ I$ (of end portion) about the centerline

$$I_a = \frac{(23.33)^3}{12} + 1.4\left(12.5 - \frac{0.83}{2}\right)^2 = 1263$$

neglecting the moment of inertia of the short end segments about their own centroid.

$$\text{Stiffness factor } k = 25\left[\frac{1}{24.72} + \frac{1.0 \times 12.5(12.5)}{1263}\right]$$

$$= 1.01 + 3.09 = 4.1$$

$$\text{Carryover factor} = \frac{3.09 - 1.01}{4.1} = 0.509$$

Therefore, slab stiffness is

$$K_s = \frac{4.1E \times 10{,}240}{(25 \times 12)} = 140E$$

4. Determine the column stiffness, K_c:

$$K_c = k'\left(\frac{EI_c}{l_c}\right) \times 2$$

for columns above and below the slab.

k' = column stiffness factor

$$l_c = 12\text{ ft} \qquad I_c = \frac{(20)^4}{12} = 13{,}333\text{ in.}^4$$

The stiffness factor, k', can be determined as follows:

$$k' = l_c\left(\frac{1}{A_a} + \frac{Mc}{I_a}\right)$$

For the column, $c = l_c/2$ and $M = 1.0(l_c/2) = l_c/2$.

$$A_a = l_c - h_s = 12 - \frac{8}{12} = 11.33$$

$$I_a = \frac{(l_c - h_s)^3}{12} = \frac{(11.33)^3}{12} = 121.2$$

$$k' = 12\left[\frac{1}{11.33} + \frac{(1 \times \frac{12}{2})(\frac{12}{2})}{121.2}\right] = 4.62$$

$$K_c = 4.62E \times \frac{13{,}333}{12 \times 12} \times 2 = 856E$$

In a flat-plate floor system, the column stiffness, K_c can be calculated directly as follows:

$$\frac{K_c}{E_c} = \frac{I_c}{(l_c - h_s)} + \frac{3I_c l_c^2}{(l_c - h_s)^3} \quad (17.33)$$

5. Calculate the torsional stiffness, K_t, of the slab at the side of the column:

$$K_t = \frac{\sum 9R_{cs}C}{l_2\left(1 - \frac{c_2}{l_2}\right)} \quad \text{and} \quad C = \sum\left(1 - 0.63\,\frac{x}{y}\right)\frac{x^3 y}{3}$$

In this example, $x = 8.0$ in. (slab thickness) and $y = 20$ in. (column width). See Fig. 17.17.

$$c = \left(1 - 0.63 \times \frac{8}{20}\right)\left(\frac{(8)^3 \times 20}{3}\right) = 2553 \text{ in.}^4$$

$$K_t = \frac{9E_{cs} \times 2553}{(20 \times 12)\left(1 - \frac{20}{(20-12)}\right)^3} = 124\,E_c$$

For two adjacent slabs, $K_t = 2 \times 124E_c = 248E_c$.

6. Calculate the equivalent column stiffness, K_{ec}:

$$\frac{1}{K_{ec}} = \frac{1}{\sum K_c} + \frac{1}{K_t} = \frac{1}{856E_c} + \frac{1}{248E_c}$$

or $K_{ec} = 192E_c$.

7. Moment distribution factors (D.F.): For the exterior joint,

$$\text{D.F. (slab)} = \frac{K_s}{K_s + K_{ec}} = \frac{140}{140 + 192} = 0.42$$

$$\text{D.F. (columns)} = \frac{K_{ec}}{\sum K} = 0.58$$

The columns above and below the slab have the same stiffness; therefore, the distribution factor of 0.58 is divided equally between both columns, and each takes a D.F. of 0.58/2 = 0.29. For the interior joint,

$$\text{D.F. (slab)} = \frac{K_s}{2K_s + K_{ec}} = \frac{140}{2 \times 140 + 192} = 0.295$$

$$\text{D.F. (columns)} = \frac{K_{ec}}{\sum K} = \frac{192}{2 \times 140 + 192} = 0.41$$

Each column will have a D.F. of 0.41/2 = 0.205.

8. Fixed-end moments: Because the actual L.L./D.L. is less than 0.75, the full-factored load is assumed to act on all spans.

$$\text{Fixed-end moment} = k'' w_u l_2 (L_1)^2$$

The factor k'' can be determined by the column analogy method: For a unit load $w = 1.0$ K/ft over the longitudinal span of 25 ft, the simple moment diagram is shown in Fig. 17.38*b*. The area of the bending moment diagram, considering the variation of the moment of inertia along the span, is

$$\text{Total area}\,(A_m) = A_1 + A_2 + 2A_3$$
$$= \frac{2}{3} \times 23.33(78.1 - 10) + 23.33 \times 10$$
$$+2\left(\frac{1}{2} \times 0.83 \times 10\right)(0.84) = 1300$$

$$\text{Fixed-end moment coefficient} = \frac{A_m}{A_a l_1^2}$$

where A_a for the slab is 24.72, as calculated in step 3.

$$k'' = \frac{1300}{27.32(25)^2} = 0.084$$

It can be seen that the fixed-end moment coefficient, $k'' = 0.084$, is very close to the coefficient $\frac{1}{12} = 0.0833$ usually used to calculate the fixed-end moments in beams. This is expected, because the part of the span that has a variable moment of inertia is very small in flat plates where no column capital or drop panels are used. In this example, only parts AB and CD, each equal to 0.83 ft, have a higher moment of inertia than I_s. In flat plates where the ratio of the span to column width is high, say, at least 20, the coefficient 0.0833 may be used to calculate approximately the fixed-end moments. Fixed-end moment (due to w_u = 276 psf) = $0.084(0.245)(20)(25)^2$ = 256 K·ft. The factors K, K'_s, and K'' can be obtained from tables prepared by Simmonds and Misic [18] to meet the ACI requirements for the equivalent frame method.

9. Moment distribution can be performed on half the frame due to symmetry. Once the end negative moments are computed, the positive moments at the center of any span can be obtained by subtracting the average value of the negative end moments from the simple beam positive moment. The moment distribution is shown in Fig. 17.39. The final bending moments and shear forces are shown in Fig. 17.40.
10. Slabs can be designed for the negative moments at the face of the columns as shown in Fig. 17.40.

Example 17.14: SI Units

Use the direct design method to design a typical interior flat slab with drop panels to carry a dead load of 8.6 kN/m^2 and a live load of 11 kN/m^2. The floor system consists of six panels in each direction, with a panel size of 6.0 by 5.4 m. All panels are supported by 0.4-m-diameter columns with 1.0-m-diameter column capitals. The story height is 3.0 m. Use f'_c = 28 MPa and f_y = 400 MPa.

Solution

1. All the ACI limitations to using the direct design method are met. Determine the minimum slab thickness, h_s, using Eqs. 17.1 and 17.2. The diameter of the column capital equals 1.0 m. The equivalent square column section of the same area will have a side of $\sqrt{\pi r^2} = \sqrt{\pi(500)^2}$ = 885 mm or 900 mm.

$$\text{Clear span (long direction)} = 6.0 - 0.19 = 5.1\ \text{m}$$
$$\text{Clear span (short direction)} = 5.4 - 0.9 = 4.5\ \text{m}$$

Because no beams are used $\alpha_{fm} = 0$, $\beta_s = 1.0$, and β = 6.0 m/5.4 m = 1.11. From Table 17.1, minimum $h_s = l_n/33$ = 5100/33 = 155 mm, but because a drop panel is used, h_s may be reduced by 10% if drop panels extend a distance of at least $l/6$ in each direction from the

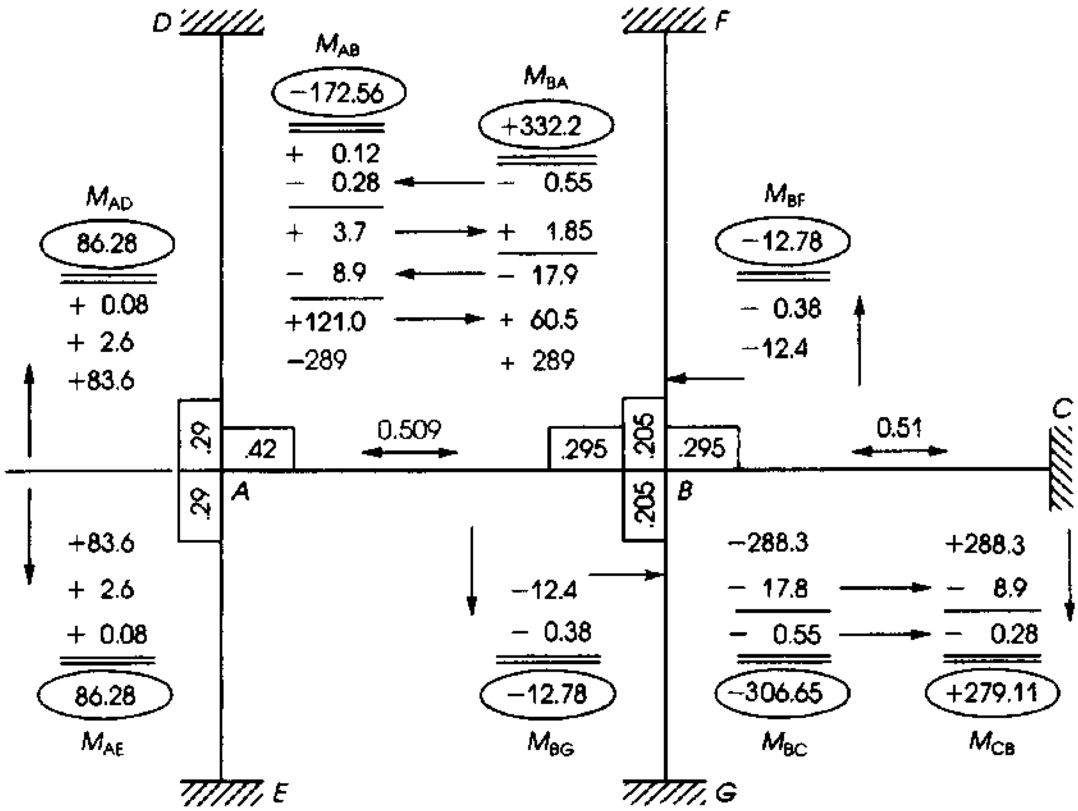


Figure 17.39 Example 17.13: Analysis by moment distribution. All moments are in K-ft.

centerline of support and project below the slab a distance of at least $h_s/4$. Therefore, use a slab thickness $h_s = 0.9 \times 155 \times 140$ mm and a drop panel length and width as follows:

$$\text{Long direction } \frac{l_1}{3} = \frac{6.0}{3} = 2.0 \text{ m}$$

$$\text{Short direction } \frac{l_2}{3} = \frac{5.4}{3} = 1.8 \text{ m}$$

The thickness of the drop panel is $1.25h_s = 1.25 \times 140 = 175$ mm. Increase drop panel thickness to 220 mm to provide adequate thickness for punching shear and to avoid the use of a high percentage of steel reinforcement. All dimensions are shown in Fig. 17.41.

2. Calculate factored loads:

$$w_u = 1.2 \times 8.6 + 1.6 \times 11 = 28 \text{ kN/m}^2$$

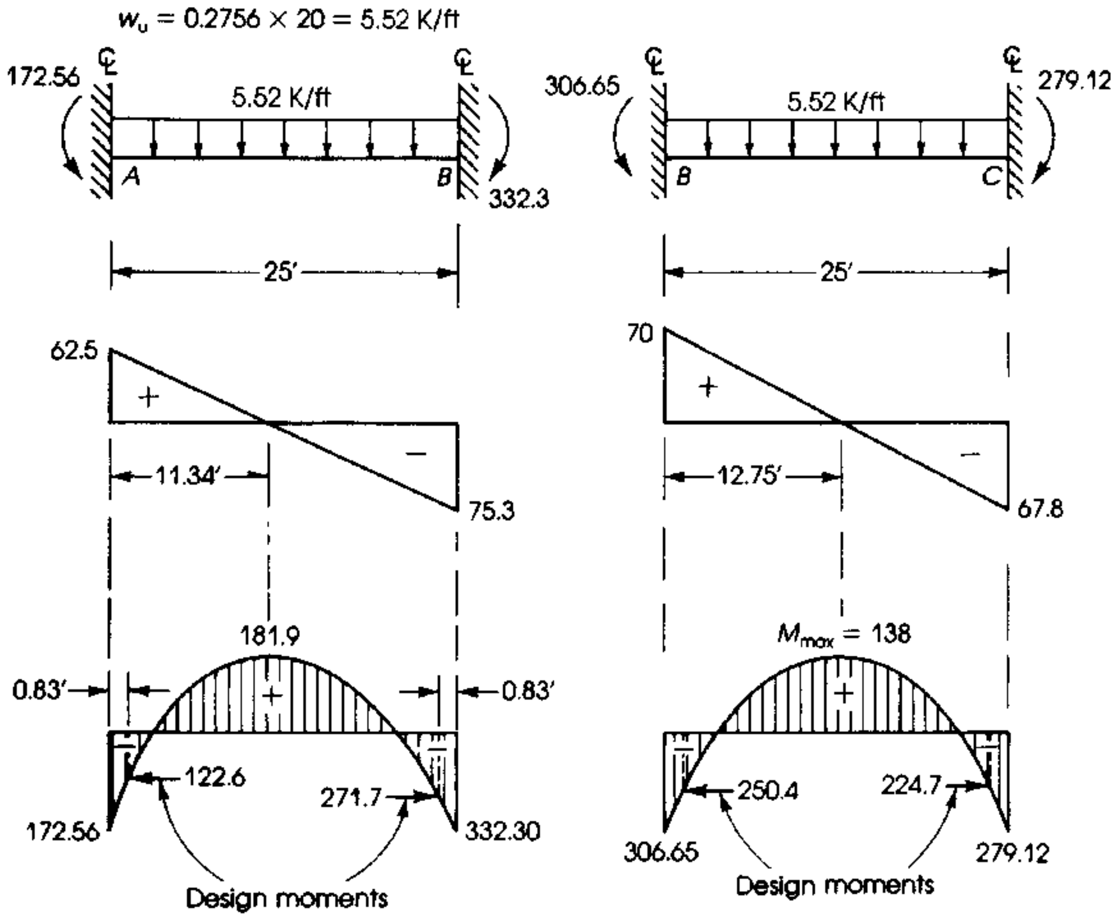
3. Check two-way shear, first in the drop panel: The critical section is at a distance $d/2$ around the column capital. Let $d = 220 - 30$ mm $= 190$ mm. Diameter of shear section $= 1.0$ m $+ d = 1.19$ m

$$V_u = 28\left[6.0\times5.4 - \frac{\pi}{4}(1.19)^2\right] = 876 \text{ kN}$$

$$b_o = 2\pi\left(\frac{1.19}{2}\right) = 3.74 \text{ m}$$

$$\phi V_c = \phi \times 0.33 \times \sqrt{f_c'}b_o d$$

$$= \frac{0.75\times0.33}{1000}\sqrt{28}\times3740\times190 = 930 \text{ kN}$$



$$V_A = 5.52 \times 12.5 - \tfrac{1}{25}(332.3 - 172.56) = 62.5\ \text{K}$$

$$V_B\ (\text{left}) = 5.52 \times 12.5 + \tfrac{1}{25}(332.3 - 172.56) = 75.3\ \text{K}$$

$$V_B\ (\text{right}) = 5.52 \times 12.5 + \tfrac{1}{25}(306.65 - 279.12) = 70\ \text{K}$$

$$V_C = 5.52 \times 12.5 - \tfrac{1}{25}(306.65 - 279.12) = 67.8\ \text{K}$$

Figure 17.40 Example 17.13: equivalent frame method — final bending moments and shear forces. (Slabs can be designed for the negative moments at the face of the columns as shown.)

which is greater than V_u of 876 kN. Then check the two-way shear in the slab; the critical section is at a distance $d/2$ outside the drop panel.

$$d(\text{slab}) = 140 - 30 = 110\ \text{mm}$$

$$\text{Critical area} = (2.0 + 0.11)(1.8 + 0.11) = 4.03\ \text{m}^2$$

$$b_o = 2(2.11 + 1.91) = 8.04\ \text{m}$$

$$V_u = 28(6\times5.4 - 4.03) = 794\ \text{kN}$$

$$\phi V_c = \frac{0.75 \times 0.33}{1000}\sqrt{21} \times 8040 \times 110 = 1003\ \text{kN} > V_u$$

One-way shear is not critical.

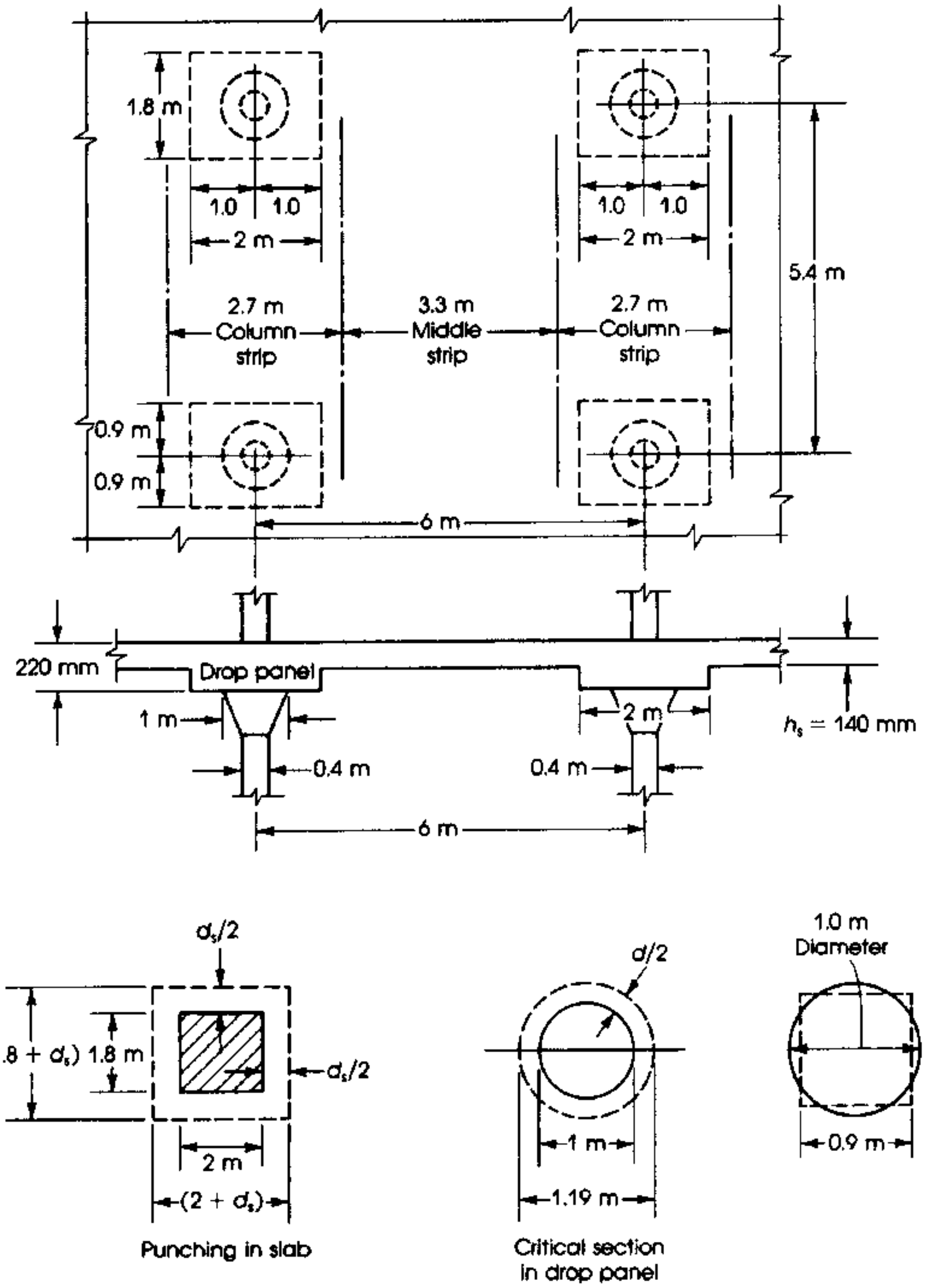


Figure 17.41 Example 17.14: interior flat slab with drop panel.

4. Calculate the total static moments in the long and short directions:

$$M_{\mathrm{ol}} = \frac{w_u}{8} l_2 l_{n1}^2 = \frac{28}{8}(5.4)(5.1)^2 = 491.6 \text{ kN·m}$$

$$M_{\mathrm{os}} = \frac{w_u}{8} l_1 l_{n2}^2 = \frac{28}{8}(6)(4.5)^2 = 425.2 \text{ kN·m}$$

Table 17.14 Design of an Interior Flab Slab With Drop Panels

	$M_o = 491.6$ kN·m $M_o = +0.35M_o = -319.5$ kN·m $M_p = +0.35M_o = +172.1$ kN·m			
Long Direction	**Column Strip**		**Middle Strip**	
Moment factor	$0.75M_o$	$0.60M_p$	$0.25M_n$	$0.40M_p$
M_u (kN·m)	-239.6	± 103.3	-79.9	± 68.8
d (mm)	190	110	110	110
Strip width b (m)	2.7	2.7	2.7	2.7
$R_u = \dfrac{M_u}{bd^2}$ (MP$_a$)	2.46	3.16	2.44	2.10
Steel ratio, ρ (%)	0.71	0.93	0.7	0.6
$A_s = \rho bd$ (mm^2)	3642	2762	2079	1782
Min. $A_s = 0.0018$bh (mm^2)	1070	680	680	680
Bars selcted (straight bars)	18×16 mm	14×16 mm	20×12 mm	16×12 mm
Spacing (mm)	150	193	135	170
	$M_o = 425.2$ kN·m $M_n = -0.65M_o = -276.4$ kN·m $M_p = +0.35M_o = +148.8$ kN·m			
Short Direction	**Column Strip**		**Middle Strip**	
Moment factor	$0.75M_n$	$0.60M_p$	$0.25M_n$	$0.40M_p$
M_u (kN·m)	-207.3	± 89.3	-69.1	± 59.5
d (mm)	180	100	100	100
Strip width b (m)	2.7	2.7	3.3	3.3
$R_u = \dfrac{M_u}{bd^2}$ (MP$_a$)	2.37	3.30	2.10	1.80
Steel ratio, ρ (%)	0.69	1.00	0.6	0.5
$A_s = \rho\ bd$ (mm^2)	3353	2700	1980	1650
Min. $A_s = 0.0018bh$ (mm ^{2})	1070	680	832	832
Bars selected (straight bars)	18×16 mm	14×16 mm	18×12 mm	16×12 mm
Spacing (mm)	150	195	185	205

Because $l_2 < l_1$, the width of the column strip in the long direction is $2(0.25 \times 5.4) = 2.7$ m. The width of the column strip in the short direction is 2.7 m. Assuming that the steel bars are 12 mm in diameter and those in the short direction are placed on top of the bars in the long direction, then the effective depth in the short direction will be about 10 mm less than the effective depth in the long direction. The d values and the design procedure are shown in Table 17.14. Minimum lengths of the selected reinforcement bars should meet the ACI Code length requirements shown in Fig. 17.16. Note that all steel ratios are less than ρ_{max}. Thus, $\phi = 0.9$.

5. The column stiffness is

$$\text{Ratio } \frac{\text{D.L.}}{\text{L.L.}} = \frac{8.6}{11} = 0.782 \text{ and } \frac{l_2}{l_2} = 1.11$$

Determine α_{min} from Table 17.7, taking into account that the relative beam stiffness is 0 because no beams are used. By interpolation, $\alpha_{min} = 1.15$. An approximate method is used here to determine the stiffness of the column with its capital.

I_s (moment of inertia of slab, short direction)

$$= 6000\frac{(140)^3}{12} = 1372 \times 10^6 \text{mm}^4$$

$$K_s = \frac{4E_cI_s}{l_2} = \frac{4E_c \times 1372 \times 10^6}{5400} = 1016 \times 10^3 E_c$$

I_c (for circular column, diameter 400 mm)

$$= \frac{\pi D^4}{64} = \frac{\pi}{64}(400)^4 = 1257 \times 10^6 \text{mm}^4$$

$$K_c = \frac{4E_cI_c}{l_c} = \frac{4E_c \times 1257 \times 10^6}{3000\text{mm}} = 1676 \times 10^3 E_c$$

Ratio of column stiffness/slab stiffness

$$= \frac{K_c}{K_s} = \frac{1676 \times 10^3}{1016 \times 10^3} = 1.65$$

which is greater than $\alpha_{\min}$ of 1.15. If I_s in the long direction is used, the calculated ratio of column to slab stiffness will be greater than 1.65. Therefore, the column is adequate.

6. Determine the balanced moment in the column and check the shear stresses in the slab, as explained in Examples 17.8 and17.9.

SUMMARY

Sections 17.1–17.5

1. A two-way slab is one that has a ratio of length to width less than 2. Two-way slabs may be classified as flat slabs, flat plates, waffle slabs, or slabs on beams.
2. The ACI Code specifies two methods for the design of two-way slabs: the direct design method and the equivalent frame method. In the direct design method, the slab panel is divided (in each direction) into three strips, one in the middle (referred to as the *middle strip*) and one on each side (referred to as *column strips*).

Section 17.6

To control deflection, the minimum slab thickness, h, is limited to the values computed by Table 17.1 or Eqs. 17.1 and 17.2 and as explained in Examples 17.1 and 17.2.

Section 17.7

For two-way slabs without beams, the shear capacity of the concrete section in one-way shear is

$$V_c = 2\lambda\sqrt{f'_c}bd$$

The shear capacity of the concrete section in two-way shear is

$$V_c = \left(2 + \frac{4}{\beta_c}\right)\lambda\sqrt{f'_c}b_od \le 4\sqrt{f'_c}b_od$$

When shear reinforcement is provided, $V_n \le 6\sqrt{f'_c}b_od$.

Section 17.8

In the direct design method, approximate coefficients are used to compute the moments in the column and middle strips of two-way slabs. The total factored moment is

$$M_o = (w_u l_2)\frac{l_1^2}{8} \tag{17.11}$$

The distribution of M_o into negative and positive span moments is given in Fig. 17.14. A summary of the direct design method is given in Section 17.8.8. The modified stiffness method is explained in Section 17.8.7.

Sections 17.9–17.11

1. Unbalanced loads on adjacent panels cause a moment in columns that can be computed by Eq. 17.22.
2. Approximately 60% of the moment transferred to both ends of a column at a joint is transferred by flexure, M_f, and 40% is transferred by eccentric shear, M_v. The fraction of the unbalanced moment transferred by flexure, M_f, is $\gamma_f M_u$, where γ_f is computed from Eq. 17.25. The shear stresses produced by M_v must be combined with shear stresses produced by the shearing force V_u.
3. Waffle slabs are covered in Section 17.11.

Section 17.12

1. In the equivalent frame method, the building is divided into equivalent frames in two directions and then analyzed for all conditions of loadings. Example 17.13 explains this procedure.
2. Example 17.14 is an example of a two-way flat slab with drop panel (SI units).

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PROBLEMS

17.1 (Flat plates) Determine the minimum slab thickness according to the ACI Code for the flat-plate panels shown in Fig. 17.42 and Table 17.15. The floor panels are supported by 24 × 24-in. columns,

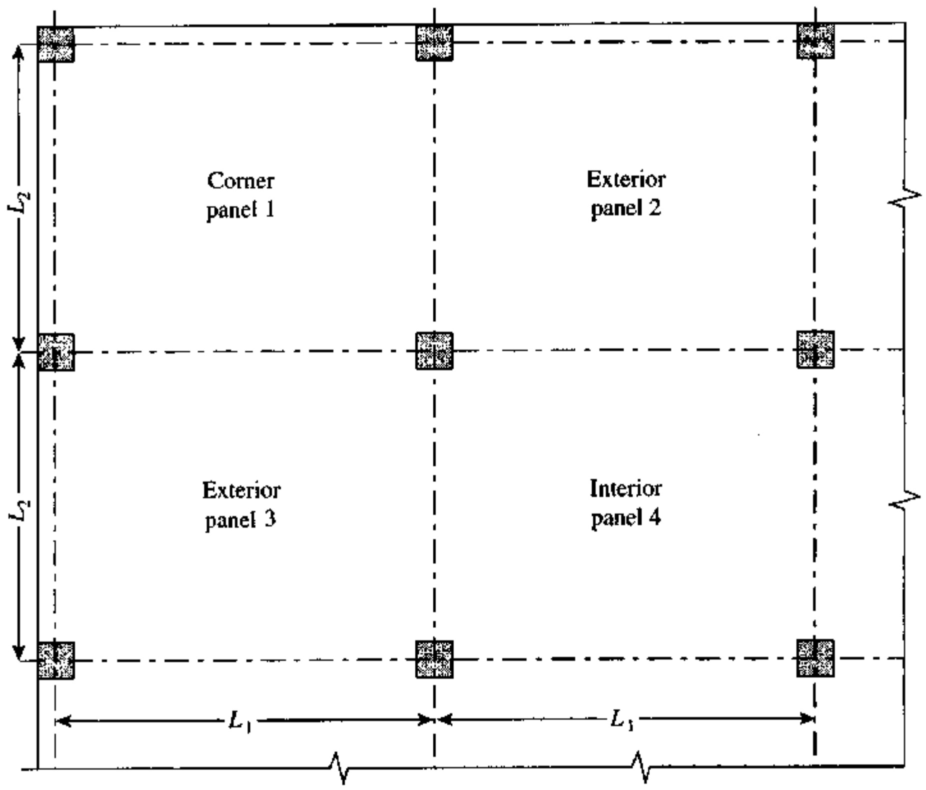


Figure 17.42 Problem 17.1.

Table 17.15 Problem 17.1

Number (Flat Plate)	Panel Dimensions (ft)		Panel Numbers
	L_1	L_2	
(a)	20	20	1 and 4
(b)	24	24	2 and 4
(c)	26	26	3 and 4
(d)	20	16	1 and 2
(e)	24	20	3 and 4
(f)	26	22	1 and 4
(g)	30	24	1 and 2
(h)	30s	30	1 and 4

12 ft long, with no edge beams at the end of the slab. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, dead load (excluding self-weight) = 55 psf, and live load = 120 psf.

17.2 (Flat plates) Use the direct design method to design the interior flat-plate panel (no. 4) of Problems 17.1*a*, *b*, *c*, and *e*, using the data given earlier. Check the shear and moment transfer at an interior column. Draw sketches showing the reinforcement distribution and the shear stresses.

17.3 (Flat plates) Repeat Problem 17.2 for the exterior panel no. 3. Check the shear and moment transfer at the exterior column. If shear stresses are not adequate, use shear reinforcement involving stirrups.

17.4 (Flat slabs with drop panels) Determine the minimum slab and drop panel thicknesses according to the ACI Code for the slabs shown in Fig. 17.42 and Table 17.15. The floor panels are supported by 24 × 24 -in. columns with no edge beams. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, additional dead load (excluding self-weight) = 60 psf, and live load = 120 psf.

17.5 (Flat slabs) Use the direct design method to design the interior flat slab panel no. 4, of Problem 17.4*a*, *b*, *c*, and *e*, using the data given in Problem 17.4. Check the shear and moment transfer at an interior column. Draw sketches showing the reinforcement distribution and the shear stresses. Use a 4-ft-column capital diameter for part *c* only.

17.6 (Flat slabs) Repeat Problem 17.5 for the exterior panel no. 3.

17.7 (Slabs on beams) Redesign the slabs in Problem 17.2, using the same data when the slabs are supported by beams on all four sides. Each beam has a width $b_w = 14$ in. and a projection below the bottom of the slab of 18 in.

17.8 (Slabs on beams) Redesign the slabs in Problem 17.7 as exterior panels.

17.9 (Waffle slabs) Repeat Example 17.12 when the spans are (a) 36 ft and (b) 42 ft. Use the same data and 24 × 24-in. columns.

17.10 (Waffle slabs) Redesign the waffle slabs in Problem 17.9 as exterior panels.

17.11 (Equivalent frame method) Redesign the flat-plate floor system of Problem 17.2*a* and *b* using the equivalent frame method.

17.12 (Equivalent frame method) Redesign the waffle slabs of Problem 17.9 using the equivalent frame method.

CHAPTER 18

STAIRS

Office building under construction, Chicago, Illinois.

18.1 INTRODUCTION

Stairs must be provided in almost all buildings, either low-rise or high-rise, even if adequate numbers of elevators are provided. Stairs consist of rises, runs (or treads), and landings. The total steps and landings are called a *staircase*. The *rise* is defined as the vertical distance between two steps, and the *run* is the depth of the step. The *landing* is the horizontal part of the staircase without rises (Fig. 18.1).

The normal dimensions of the rises and runs in a building are related by some empirical rules.

$$\text{Rise} + \text{run} = 17 \text{ in}$$

$$2 \times \text{rise} + \text{run} = 25 \text{ in. (635 mm)}$$

$$\text{rise} \times \text{run} = 75 \text{ in.}^2 \ (0.05 \text{ m}^2)$$

The rise depends on the use of the building. For example, in public buildings the rise is about 6 in., whereas in residential buildings it varies between 6 and 7.5 in. The run is about 1 ft in public buildings and varies between 9 in. and 12 in. in residential buildings. In general, a rise should not exceed 8 in. or be less than 4 in., and the number of rises is obtained by dividing the structural floor-to-floor dimension by the assumed rise.

The finishing on the stairs varies from troweling Alundum grits to adding asphalt tiles, terrazzo tiles, marble, or carpets. In addition to dead loads, stairs must be designed for a minimum live load of 100 psf.

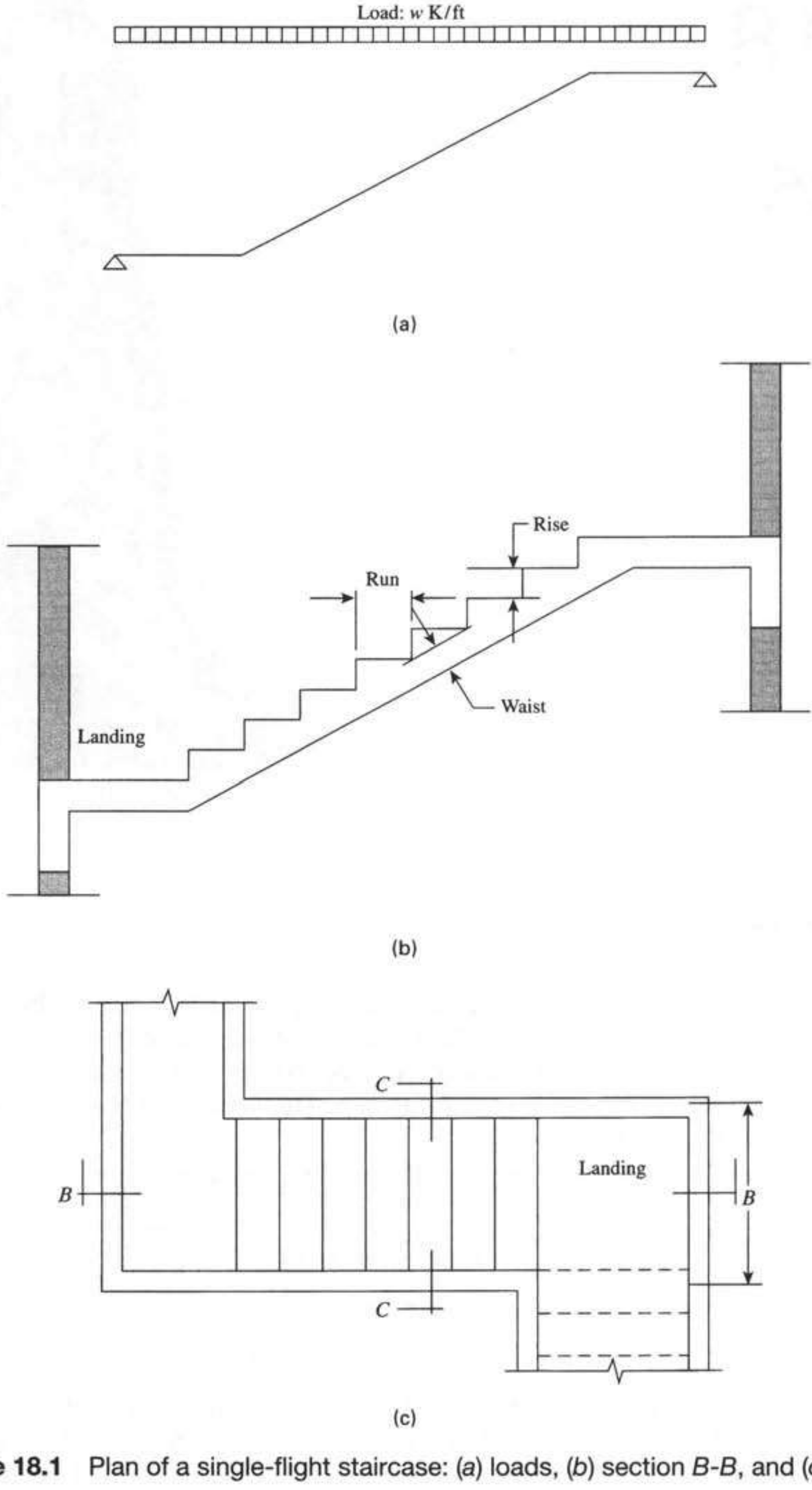


Figure 18.1 Plan of a single-flight staircase: (*a*) loads, (*b*) section *B-B*, and (*c*) plan.

18.2 TYPES OF STAIRS

There are different types of stairs, which depend mainly on the type and function of the building and on the architectural requirements. The most common types are as follows.

1. *Single-flight stairs:* The structural behavior of a flight of stairs is similar to that of a one-way slab supported at both ends. The thickness of the slab is referred to as the waist (Fig. 18.1). When the flight of stairs contains landings, it may be more economical to provide beams at B and C between landings (Fig. 18.2). If such supports are not provided, which is quite common, the span of the staircase will increase by the width of two landings and will extend between A and D. In residential buildings, the landing width is in the range of 4 to 6 ft, and the total distance between A and D is about 20 ft.

   An alternative method of supporting a single flight of stairs is to use stringers, or edge beams, at the two sides of the stairs; the steps are then supported between the beams (Fig. 18.3).
2. *Double-flight stairs:* It is more convenient in most buildings to build the staircase in double flights between floors. The types commonly used are quarter-turn (Fig. 18.4) and closed-or open-well stairs, as shown in Fig. 18.5. For the structural analysis of the stairs, each flight is treated as a single flight and is considered supported on two or more beams, as shown in Fig. 18.2. The landing extends in the transverse direction between two supports and is designed as a one-way slab. In the case of open-well stairs, the middle part of the landing carries a full load, whereas the two end parts carry half-loading only, as shown in Fig. 18.5(*d*). The other half-loading is carried in the longitudinal direction by the stair flights, sections A-A and B-B.
3. *Three or more flights of stairs:* In some cases, where the overall dimensions of the staircase are limited, three or four flights may be adopted (Fig. 18.6). Each flight will be treated separately, as in the case of double-flight staircases.
4. *Cantilever stairs:* Cantilever stairs are used mostly in fire-escape stairs, and they are supported by concrete walls or beams. The stairsteps may be of the full-flight type, projecting from one side of the wall, the half-flight type, projecting from both sides of the supporting wall, or of the semispiral type, as shown in Fig. 18.7. In this type of stairs, each step acts as a cantilever, and the main reinforcement is placed in the tension side of the run and the bars are anchored within the concrete wall. Shrinkage and temperature reinforcement is provided in the transverse direction.

   Another form of a cantilever stair is that using open-riser steps supported by a central beam, as shown in Fig. 18.8. The beam has a slope similar to the flight of stairs and receives the steps on its horizontally prepared portions. In most cases, precast concrete steps are used, with special provisions for anchor bolts that fix the steps into the beam.
5. *Precast flights of stairs:* The speed of construction in some projects requires the use of precast flights of stairs (Fig. 18.8). The flights may be cast separately and then fixed to cast-in-place landings. In other cases, the flights, including the landings, are cast and then placed in position on their supporting walls or beams. They are designed as simply supported one-way slabs with the main reinforcement at the bottom of the stair waist. Adequate reinforcement must be provided at the joints, as shown in Fig. 18.9.

   Provisions must be made for lifting and handling the precast stair units by providing lifting holes or inserting special lifting hooks into the concrete. Special reinforcement must be provided at critical locations to account for tensile stresses that will occur in the stairs from the lifting and handling process.

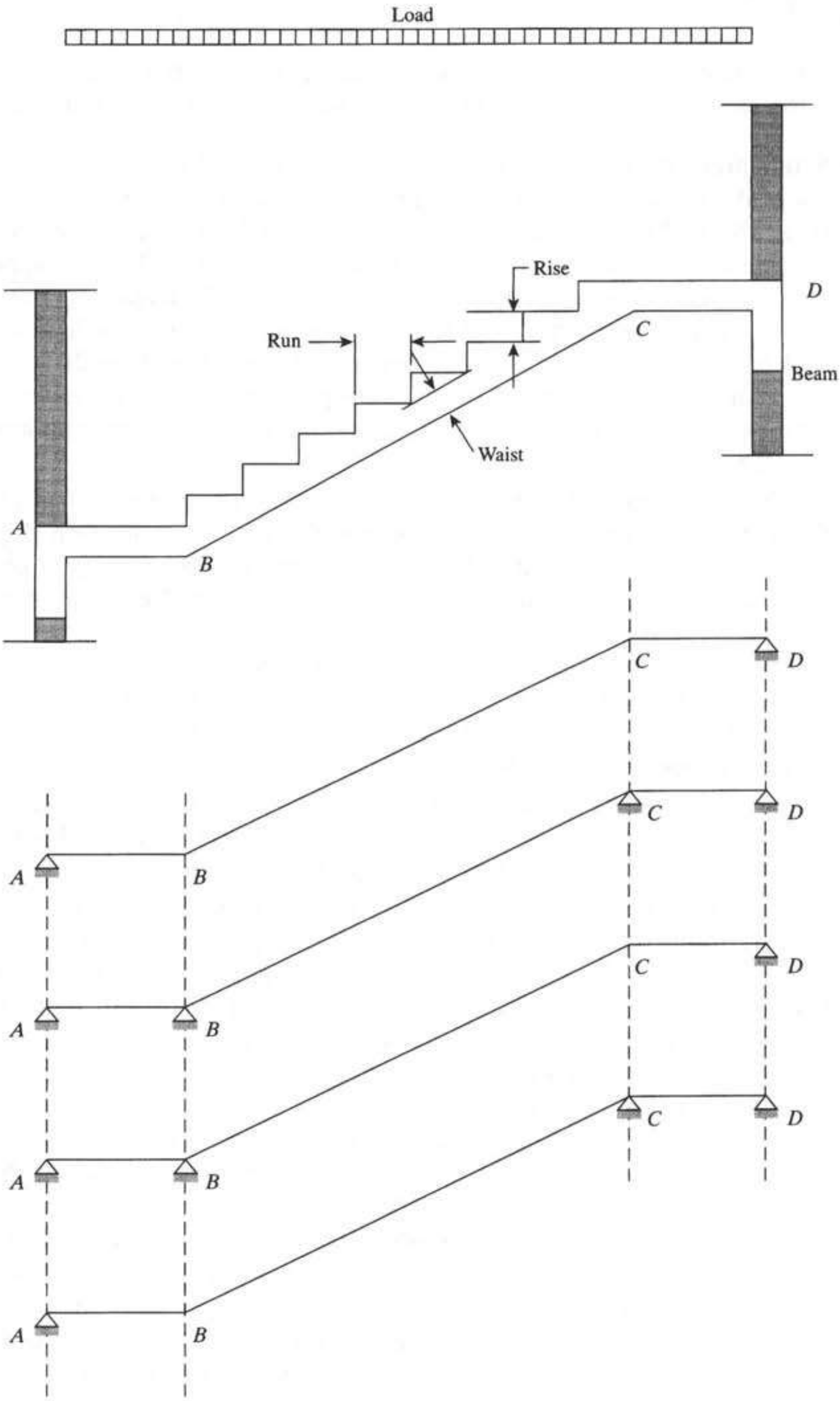


Figure 18.2 Supporting systems of one flight.

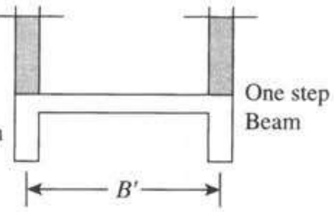


Figure 18.3 Steps supported by stringer beams.

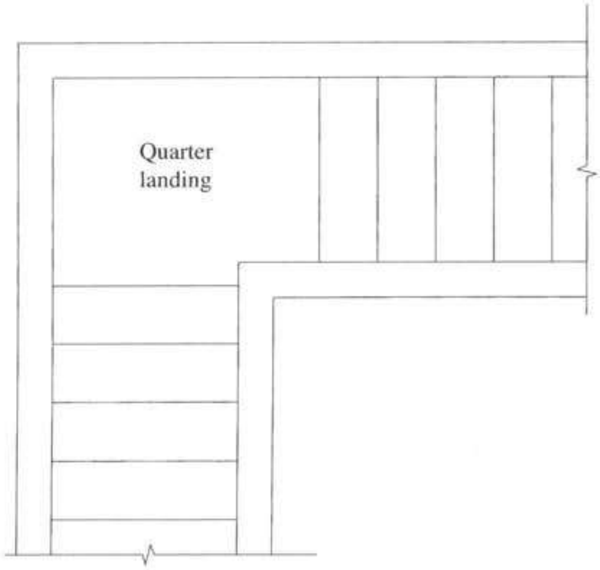


Figure 18.4 Quarter-turn staircase.

6. *Free-standing staircase:* In this type of stairs, the landing projects into the air without any support at its end (Fig. 18.10). The stairs behave in a springboard manner, causing torsional stresses in the slab.

Three systems of loading must be considered in the design of this type of stairs, taking into consideration that torsional moments will develop in the slab in all cases:

a. When the live load acts on the upper flight and half the landing only (Fig. 18.11), the upper flight slab will be subjected to tensile forces in addition to bending moments, whereas the lower flight will be subjected to compression forces, which may cause buckling of the slab.

b. When the live load acts on the lower flight and half the landing only (Fig. 18.12), the upper flight slab will be subjected to tensile forces, whereas the lower flight will be subjected to bending moment and compression forces.

c. When the live load acts on both upper and lower flights, the loading of one flight will cause the twisting of the other. The torsional stresses developed in the stairs require adequate reinforcement in both faces of the stair slabs and the landing. Transverse reinforcement in the slab and the landing must be provided in both faces of the concrete in the shape of closed U-bars lapping at midwidth of the stairs. Typical reinforcement details are shown in Fig. 18.13.

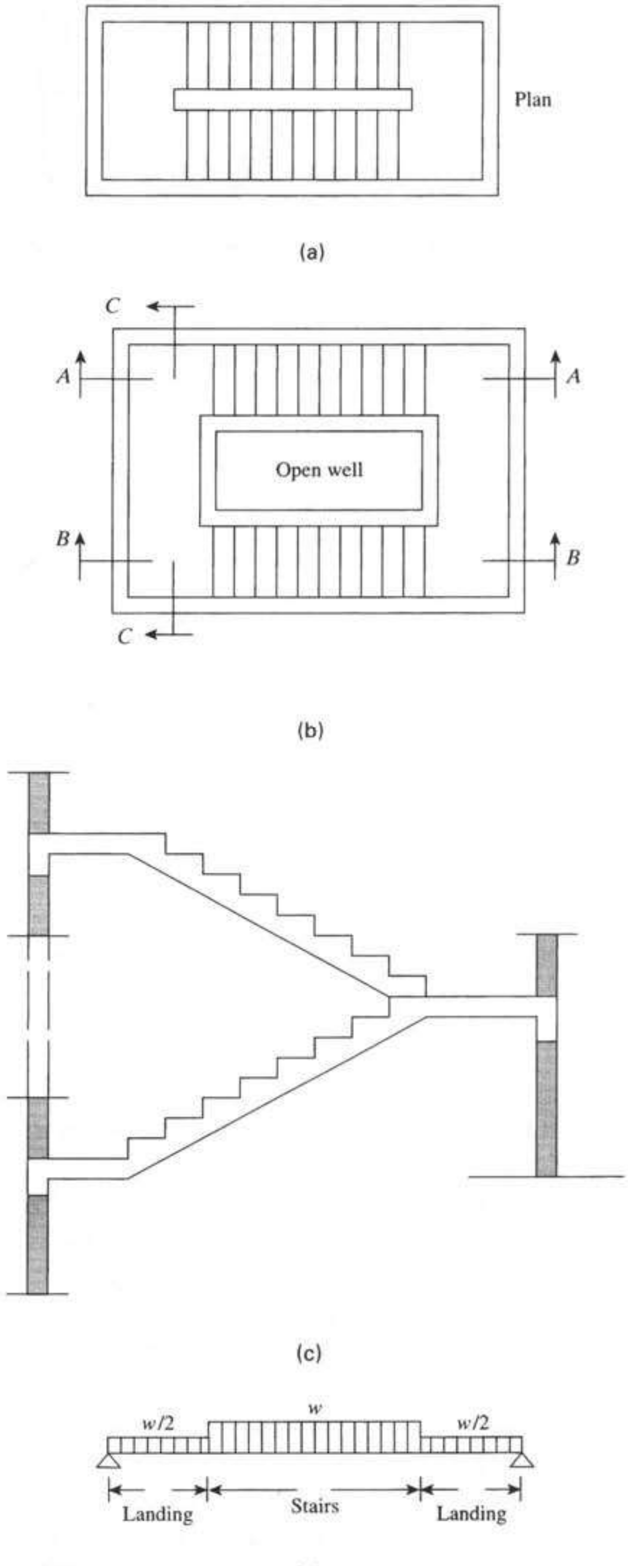
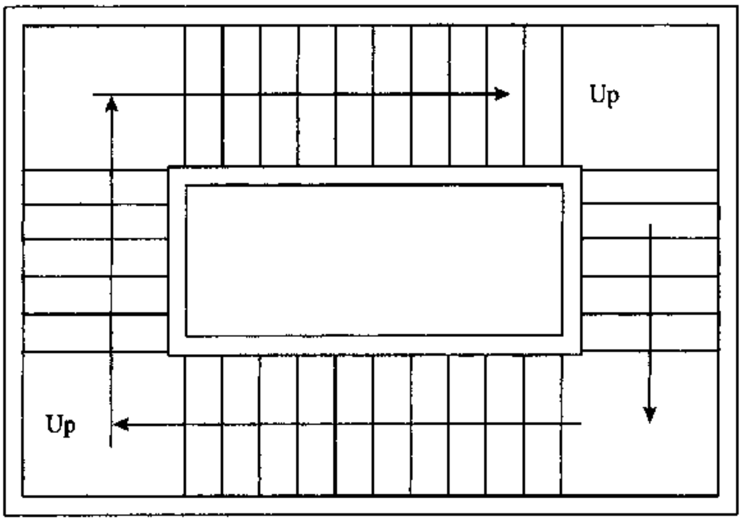
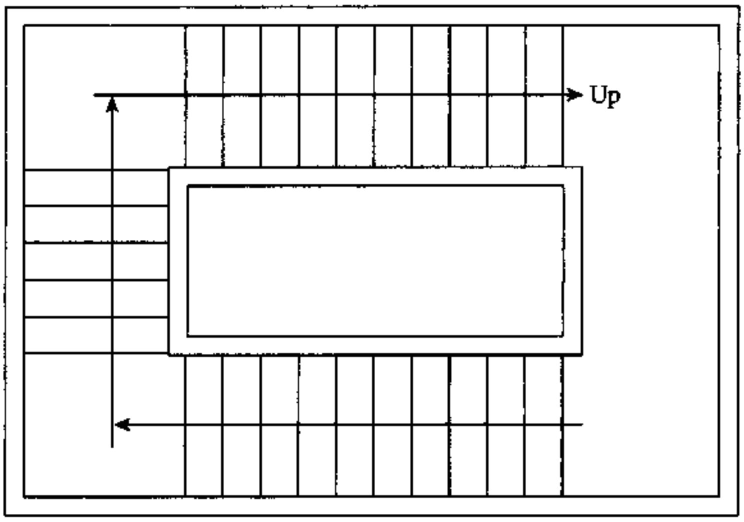


Figure 18.5 Double-flight stairs: (*a*) closed-well staircase, (*b*) open-well staircase, (*c*) section *B-B*, and (*d*) section *C-C*.



Four-stair flight



Three-stair flight

Figure 18.6 Three- and four-stair flights.

This type of stairs is favored by architects and sometimes called a pliers-shaped staircase or jackknife staircase.

A study was made to determine the effect of the following parameters on the free-standing staircases forces and moments considering a live load of 100 psf (Figs. 18.10 and 18.13).

1. *The width of the stairs (Fig. 18.10).* An increase in the width from 4 to 10 ft, will increase the forces and moments sharply. For example, the torsional moment along the flight increases by about 1,400%. Therefore, it is desirable to restrict the flight width between 4.0 and 6.0 ft. Other moments increase by about 450%.
2. *The span length* L. An increase in the span L will increase the forces and moments in the stair flight and landing significantly. For example, if L is increased from 8 ft to

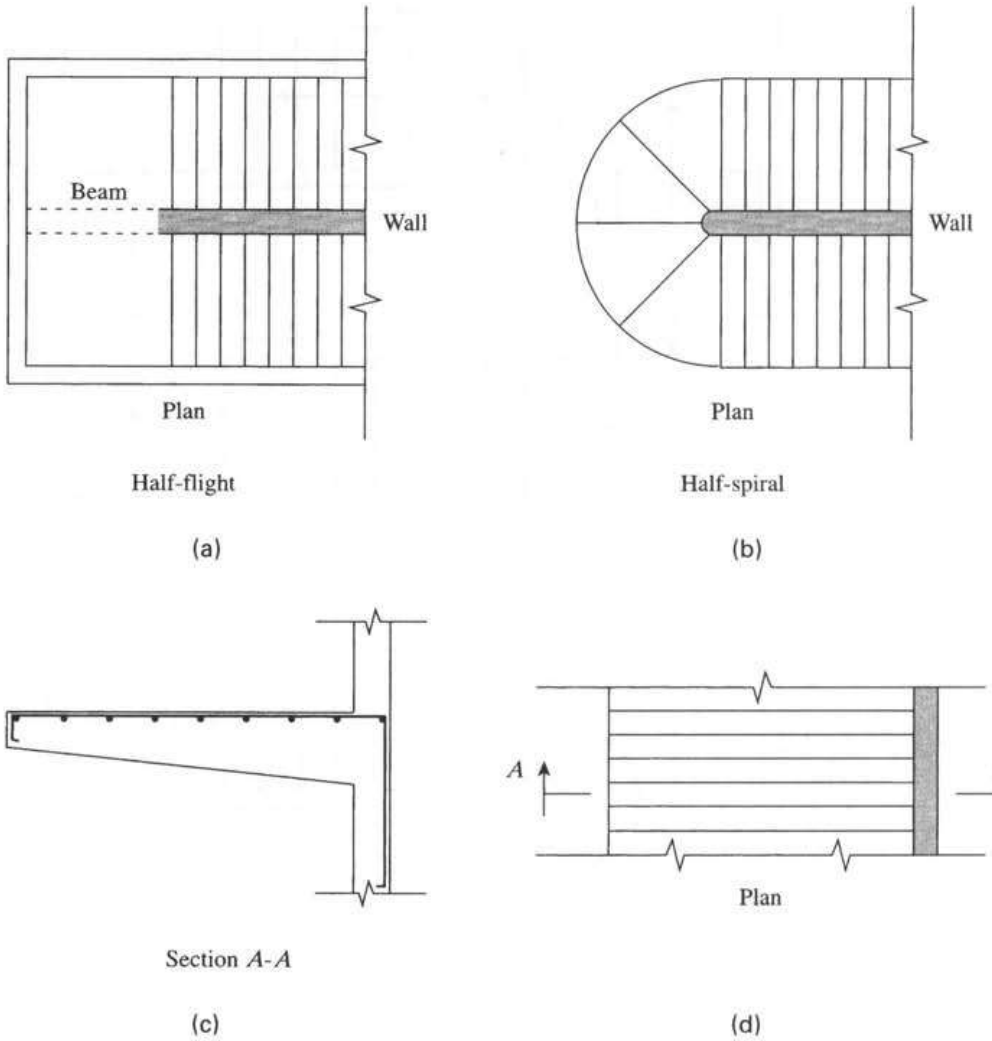


Figure 18.7 Steps projecting from one or two sides of the supporting wall.

16 ft, the shearing forces at the top edge of the stairs increases by about 230%. Moments increase by about 100% to 150%.

3. *The total flight height* h. If h is increased from 10 ft to 16 ft, the shearing force at the top edge increases by about 150%. Moments increase by about 50 to 100%.
4. *The flight slab thickness* t. This parameter has the least effect on forces and moments. For example, if t is increased from 6 to 10 in., the moments increase by about 25% and the shearing force by about 20%.
5. For practical design, the parameters may be chosen as follows: flight width between 4- and 6 ft, horizontal span L between 9- and 12 ft, total flight height between 10- and 15 ft, and slab thickness between 6- and 10 in.

The above information is a guide to help the designer to choose the right parameters for an economical design.

7. *Run-riser stairs:* Run-riser stairs are stepped underside stairs that consist of a number of runs and risers rigidly connected without the provision of the normal waist slab (Fig. 18.14*a*). This type of stairs has an elegant appearance and is sometimes favored by architects. The

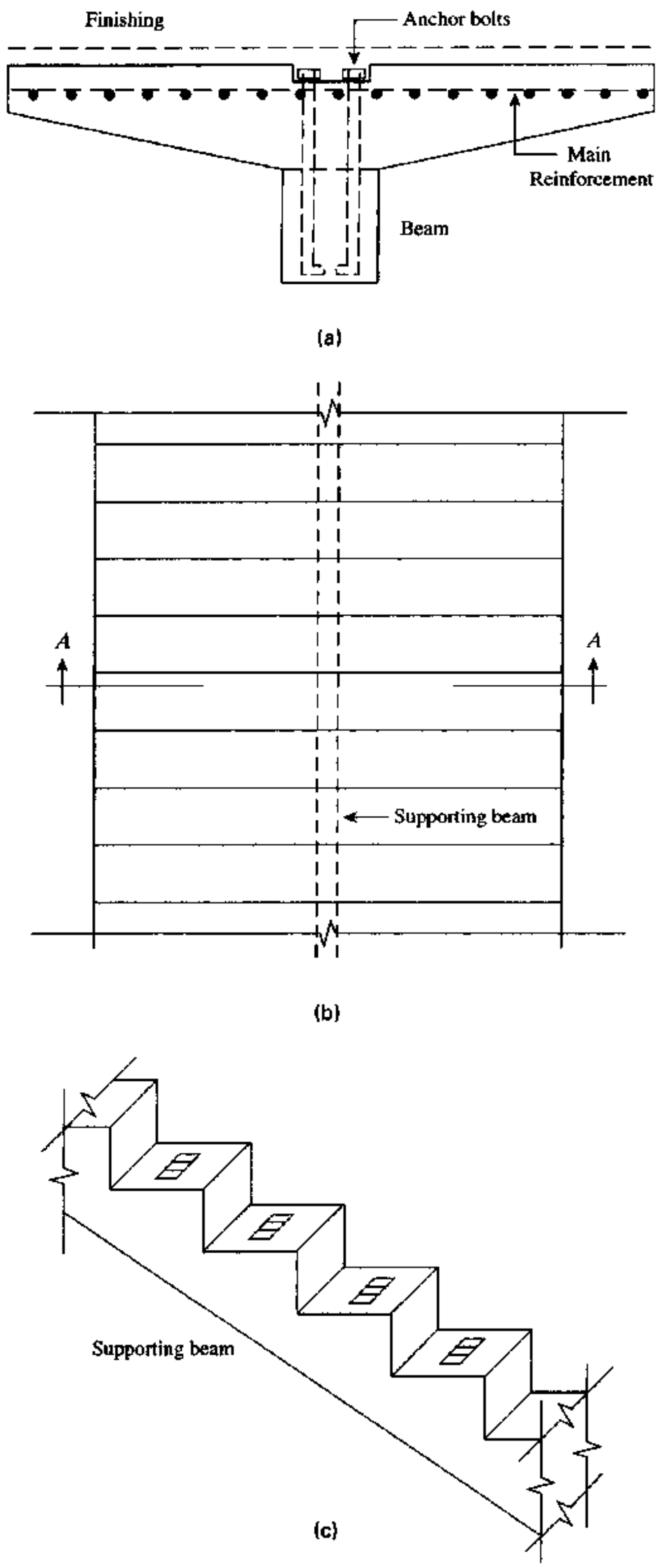


Figure 18.8 Precast cantilever stair supported by central beam: (*a*) section *A-A*, (*b*) part plan, and (*c*) supporting beam.

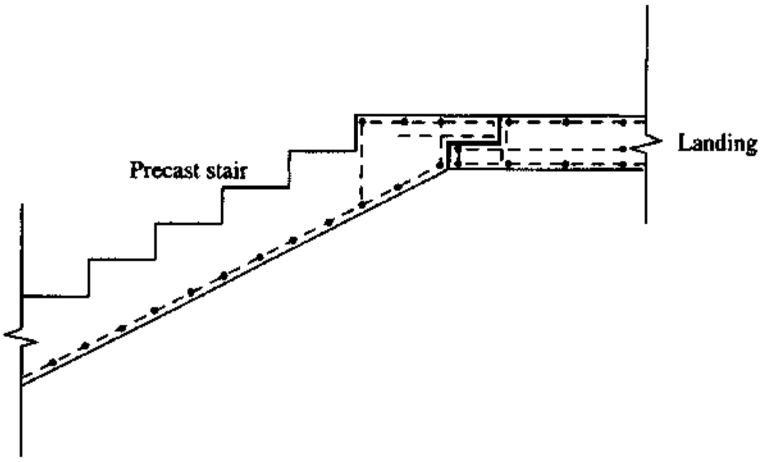


Figure 18.9 Joint of a precast concrete flight of stairs.

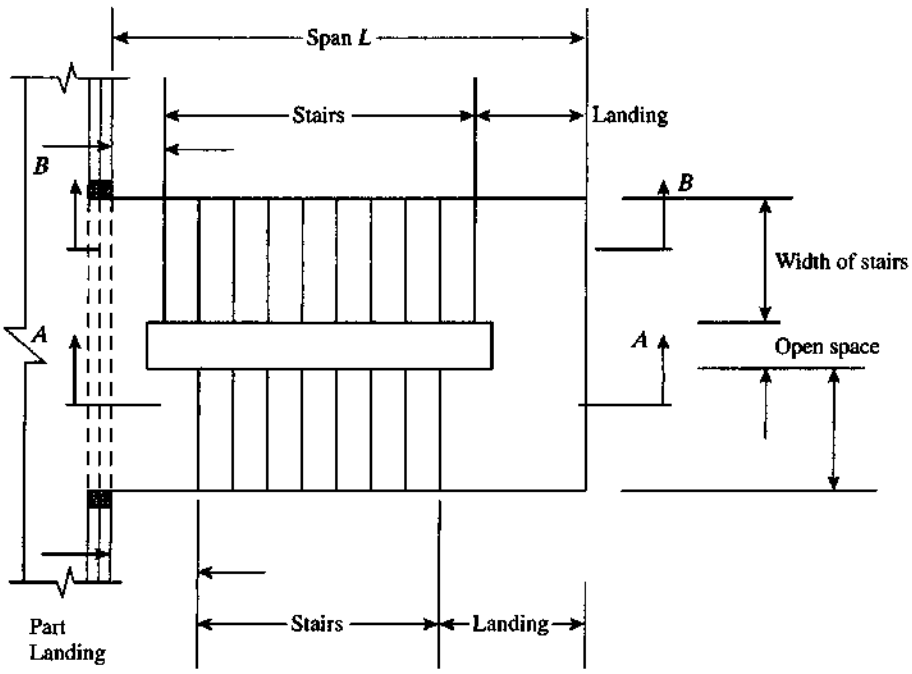


Figure 18.10 Plan of a free-standing staircase.

structural analysis of run-riser stairs can be simplified by assuming that the effect of axial forces is negligible and that the load on each run is concentrated at the end of the run (Fig. 18.14*b*). For the analysis of a simply supported flight of stairs, consider a simple flight of two runs, ABC, subjected to a concentrated load P at B' (Fig. 18.14*b*). Because joints B and B' are rigid, the moment at joint B is equal to the moment at B', or

$$M_B = M'_B = \frac{PS}{2}$$

where S is the width of the run. The moment in rise, BB', is constant and is equal to $PS/2$.

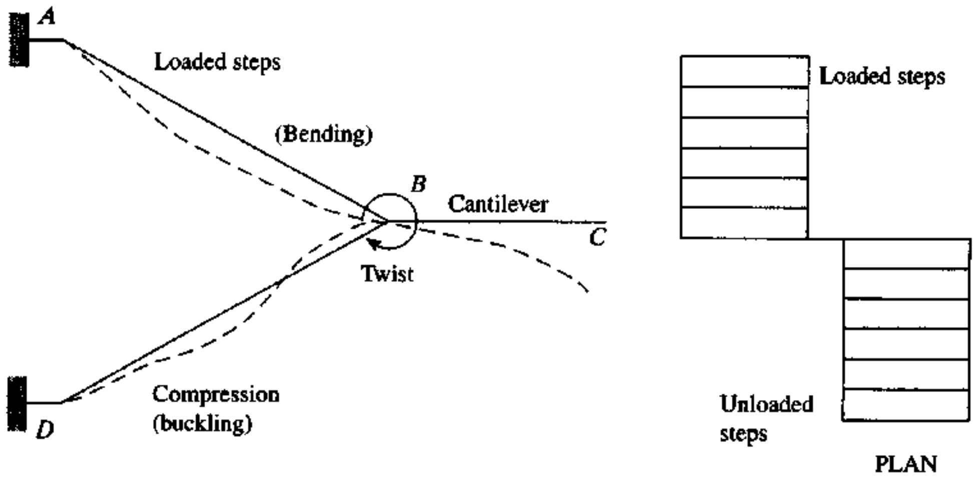


Figure 18.11 Case 1, *ABC* loaded.

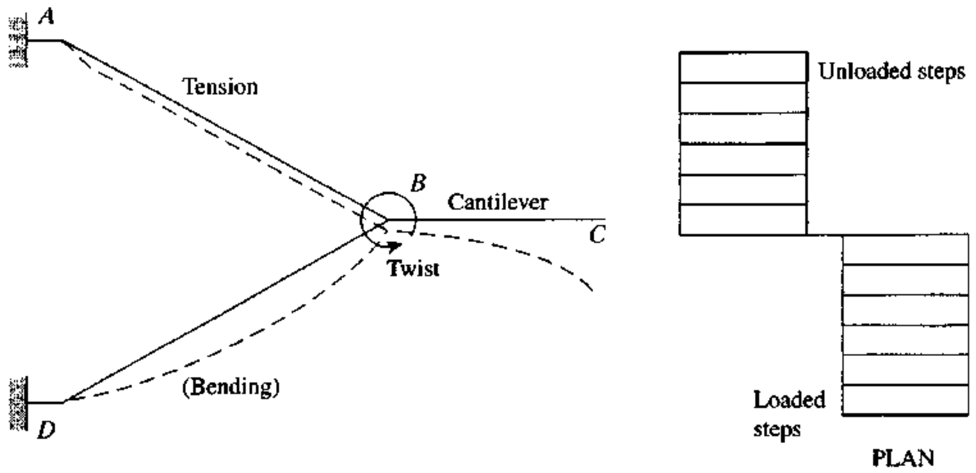


Figure 18.12 Case 2, *DBC* loaded.

When the rise is absent, the stairs, *ABC*, act as a simply supported beam, and the maximum bending moment occurs at midspan with value

$$M_B = \frac{PL}{4} = \frac{PS}{2}$$

For a flight of stairs that consists of a number of runs and risers, the same approach can be used; the bending moment diagram is shown in Fig. 18.15*a*. The moment in BB' is constant and is equal to the moment at joint B, or $2PS$. Similarly, $M_C = M'_C = 3PS$, $M_D = M'_D = 3PS$, and $M_E = M'_E = 2PS$

If a landing is present at one or both ends, the load on the landing practically may be represented by concentrated loads similar to the runs. The structural analysis may also be performed by considering a load uniformly distributed on the flight of stairs. The moment in every riser is constant and is obtained from the bending moment diagram of a simply supported beam subjected to a uniform load (Fig. 18.15*b*). Example 18.3 illustrates the design of a staircase using the two assumptions of concentrated loads and uniform loads.

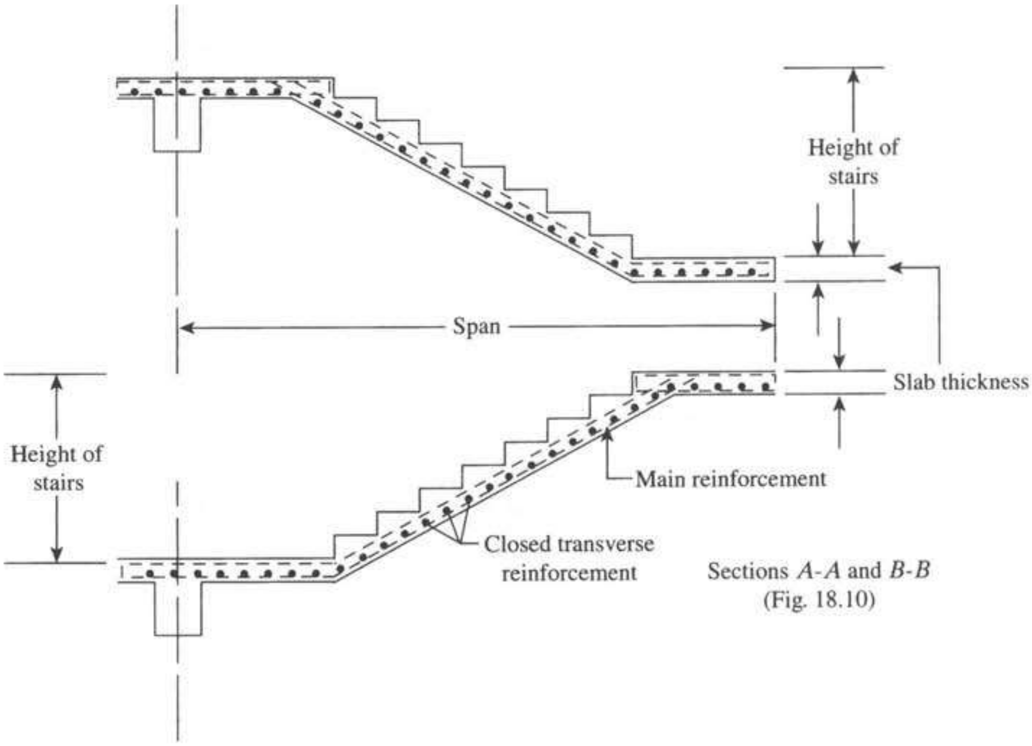


Figure 18.13 Section of a free-standing staircase.

Free-standing staircase.

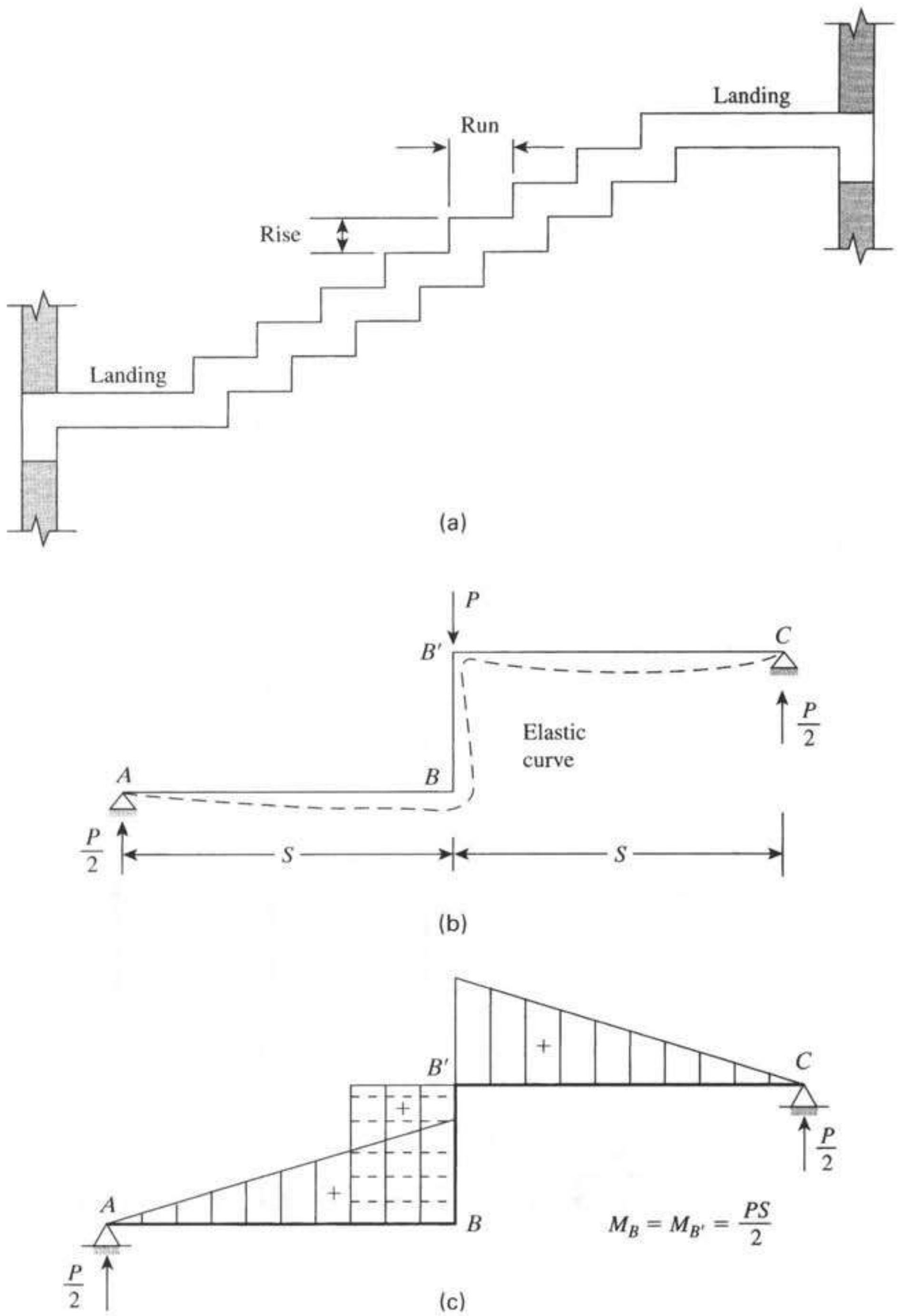


Figure 18.14 Run-riser staircase: (*a*) cross-section, (*b*) elastic curve, and (*c*) bending moment diagram.

If the stair flight is fixed or continuous at one or both ends, the moments can be obtained using any method of structural analysis. To explain this case, consider a flight of stairs that consists of two runs and is fixed at both ends (Fig. 18.16*a*). The moments at the fixed ends, A and B, due to a concentrated load at B are equal to $PL/8 = PS/4$. This result is obtained by assuming that the rise does not exist and the stairs, ABC, act as a

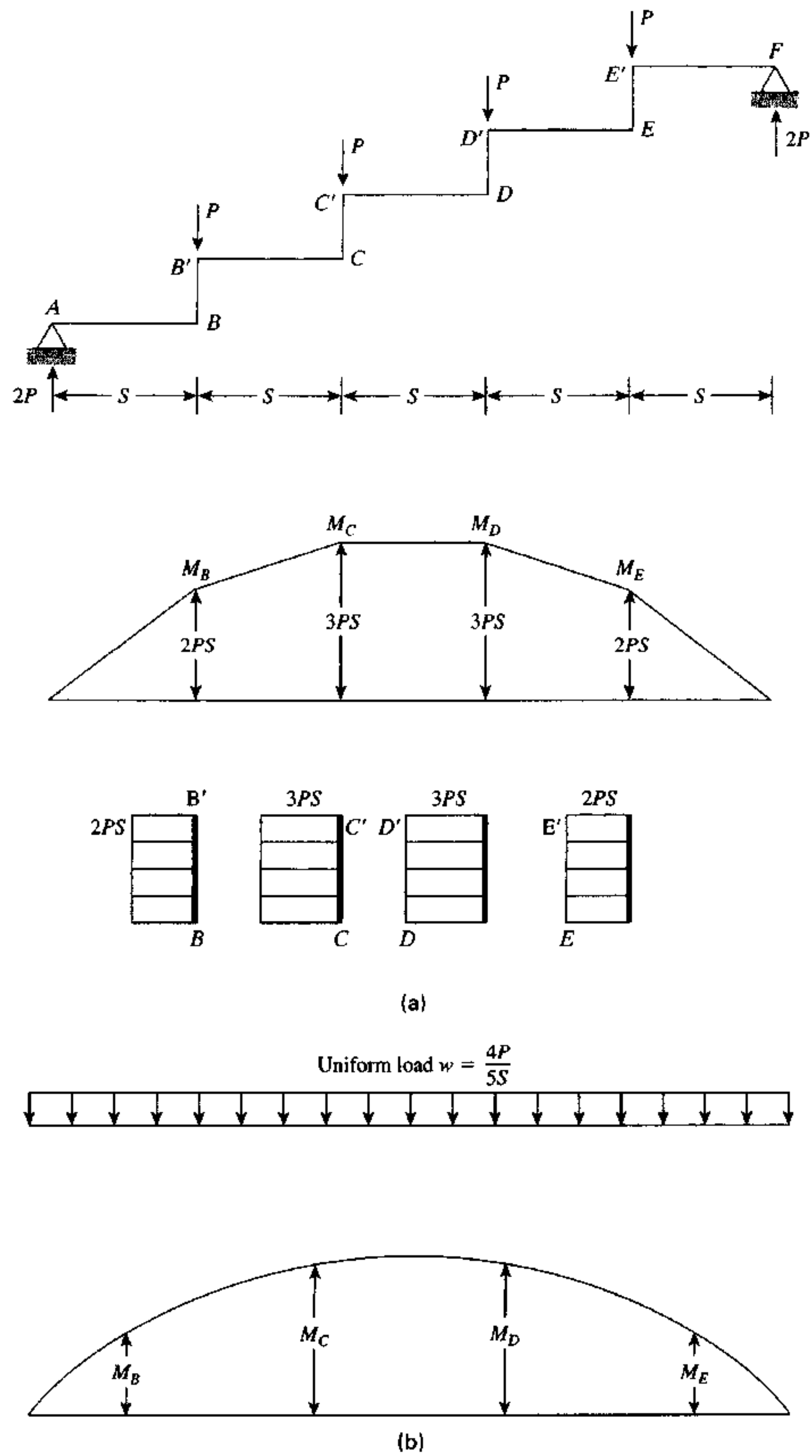
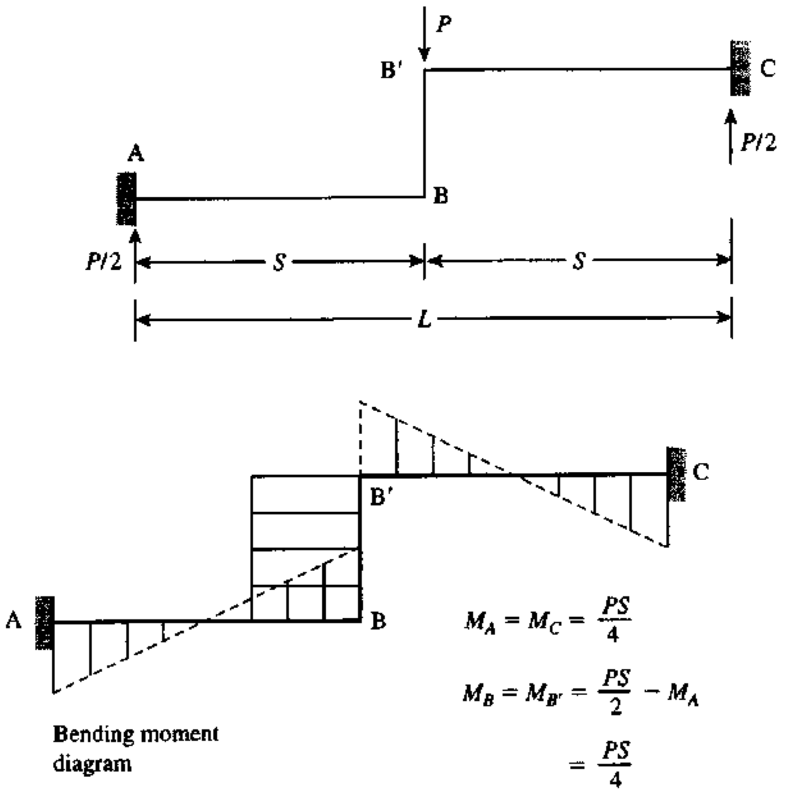
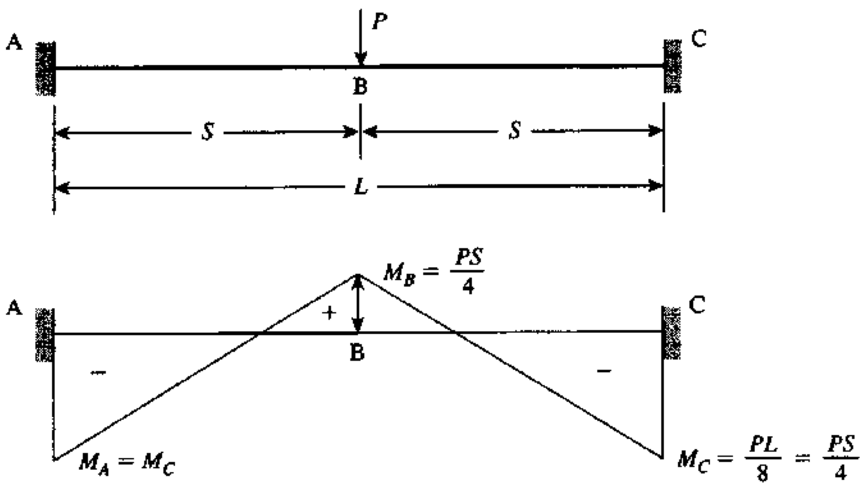


Figure 18.15 Distribution of moments: (*a*) bending moment due to concentrated loads and (*b*) bending moment due to uniform load.



(a)



(b)

Figure 18.16 Fixed-end staircase: (*a*) loaded steps and (*b*) loaded beam.

fixed-end beam subjected to a concentrated load at midspan (Fig. 18.16*b*). The moment at midspan, section B, is equal to

$$\frac{PL}{4} - M_A = \frac{PS}{2} - \frac{PS}{4} = \frac{PS}{4}$$

The bending moment of a flight of stairs with one riser is shown in Fig. 18.16*a*. Note that the moment in the riser BB' is constant, and $M_B = M'_B = PS/4$.

For a symmetrical stair flight, fixed at both ends and subjected to a number of concentrated loads at the node of each run, the moment at the fixed end can be calculated as follows:

$$M \text{ (fixed end)} = \frac{PS}{12}(n^2 - 1)$$

where

P = concentrated load at the node of the run
S = width of run
n = number of runs

When $n = 2$, then

$$M \text{ (fixed end)} = \frac{PS}{12}(4 - 1) = \frac{PS}{4}$$

which is the same result obtained earlier.

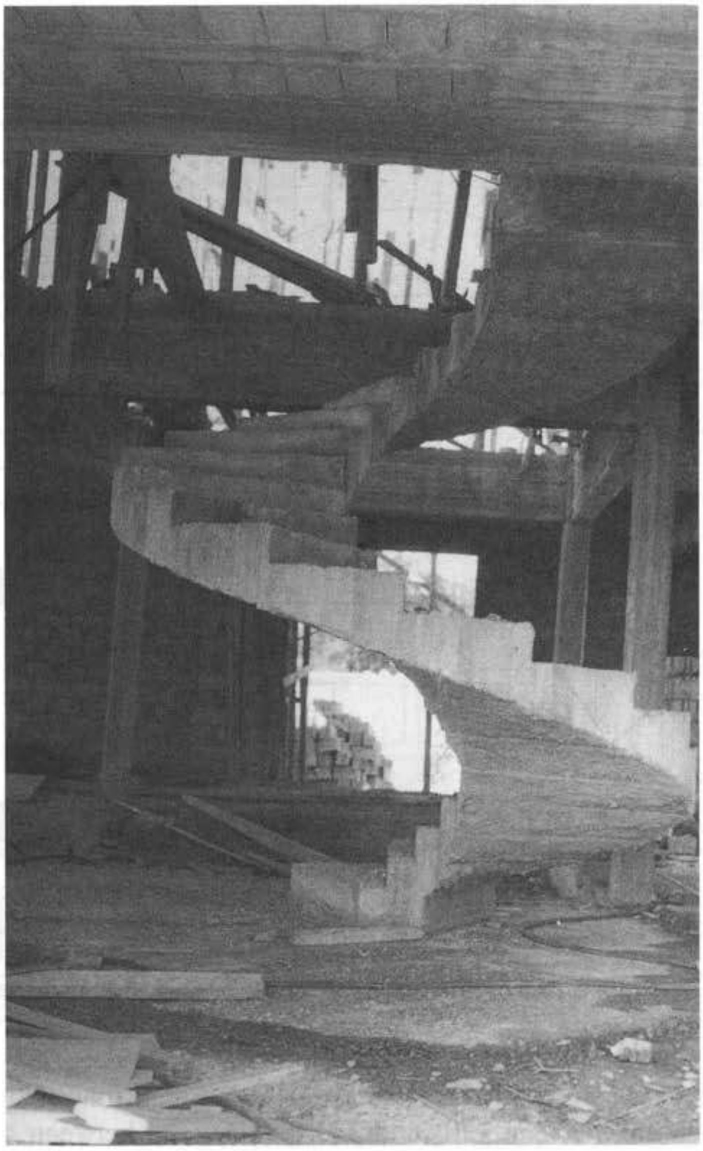
If a landing is present at one or both ends, the load on the landing may be represented by concentrated loads at spacing S.

8. *Helical stairs (open-spiral stairs):* A helical staircase is a three-dimensional structure, which usually has a circular shape in plan (Fig. 18.17). It is a distinctive type of stairs used mainly in entrance halls, theater foyers, and special low-rise office buildings. The cost of a helical stair is much higher than that of a normal staircase.

The stairs may be supported at some edges within adjacent walls or may be designed as a free-standing helical staircase, which is most popular. The structural analysis of helical staircases is complicated and was discussed by Morgan [1] and Scordelis [2] using the principles of strain energy. Design charts for helical stairs are also prepared by Cusens and Kuang [3]. Under load, the flight slab will be subjected to torsional stresses throughout. The upper landing will be subjected to tensile stresses, whereas compressive stresses occur at the bottom of the flight. The forces acting at any section may consist of vertical moment, lateral moment, torsional moment, axial force, shearing force across the waist of the stairs, and radial horizontal shearing force. The main longitudinal reinforcement consists of helical bars placed in the concrete waist of the stairs and runs from the top landing to the bottom support. The transverse reinforcement must be in a closed stirrup form to resist torsional stresses or in a U-shape lapped at about the midwidth of the stairs.

A study was made to determine the effect of the following parameters on the forces and moments that develop on helical staircases. These parameters are:

1. The total arc subtended by the helix with an angle that normally ranges from 240° to 360°. Referring to Fig. 18.17, for 16 equal runs at 20° pitch, the total arc equals 320°. If the arc is increased from 240° to 360° the vertical moment may increase by about 1,200% for a live load of 100 psf. Other forces increase appreciably.



Reinforced concrete helical staircase.

2. The width of stairs that normally ranges from 4 to 8 ft. All other parameters are constant. The increase of stair width by 100%, from 4 to 8 ft, increases the torsional moment by about 700%.
3. Variation in the interior and exterior radii (R_i and R_e) keeping the stair width of 6 ft constant. The increase in R_e (from 9 to 12 ft) and R_i (from 3 to 6 ft) with a ratio of R_e/R_i that varies between 3 and 2, increases the lateral moment by about 230%.
4. The thickness of stair slab is not as critical as the other parameters. For a variation in slab thickness between 6 and 12 in., the lateral moment increases by about 70%, while the torsional moment increases by about 170%.
5. The total height of the helical stair, h, has the least effect on all forces (for h between 9 and 15 ft) The increase in lateral moment is about 70% and in torsional moment is about 40%. Other forces decrease by about 80%.

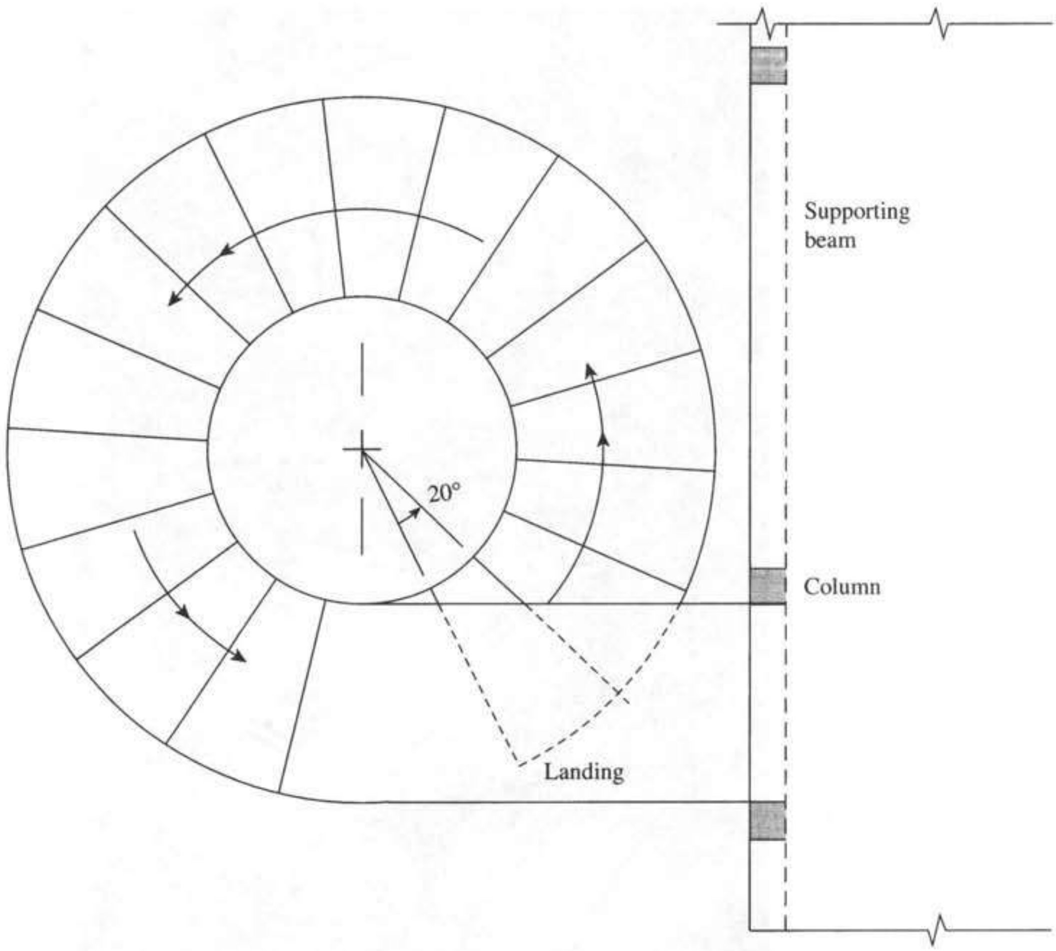


Figure 18.17 Plan of a helical staircase (16 equal runs at 20° pitch).

6. Based on this study, the possible practical dimensions may be chosen as follows: Total subtended arc between 120° and 320°, stair width between 4 and 6 ft, stairs slab thickness between 6 and 10 in., and stair height between 10 and 15 ft.

The above information can be used as a guide to achieve a proper and economical design of helical staircase.

An alternative method of providing a helical stair is to use a central helical girder located at the midwidth of the stairs and have the steps project equally on both sides of the girder. Each step is analyzed as a cantilever, and the reinforcement bars extend all along the top of the run. Precast concrete steps may be used and can be fixed to specially prepared horizontal faces at the top surfaces of the girder.

18.3 EXAMPLES

Example 18.1

Design the cantilever stairs shown in Fig. 18.18 to carry a uniform live load of 100 psf. Assume the rise of the steps equals 6.0 in. and the run equals 12 in. Use normal-weight concrete with $f'_c = 3$ ksi and $f_y = 60$ ksi.

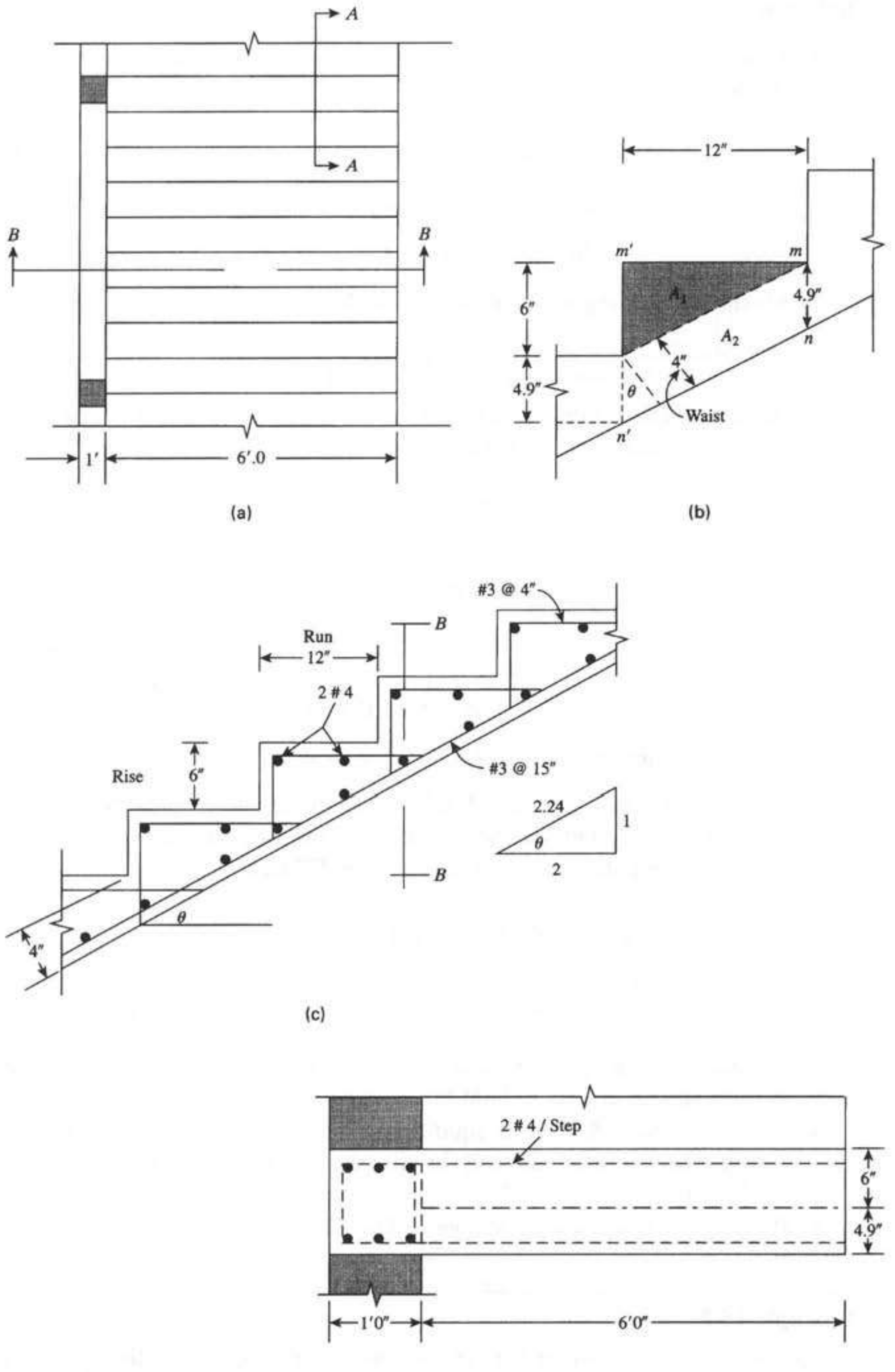


Figure 18.18 Example 18.1: cantilever stairs: (*a*) plan, (*b*) section in one step, (*c*) section *A-A*, and (*d*) section *B-B*.

Solution

1. Loads: Assume the thickness of the slab (waist) is 4.0 in. Weight of the assumed slab (areas A_1 and A_2) is

$$\text{trapezoidal area } mnn'm' = \left(\frac{4.9 + 10.9}{2 \times 12}\right)(1)(150) = 98.8 \text{ lb/ft}$$

Refer to Fig. 18.18*b*. Assume the weight of the step cover is 5 lb/ft. Total D.L. = 119 lb/ft.

$$W_u = 1.2D + 1.6L = 1.2 \times 119 + 1.6 \times 100 = 302 \text{ lb/ft}$$

2. Maximum bending moment per step is $W_u l^2/2$.

$$M_u = \frac{0.302}{2}(6)^2 = 5.44 \text{ K·ft}$$

Average thickness of a step is (10.9 + 4.9)/2 = 7.9 in. Let d = 7.9 − 0.75 (concrete cover) − 0.25 ($\frac{1}{2}$ bar diameter) = 6.9 in.

$$M_u = \phi A_s f_y \left(d - \frac{a}{2}\right) \qquad \text{Assume } a = 0.5 \text{ in.}$$

$$A_s = \frac{5.44 \times 12}{0.9 \times 60(6.9 - 0.25)} = 0.19 \text{ in.}^2$$

Check

$$a = \frac{A_s f_y}{0.85\, f'_c b} = \frac{0.19 \times 60}{0.85 \times 3 \times 12} = 0.38 \text{ in.} \qquad \text{(close to 0.5 in.)}$$

$$\text{Minimum } A_s = 0.00333(12)(6.9) = 0.28 \text{ in.}^2$$

Use two no. 4 bars per step. A smaller depth may be adopted, but to avoid excessive deflection and vibration of stairs, a reasonable depth must be chosen.

3. Check flexural shear at a distance d from the face of the support.

$$V_u = 0.315\left(6 - \frac{6.9}{12}\right) = 1.7 \text{ K}$$

$$\phi V_c = 0.75(2\lambda\sqrt{f'_c}bd) = \frac{0.75}{1000} \times 2 \times 1 \times \sqrt{3000} \times 12 \times 6.9 = 6.8 \text{ K}$$

Because $V_u < \phi V_c/2$, no shear reinforcement is required. But it is recommended to use no. 3 stirrups spaced at 4 in. to hold the main reinforcement.

4. The stairs must remain in equilibrium either by the weight of the wall or by a reinforced concrete beam within the wall. In this case, the beam will be subjected to torsional moment of 5.7 K-ft/ft.

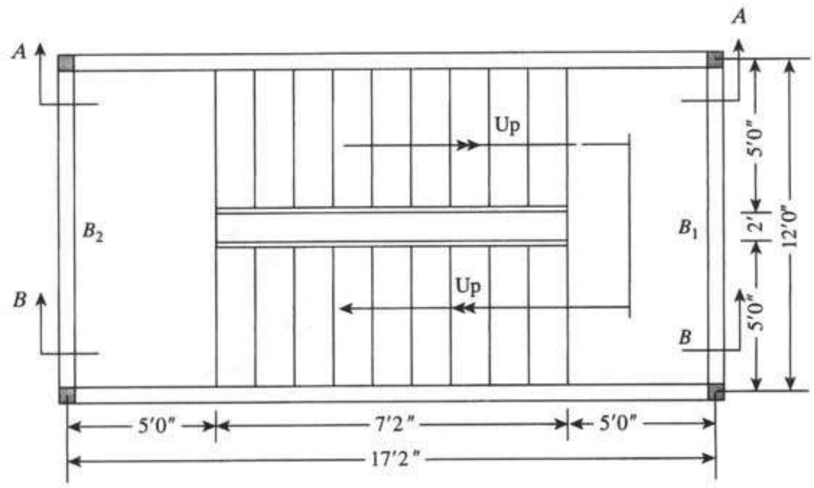
5. Reinforcement details are shown in Fig. 18.18.

Example 18.2

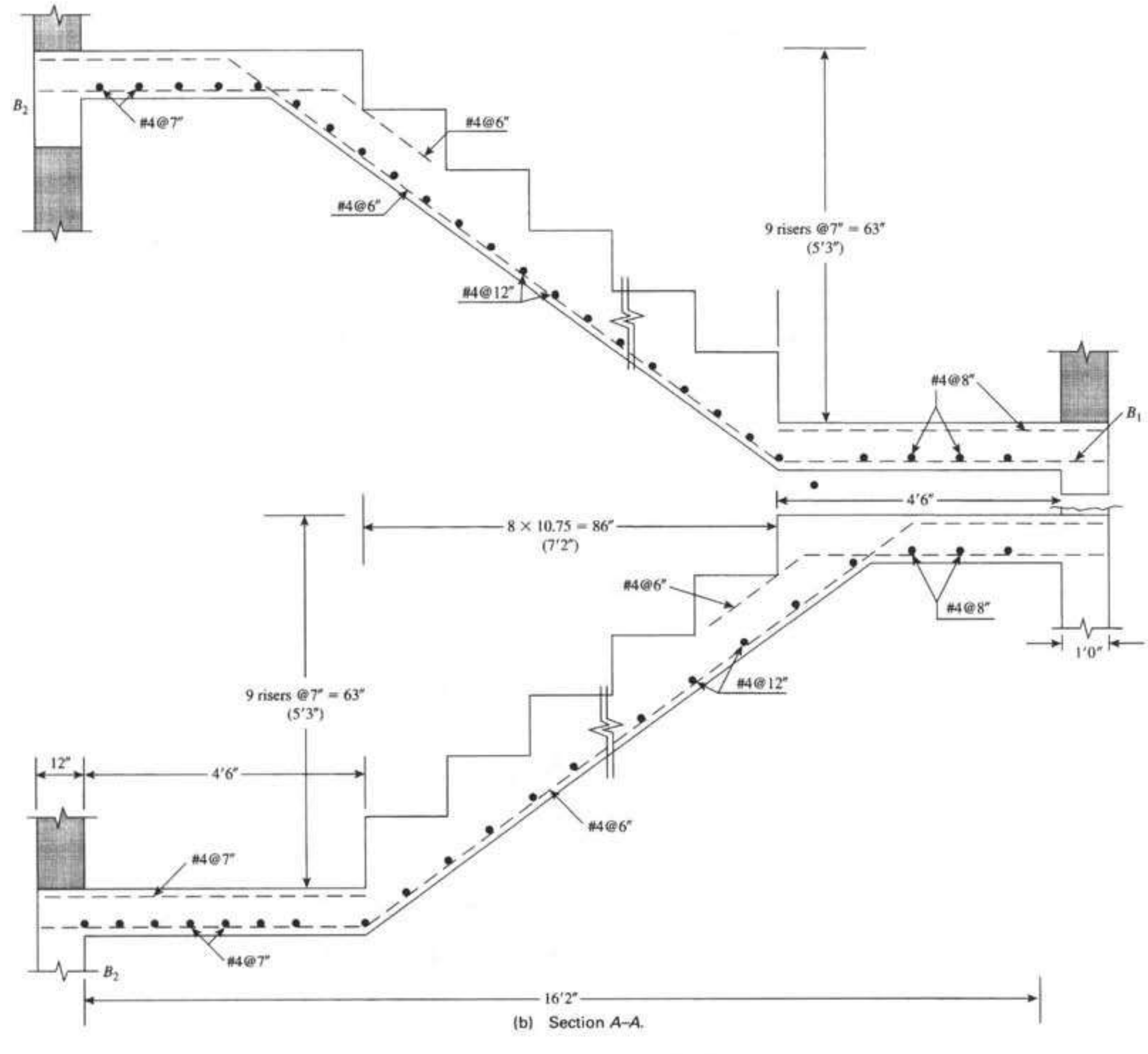
Design the staircase shown in Fig. 18.19, which carries a uniform live load of 120 psf. Assume a rise of 7.0 in. and a run of 10.75 in. Use $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Structural system: If no stringer beam is used, one of the four possible solutions shown in Fig. 18.2 may be adopted. When no intermediate supports are used, the flight of stairs will be supported at the ends of the upper and lower landings. This structural system will be adopted in this example.

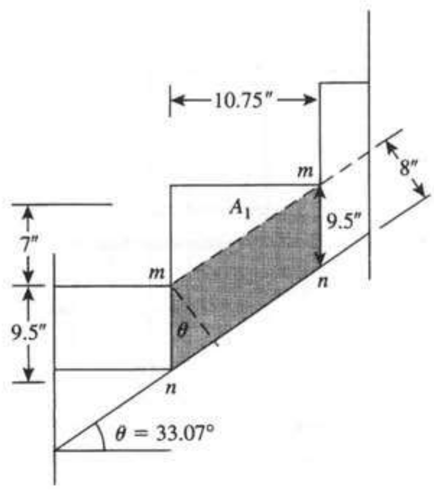


(a)

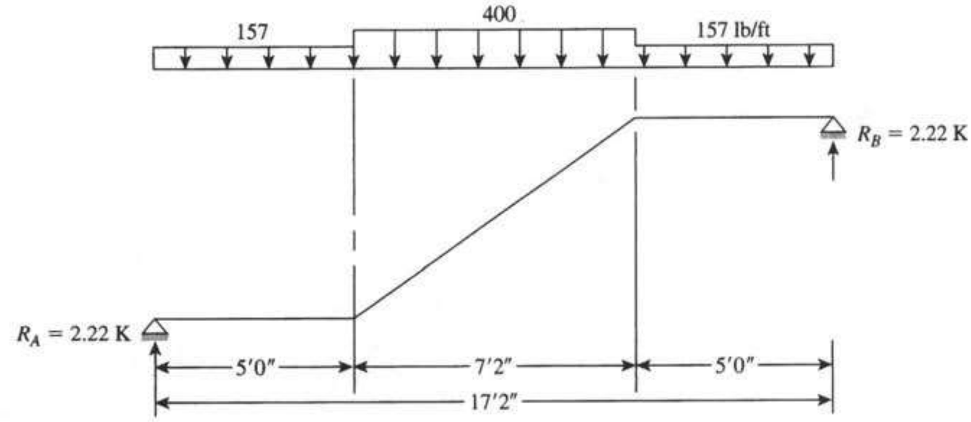


(b) Section A–A.

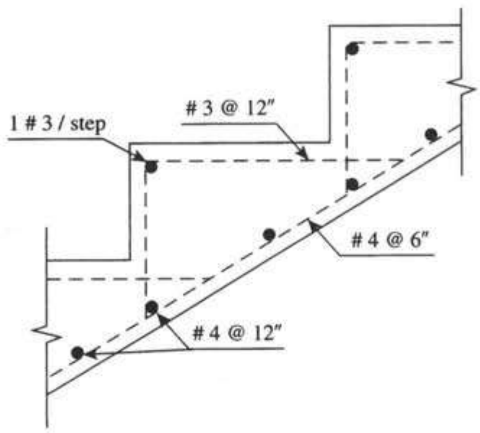
Figure 18.19 Example 18.2.



(c) Section in steps.



(d) Loads.



(e) Reinforcement details.

Figure 18.19 *(continued)*

2. Loads: Assume the thickness of the slab (waist) is 8.0 in.

$$\text{Weight of one step} = \text{trapezoidal area} \times 150 \text{ pcf}$$

$$= \left(\frac{9.5 + 16.5}{2 \times 12}\right)\left(\frac{10.75}{12}\right)(150) = 145.6 \text{ lb per step}$$

$$\text{Average weight per foot length} = 145.6 \left(\frac{12}{10.75}\right) = 162.5 \text{ lb/ft}$$

$$\text{Weight of 8 in. landing} = \frac{8}{12} \times 150 = 100 \text{ lb/ft}$$

Assume weight of step cover is 7.5 lb/ft and weight of landing = 2 lb/ft. The total D.L. on stairs is 162.5 + 7.5 = 170 lb/ft. The total D.L. on landing is 100 + 2 = 102 lb/ft.

$$W_u \text{ (on stairs)} = 1.2 \times 170 + 1.6 \times 120 = 400 \text{ lb/ft}$$

$$W_u \text{ (on landing)} = 1.2 \times 102 + 1.6 \times 120 = 314 \text{ lb/ft}$$

Because the load on the landing is carried into two directions, only half the load will be considered in each direction.

3. Calculate the maximum bending moment and steel reinforcement (Fig. 18.19*d*):

a. The moment at midspan is

$$M_u = 2.22\left(\frac{17.2}{2}\right) - (0.157 \times 5)(6.1) - (0.400)\frac{(3.6)^2}{2} = 11.71 \text{ K·ft}$$

Let $d = 8.0 - 0.75$ (concrete cover) $- 0.25$ $\left(\frac{1}{2}\text{ bar diameter}\right) = 7.0$ in.

b. $M_u = \phi A_s f_y (d - a/2)$; assume $a = 0.8$ in.

$$A_s = \frac{11.71 \times 12}{0.9 \times 60(7 - 0.4)} = 0.4 \text{ in.}^2$$

Check:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{0.4 \times 60}{0.85 \times 3 \times 12} = 0.78 \text{ in.}, \qquad c = 0.92 \text{ in.}$$

$$\text{Minimum } A_s = 0.0033 \times 12 \times 8 = 0.32 \text{ in.}^2 < 0.4 \text{ in.}^2$$

Use no. 4 bars spaced at 6 in. ($A_s = 0.4 \text{ in.}^2$). For 5-ft-wide stairs, use 10 no. 4 bars.

$$d_t = 7 \text{ in.} \qquad c = 0.92 \text{ in.}$$

Net tensile strain,

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) = 0.0198 \text{ in.}$$

$$\varepsilon_t > 0.005 \qquad \phi = 0.9$$

c. Transverse reinforcement must be provided to account for shrinkage.

$$A_s = 0.0018 \times 12 \times 8 = 0.18 \text{ in.}^2/\text{ft}$$

Use no. 4 bars spaced at 12 in. ($A_s = 0.2 \text{ in.}^2$).

d. If the slab will be cast monolithically with its supporting beams, additional reinforcement must be provided at the top of the upper and lower landings. Details of stair reinforcement are shown in Fig. 18.19.

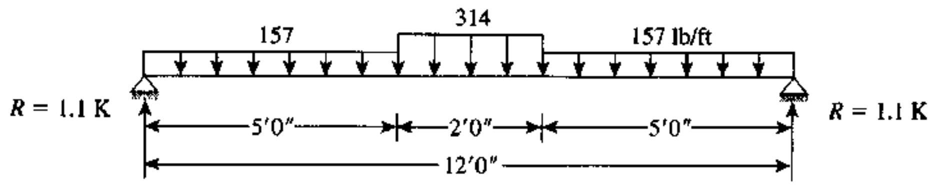


Figure 18.20 Example 18.2: loads on landing.

4. Minimum slab thickness for deflection is

$$\frac{L}{25} = \frac{17.2 \times 12}{25} = 8.26 \text{ in.}$$

(for a simply supported slab). In the case presented here, where the slab ends are cast with the supporting beams and additional negative reinforcement is provided, minimum thickness can be assumed to be

$$\frac{L}{28} = 7.4 \text{ in.} < 8 \text{ in. used}$$

5. Design of landings: Considering a 1-ft length of the landing, the load on the landing is as shown in Fig. 18.20. The middle 2 ft will carry a full load, whereas the two 5-ft lengths on each side will carry half the ultimate load.

$$\text{Maximum bending moment} = (1.1 \times 6) - (0.157 \times 5)(3.5) - (0.314)\frac{(1)^2}{2} = 3.7 \text{ K·ft}$$

Because the bars in the landing will be placed on top of the main stair reinforcement,

$$d = 8.0 - 0.75 - \frac{4}{8} - 0.25 = 6.375 \text{ in.} \qquad \text{say, } 6.3 \text{ in.}$$

Assume $a = 0.4$ in.

$$A_s = \frac{3.7 \times 12}{0.9 \times 60(6.3 - 0.2)} = 0.14 \text{ in.}^2 < A_s(\text{min}) \text{ of } 0.32 \text{ in.}^2$$

Use $A_s = 0.32 \text{ in.}^2$ Use no. 4 bars spaced at 7 in. ($A_s = 0.34 \text{ in.}^2$).

6. The transverse beams at the landing levels must be designed to carry loads from stairs (2.3 K/ft) in addition to their own weight and the weight of the wall above.
7. Check shear as usual.

Example 18.3

Design the simply supported run-riser stairs shown in Fig. 18.21 for a uniform live load of 120 psf. Use $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Loads: Assume the thickness of runs and risers is 6 in. The concentrated load at each riser is calculated as follows (refer to Fig. 18.21*b*). Due to dead load per foot depth of run,

$$P_D = \left(\frac{16}{12} \times \frac{6}{12} + \frac{1}{12} \times \frac{6}{12}\right) 150 = 106 \text{ lb}$$

Note that the node dead load on the landing is less than 106 lb but can be assumed to be equal to P_D to simplify calculations. Due to live load peer foot depth of run, $P_L = \frac{10}{12} \times (120) = 100$ lb.

$$\text{Factored load, } P_u = 1.2P_D + 1.6P_L$$

$$= 1.2 \times 106 + 1.6 \times 100 = 290 \text{ lb}$$

2. Calculate the bending moments at midspan: Loads in this example are symmetrical about midspan section B. Reaction at A, R_A is $\frac{1}{2}(15)(290) = 2175$ lb $= (7\frac{1}{2}P \times 290)$

$$\text{Moment at } B = R_A(8S) - 7P_u(4S)$$

$$= 2.175(8 \times 10) - 7(0.29)(4 \times 10) = 92.8 \text{ K·in.}$$

3. Calculate the reinforcement required at midspan section: For $h = 6$ in., $d = 6 - 1.0 = 5.0$ in.,

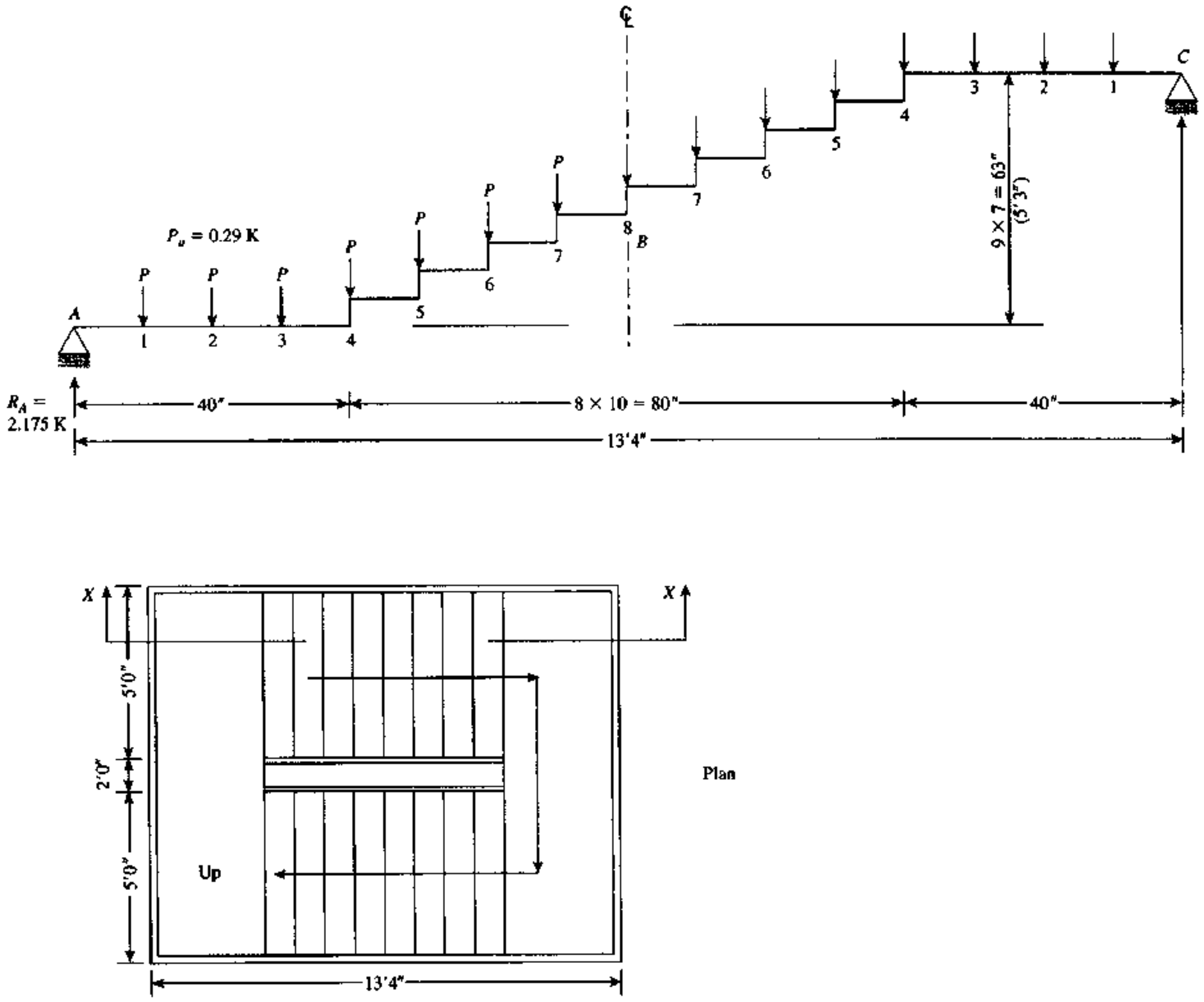
$$R_u = \frac{M_u}{bd^2} = \frac{92.8 \times 1000}{12(5.0)^2} = 309 \text{ psi}$$

For $f'_c = 3$ ksi, $f_y = 60$ ksi, and $R_u = 309$ psi, the steel ratio is $\rho = 0.0061 < \rho_{max} = 0.0135$ $(\phi = 0.9)$.

$$A_s = 0.0061 \times 12 \times 5.0 = 0.366 \text{ in.}^2$$

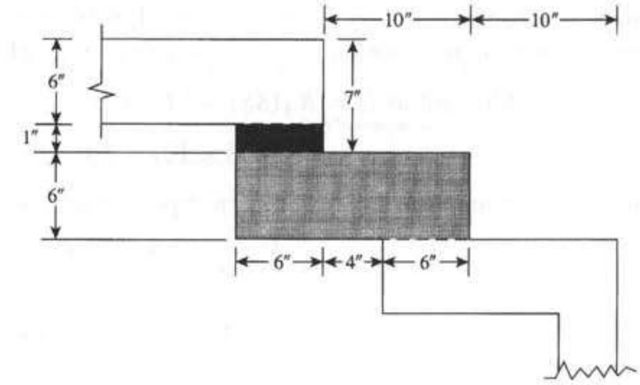
Use no. 4 bars spaced at 6 in. ($A_s = 0.39$ in.^{2}) horizontally and vertically in closed stirrup form. For distribution bars, use minimum ρ of 0.0018.

$$A_s = 0.0018 \times 16 \times 6 = 0.18 \text{ in.}^2$$

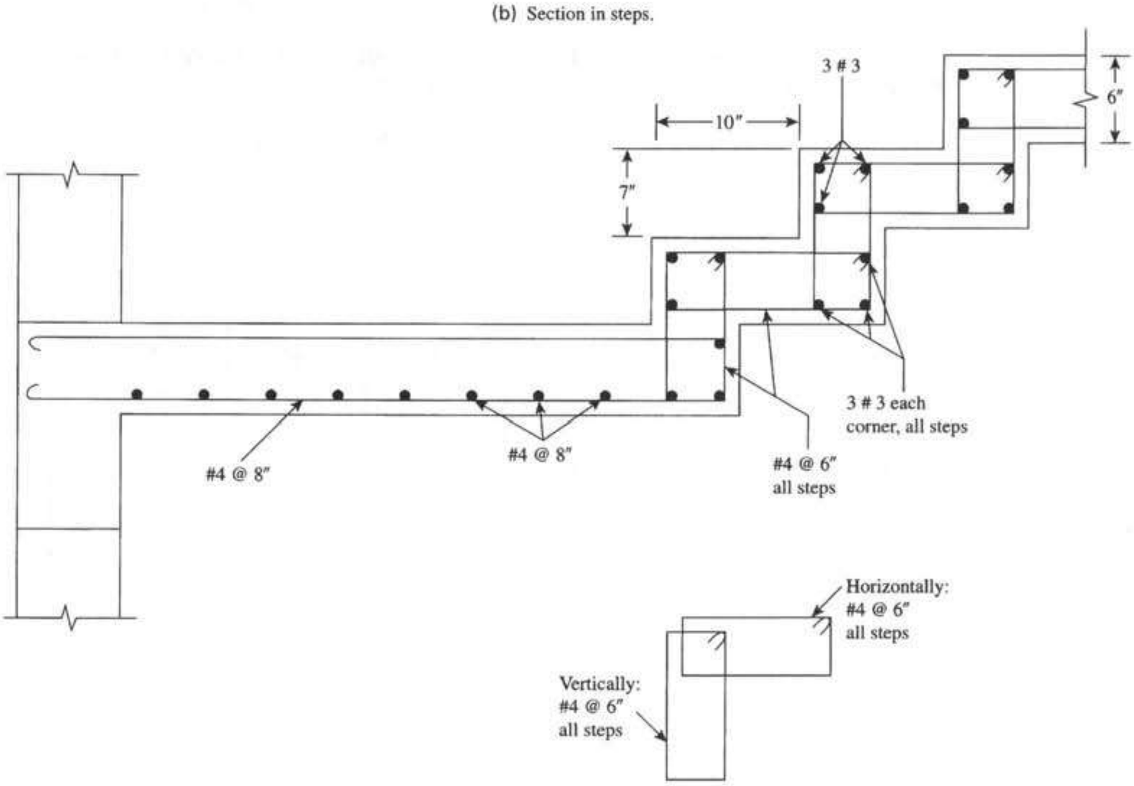


(a) Plan and section X–X

Figure 18.21 Example 18.3.



(b) Section in steps.



(c) Reinforcement details.

Figure 18.21 (*continued*)

Use no. 3 bars spaced at 6 in. ($A_s = 0.22\,\text{in.}^2$). For each step corner, use three no. 3 bars ($A_s = 0.33\,\text{in.}^2$), as shown in Fig. 18.21*c*.

4. The moments and reinforcement required for other sections can be prepared in table form, as follows:

Location	A	1	2	3	4	5	6	7	8
B.M. (K in.)	0	22	41	57	70	80	87	91	92.8
R_u (psi)	0	73	137	190	233	267	290	303	309
ρ (%)	0	0.18	0.26	0.38	0.46	0.52	0.58	0.60	0.61
A_s (in.^{2})	0	0.11	0.16	0.23	0.28	0.31	0.35	0.36	0.37

Use no. 4 bars at 8 in. for the landing and no. 4 bars at 6 in. for the steps. For distribution bars, use minimum ρ of 0.0018. For $A_s = 0.18\,\text{in.}^2$, use no. 4 bars spaced at 8 in. in the landing. Details of reinforcement are shown in Fig. 18.21*c*.

5. Check reinforcement required in the transverse direction of landing: Load per square foot on the landing is $\frac{290}{10} \times 12 = 348$ psf.

$$M_u = \frac{0.348}{8}(12)^2 \times 12 = 75 \text{ K·in.}$$

$$R_u = \frac{75 \times 1000}{12(5.0)^2} = 250 \text{ psi} \qquad \rho = 0.0049 \qquad A_s = 0.29 \text{ in.}^2$$

Use no. 4 bars spaced at 8 in. ($A_s = 0.29\,\text{in.}^2$)

6. If a uniform load is assumed to be acting on the flight of stairs, similar results will be obtained. For example, ultimate node load was calculated to be 290 lb acting over a 10-in. run width. Load per foot is $\frac{290}{10} \times 12 = 348$ lb/ft. Maximum moment is at midspan, section B:

$$M_u = \frac{0.348}{8}(13.33)^2 = 92.8 \text{ K·in.}$$

Moments at other sections can be easily calculated, and the design can be arranged in a table form, as explained in step 4.

SUMMARY

Sections 18.1–18.2

The different types of stairs are single and multiple flights, cantilever and precast concrete flights, free-standing and helical staircases, and run-riser stairs.

Section 18.3

Design examples are presented in this section.

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PROBLEMS

18.1 Design a typical flight of the staircase shown in Fig. 18.22, which is a part of a multistory building. The height between the concrete floors is 10 ft (3.0 m). The stairs are supported at the ends of the landings and carry a live load equal to 120 psf (5.75 kN/m^2); $f_c' = 3$ ksi (20 MPa) and $f_y = 60$ ksi (400 MPa)

18.2 Repeat Problem 18.1 if the stairs are supported by four transverse beams at A, B, C, and D and the live load is increased to 150 psf (7.2 kN/m^2).

18.3 The stairs shown in Fig. 18.23 are to be designed for a live load equal to 100 psf (4.8 kN/m). The stairs are supported by beams, as shown. Design the stairs and the supporting beams for $f_c' = 3$ ksi (20 MPa) and $f_y = 60$ ksi (400 MPa).

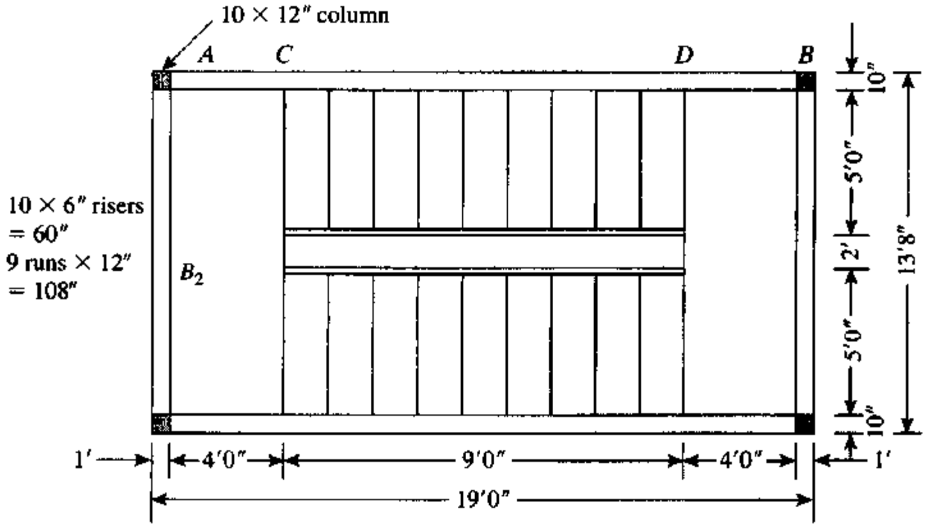


Figure 18.22 Problem 18.1.

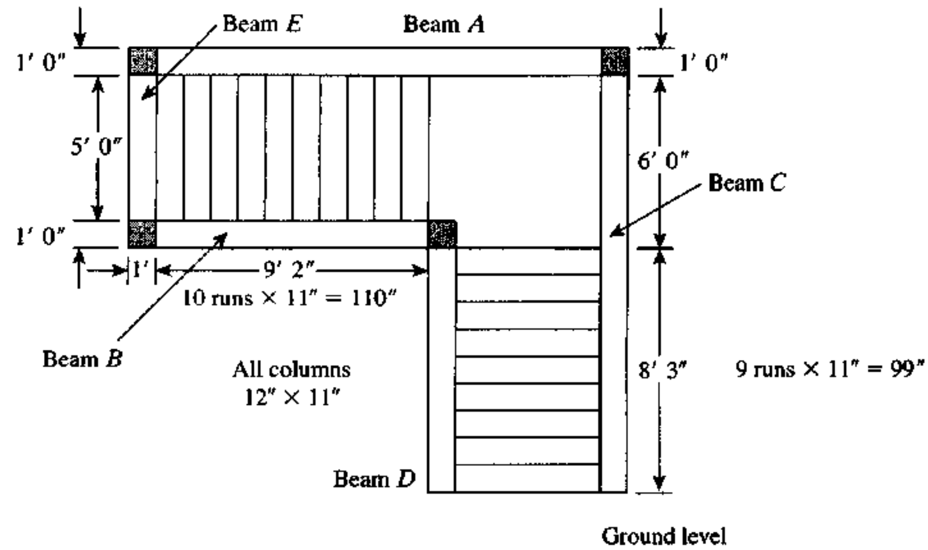


Figure 18.23 Problem 18.3.

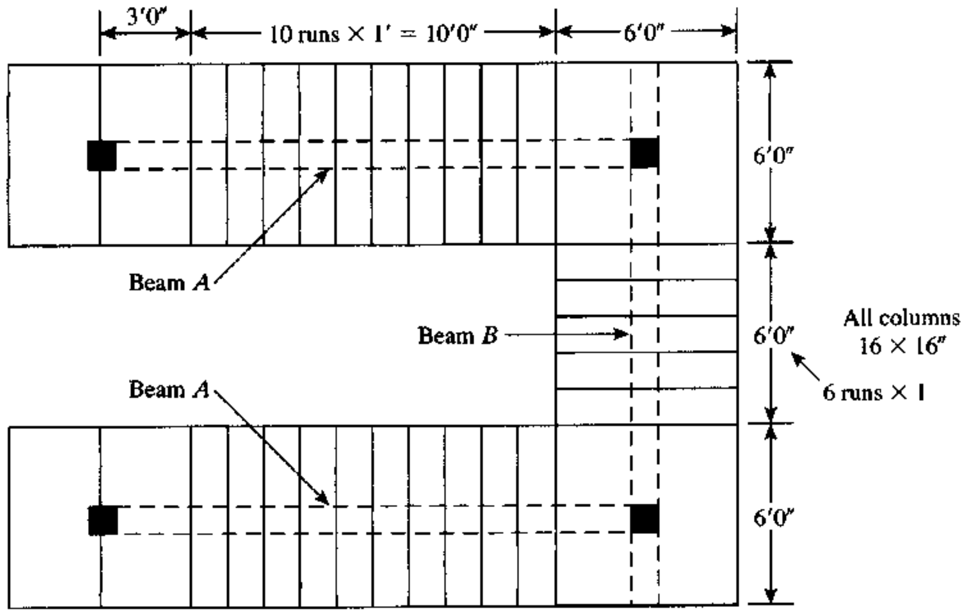


Figure 18.24 Problem 18.4.

18.4 Design a typical flight of stairs in a public building for the staircase arrangement shown in Fig. 18.24. The stairs are supported by central beams, *A* and *B*. Design only one flight and the supporting beams *A* and *B*. The runs are 1.0 ft (300 m) deep and the rises are 6.5 in. high. Use $f_c' = 3$ ksi (20 MPa), $f_y = 60$ ksi (400 MPa), and a live load equal to 80 psf (3.85 kN/m^2).

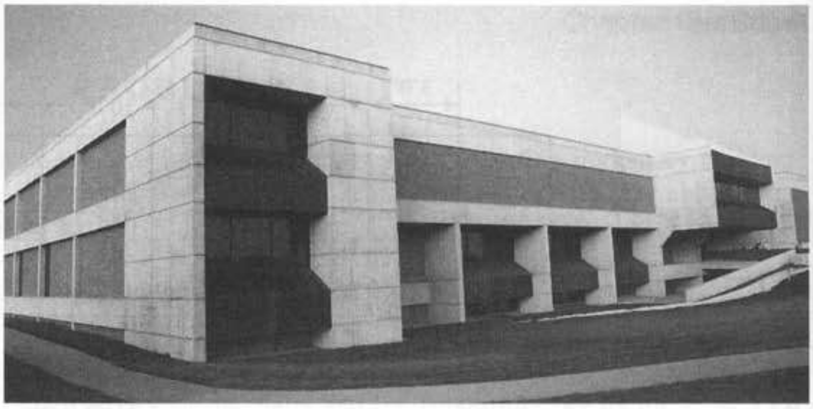
Note : Design the beams for bending moments and shear, and neglect torsional moments caused by loading one-half of the steps.

18.5 Repeat Example 18.3 if the run is 12 in. (300 mm) and the rise is 6 in. (150 mm).

18.6 Repeat Example 18.3 if the landing is 5 ft (6 × 10″), runs are 8.33 ft (10 × 10″), risers at 5.5 ft (11 × 6″), and the live load is 120 psf.

CHAPTER 19

INTRODUCTION TO PRESTRESSED CONCRETE



Library building, South Dakota State University, Brookings, South Dakota.

19.1 PRESTRESSED CONCRETE

19.1.1 Principles of Prestressing

To prestress a structural member is to induce internal, permanent stresses that counteract the tensile stresses in the concrete resulting from external loads; this extends the range of stress that the member can safely withstand. Prestressing force may be applied either before or at the same time as the application of the external loads. Stresses in the structural member must remain, everywhere and for all states of loading, within the limits of stress that the material can sustain indefinitely. The induced stresses, primarily compressive, are usually created by means of hightensile steel tendons, which are tensioned and anchored to the concrete member. Stresses are transferred to the concrete either by the bond along the surface of the tendon or by anchorages at the ends of the tendon.

To explain this discussion, consider a beam made of plain concrete, which has to resist the external gravity load shown in Fig. 19.1*a*. The beam section is chosen with the tensile flexural stress as the critical criterion for design; therefore, an uneconomical section results. This is because concrete is considerably stronger in compression than in tension. The maximum flexural tensile strength of concrete, the modulus of rupture, f_r, is equal to $7.5\lambda\sqrt{f'_c}$ (Fig. 19.1*a*).

In normal reinforced concrete design, the tensile strength of concrete is ignored and steel bars are placed in the tension zone of the beam to resist the tensile stresses, whereas the concrete resists the compressive stresses (Fig. 19.1*b*).

In prestressed concrete design, an initial compressive stress is introduced to the beam to offset or counteract the tensile stresses produced by the external loads (Fig. 19.1*c*). If the induced compressive stress is equal to the tensile stress at the bottom fibers, then both stresses cancel themselves, whereas the compressive stress in the top fibers is doubled; in this case, the whole section is in compression. If the induced compressive stress is less than the tensile stress at the bottom fibers, these fibers will be in tension, whereas the top fibers are in compression.

In practice, a concrete member may be prestressed in one of the following methods.

1. *Posttensioning:* In posttensioning, the steel tendons are tensioned after the concrete has been cast and hardened. Posttensioning is performed by two main operations: tensioning the steel wires or strands by hydraulic jacks that stretch the strands while bearing against the ends of the member and then replacing the jacks by permanent anchorages that bear on

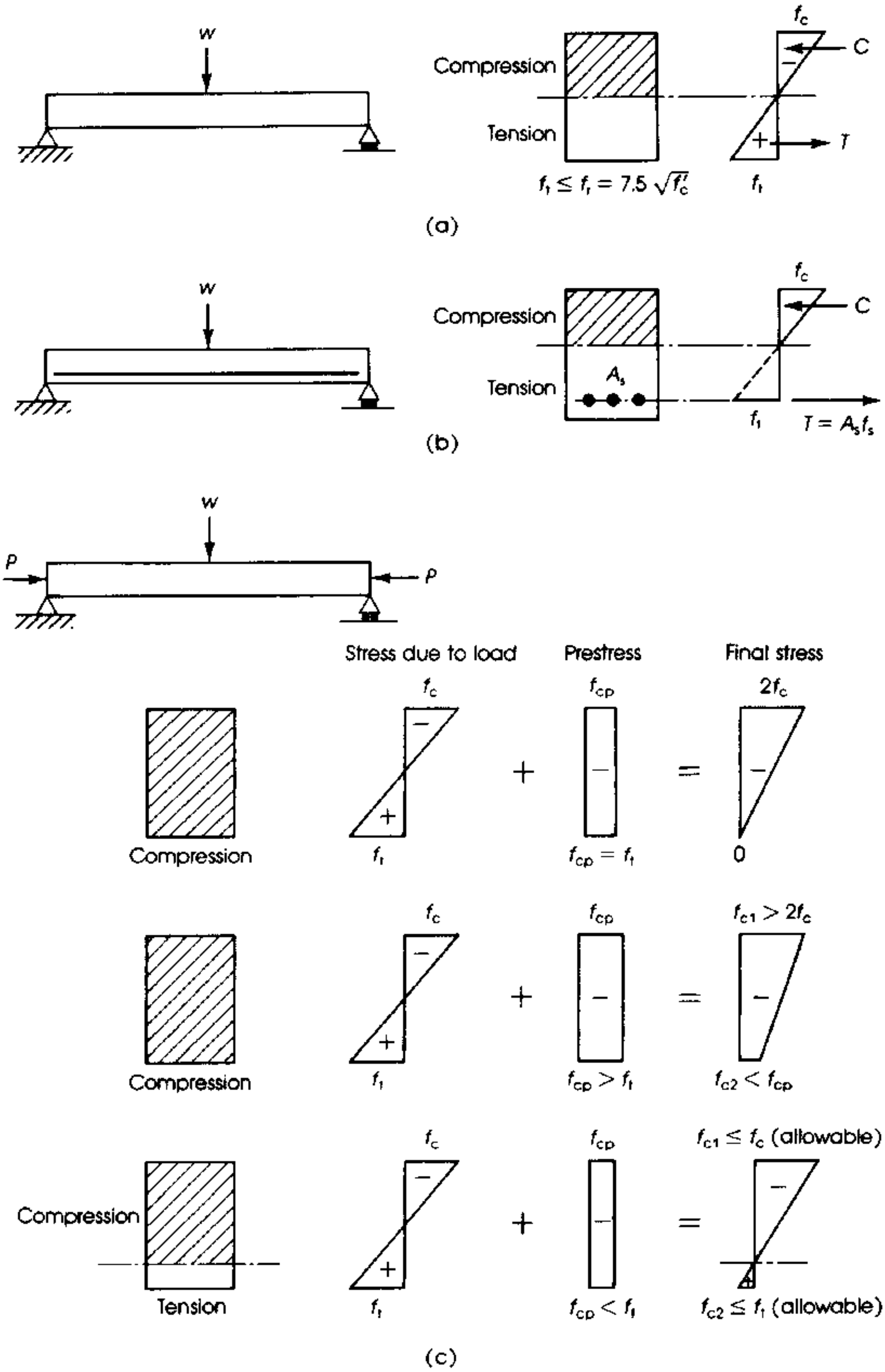


Figure 19.1 Effect of prestressing: (*a*) plain concrete, (*b*) reinforced concrete, and (*c*) prestressed concrete.

the member and maintain the steel strands in tension. A tendon is generally made of wires, strands, or bars. Wires and strands can be tensioned in groups, whereas bars are tensioned one at a time. In the posttensioning process, the steel tendons are placed in the formwork before the concrete is cast and the tendons are prevented from bonding to the concrete by waterproof paper wrapping or a metal duct (sheath). Tendons bonded to the concrete are called bonded tendons. Unbonded tendons, left without grout or coated with grease, have no bond throughout the length of the tendon.

2. *Pretensioning:* In pretensioning, the steel tendons are tensioned before the concrete is cast. The tendons are temporarily anchored against some abutments and then cut or released after the concrete has been placed and hardened. The prestressing force is transferred to the concrete by the bond along the length of the tendon. Pretensioning is generally done in precasting plants in permanent beds, which are used to produce pretensioned precast concrete elements for the building industry.
3. *External prestressing:* In external prestressing, the prestressing force is applied by flat jacks placed between the concrete member ends and permanent rigid abutments. The member does not contain prestressing tendons, as in the previous two methods (also called internal prestressing). External prestressing is not easy in practice because shrinkage and creep in concrete tend to reduce the induced compressive stresses unless the prestressing force can be adjusted.

The profile of the tendons may be straight, curved (bent), or circular, depending on the design of the structural member. Straight tendons are generally used in solid and hollow-cored slabs, whereas bent tendons are used in beams and most structural members. Circular tendons are used in circular structures such as tanks, silos, and pipes. The prestressing force may be applied in one or more stages, either to avoid overstressing concrete or in cases when the loads are applied in stages. In this case, part of the tendons are fully prestressed at each stage.

A considerable number of prestressing systems have been devised, among them Freyssinet, Magnel Blaton, B.B.R.V., Dywidag, CCL, Morandi, VSL, Western Concrete, Prescon, and INRYCO. The choice of the prestressing system for a particular job can sometimes be a problem. The engineer should consider three main factors that govern the choice of the system:

1. The magnitude of the prestressing force required
2. The geometry of the section and the space available for the tendons
3. Cost of the prestressing system (materials and labor)

The following example illustrates some of the features of prestressed concrete.

Example 19.1

For the simply supported beam shown in Fig. 19.2, determine the maximum stresses at midspan section due to its own weight and the following cases of loading and prestressing:

1. A uniform live load of 900 lb/ft
2. A uniform live load of 900 lb/ft and an axial centroidal longitudinal compressive force of $P = 259.2\,\text{K}$
3. A uniform live load of 2100 lb/ft and an eccentric longitudinal compressive force $P = 259.2\,\text{K}$ acting at an eccentricity $e = 4\,\text{in.}$
4. A uniform live load of 2733 lb/ft and an eccentric longitudinal compressive force $P = 259.2\,\text{K}$ acting at the maximum practical eccentricity for this section ($e = 6\,\text{in.}$)

5. The maximum live load when $P = 259.2\,\text{K}$ acting at $e = 6\,\text{in}$.
   Use $b = 12\,\text{in.}$, $h = 24\,\text{in.}$, normal-weight concrete with $f_c' = 4500$ psi, and an allowable $f_c' = 2050$ psi.

Solution

1. Stresses due to dead and live loads only are Self-weight of beam $= (1 \times 2) \times 150 = 300\,\text{lb/ft}$

$$\text{Dead-load moment } M_{\text{D.L.}} = \frac{wL^2}{8} = \frac{0.300(24)^2}{8} = 21.6\ \text{K·ft}$$

Stresses at the extreme fibers are

$$\sigma = \frac{Mc}{I} = \frac{M(h/2)}{bh^3/12} = \frac{6M}{bh^2}$$

$$\sigma_D = \frac{6 \times 21.6 \times 12{,}000}{12(24)^2} = \pm 225\ \text{psi}$$

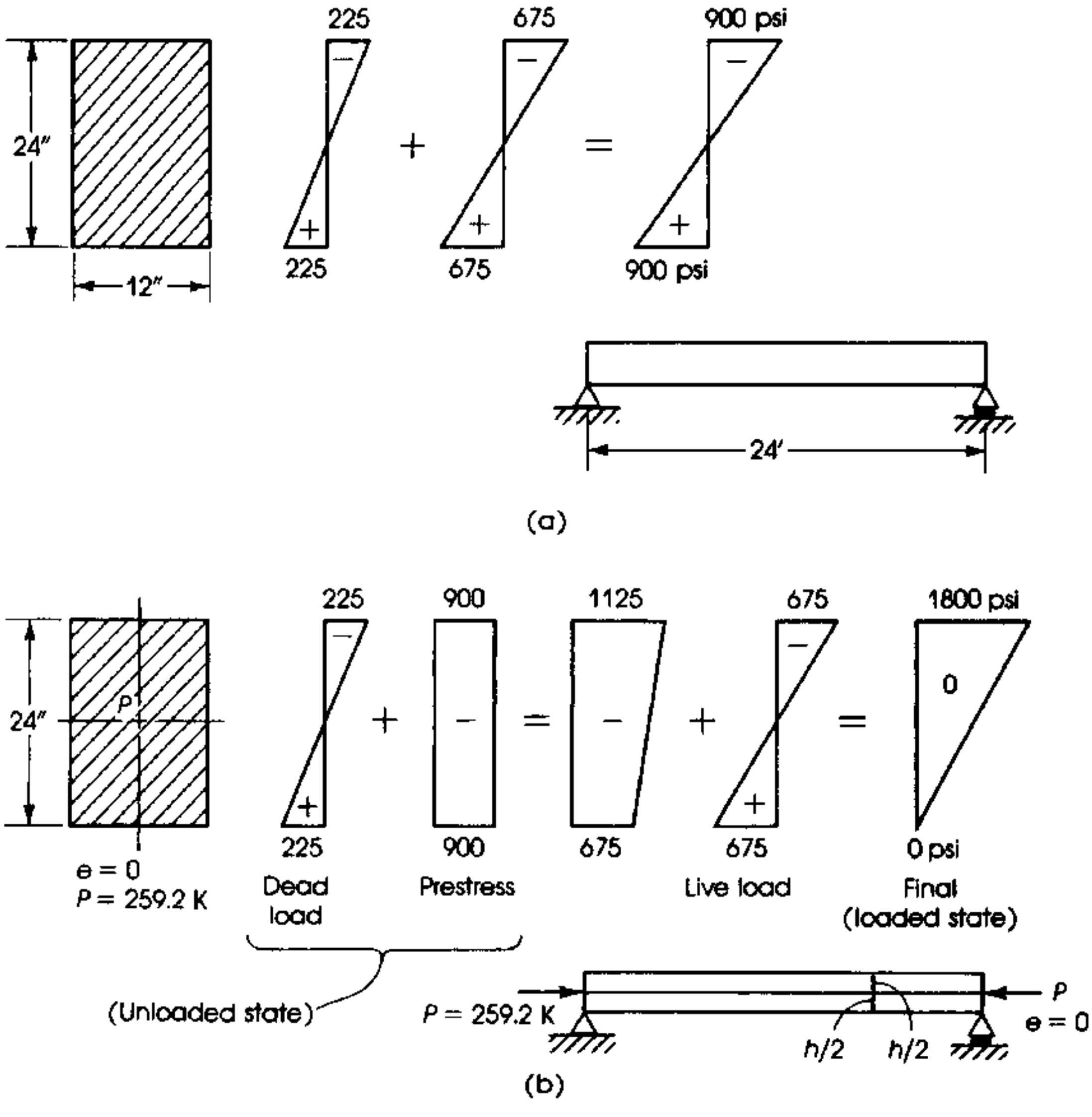


Figure 19.2 Example 19.1.

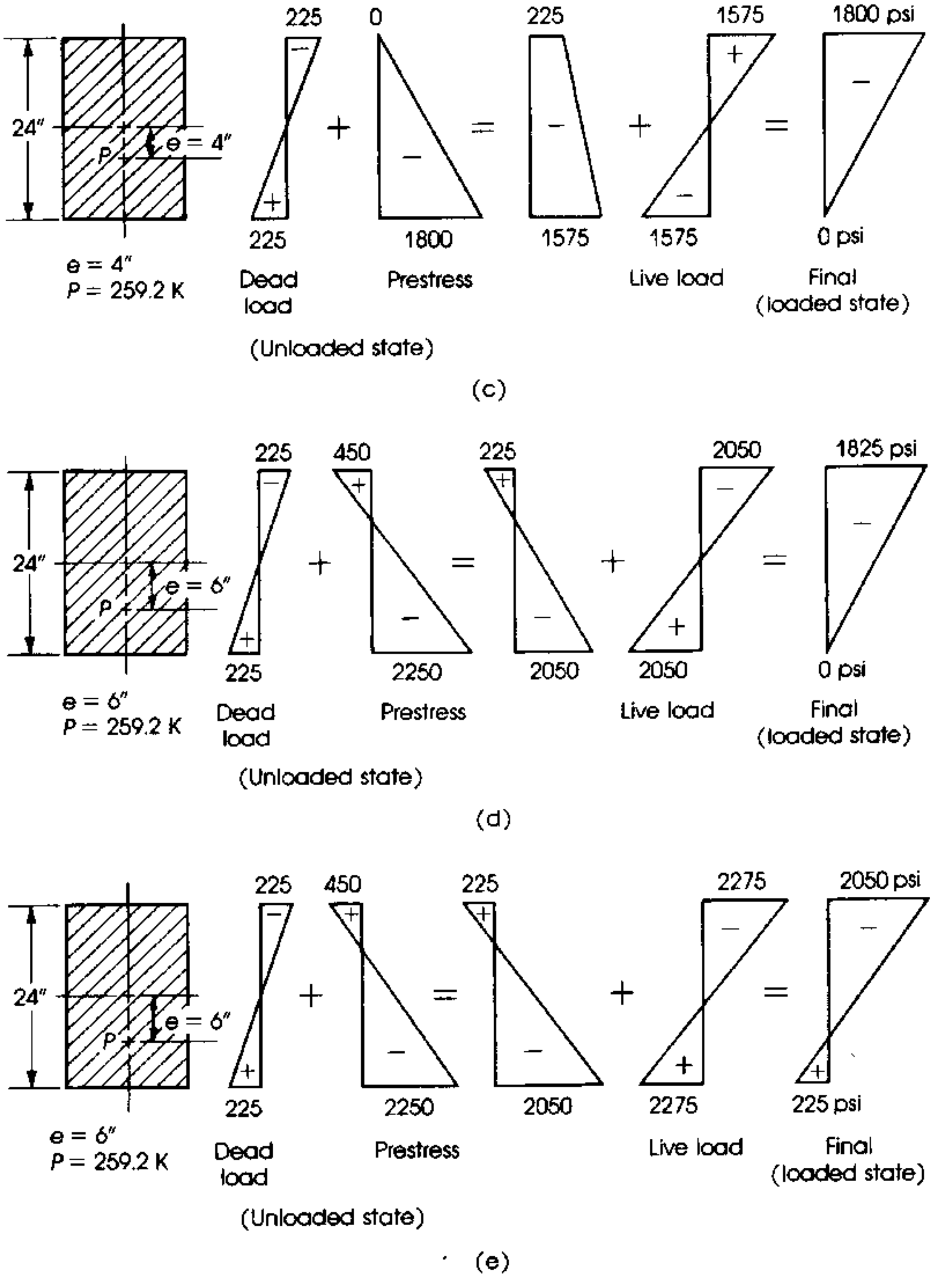


Figure 19.2 (*continued*)

Stresses due to the live load $L_1 = 900$ lb/ft are

$$M_{\text{L.L.}} = \frac{0.9(24)^2}{8} = 64.8 \text{ K·ft}$$

$$\sigma_{L_1} = \frac{6M}{bh^2} = \frac{6 \times 64.8 \times 12{,}000}{12(24)^2} = \pm 675 \text{ psi}$$

Adding stresses due to the dead and live loads (Fig. 19.2*a*) gives

$$\text{Top stress} = -225 - 675 = -900 \text{ psi} \quad \text{(compression)}$$

$$\text{Bottom stress} = +225 + 675 = +900 \text{ psi} \quad \text{(tension)}$$

The tensile stress is higher than the modulus of rupture of concrete, $f_r = 7.5\lambda\sqrt{f'_c} = 503$ psi; hence, the beam will collapse.

2. In the case of stresses due to uniform prestress, if a compressive force $P = 259.2$ K is applied at the centroid of the section, then a uniform stress is induced on any section along the beam.

$$\sigma_P = \frac{P}{\text{area}} = \frac{259.2 \times 1000}{12 \times 24} = -900 \text{ psi} \quad \text{(compression)}$$

Final stresses due to live and dead loads plus prestress load at the top and bottom fibers are 1800 psi and 0, respectively (Fig. 19.2*b*). In this case, the prestressing force has doubled the compressive stress at the top fibers and reduced the tensile stress at the bottom fibers to 0. The maximum compressive stress of 1800 psi is less than the allowable stress of 2050 psi.

3. For stresses due to an eccentric prestress ($e = 4$ in.), if the prestressing force $P = 259.2$ K is placed at an eccentricity of $e = 4$ in. below the centroid of the section, the stresses at the top and bottom fibers are calculated as follows. Moment due to eccentric prestress is Pe:

$$\begin{aligned} \sigma_P &= -\frac{P}{A} \pm \frac{(Pe)c}{I} = -\frac{P}{A} \pm \frac{6(Pe)}{bh^2} \\ &= -\frac{259.2 \times 1000}{12 \times 24} \pm \frac{6(259.2 \times 1000 \times 4)}{12(24)^2} \\ &= -900 \pm 900 \\ &= -1800 \text{ psi} \end{aligned}$$

at the bottom fibers and $\sigma_P = 0$ at the top fibers. Consider now an increase in the live load of $L_2 = 2100$ lb/ft:

$$M_{\text{L.L.}} = \frac{2.1 \times (24)^2}{8} = 151.2 \text{ K·ft}$$

$$\sigma_{L_2} = \frac{6(151.2 \times 12{,}000)}{12(24)^2} = \pm 1575 \text{ psi}$$

Final stresses due to the dead, live, and prestressing loads at the top and bottom fibers are 1800 psi and 0, respectively (Fig. 19.2*c*). Note that the final stresses are exactly the same as those of the previous case when the live load was 900 lb/ft; by applying the same prestressing force but at an eccentricity of 4 in., the same beam can now support a greater live load (by 1200 lb/ft).

4. For stresses due to eccentric prestress with maximum eccentricity, assume that the maximum practical eccentricity for this section is at $e = 6$ in., leaving a 2-in. concrete cover; then the bending moment induced is $Pe = 259.2 \times 6 = 1555.2$ K·in. $= 129.6$ K·ft. Stresses due to the prestressing force are

$$\begin{aligned} \sigma_P &= -\frac{259.2 \times 1000}{12 \times 24} \pm \frac{6 \times (129.6 \times 12{,}000)}{12(24)^2} \\ &= -900 \pm 1350 \text{ psi} \\ &= -2250 \text{ psi} \quad \text{and} \quad +450 \text{ psi} \end{aligned}$$

Increase the live load now to $L_3 = 2733$ lb/ft. The stresses due to the live load, L_3, are

$$M_{\text{L.L.}} = \frac{2.733 \times (24)^2}{8} = 196.8 \text{ K·ft}$$

$$\sigma_{L_3} = \frac{6(196.8 \times 12{,}000)}{12(24)^2} = \pm 2050 \text{ psi}$$

The final stresses at the top and bottom fibers due to the dead load, live load (L_3), and the prestressing force are 1825 psi and 0, respectively (Fig. 19.2*d*). Note that the final stresses are about the same as those in the previous cases, yet the live load has been increased to 2733 lb/ft. A tensile stress of 225 psi is developed when the prestressing force is applied on the beam. This stress is less than the modulus of rupture of concrete, $f_r = 503$ psi; hence, cracks will not develop in the beam.

5. The maximum live load when the eccentric force P acts at $e = 6$ in. is determined as follows. In the previous case, the final compressive stress is equal to 1825 psi, which is less than the allowable stress of 2050 psi. Therefore, the live load may be increased to $L_4 = 3033$ lb/ft.

$$M_{\text{L.L.}} = \frac{3.033 \times (24)^2}{8}$$

$$\sigma_{L_4} = \frac{6(218.4 \times 12{,}000)}{12(24)^2} = \pm 2275 \text{ psi}$$

Final stresses due to the dead load, live load (L_4), and the prestressing force are −2050 psi and ±225 psi (Fig. 19.2*e*). The compressive stress is equal to the allowable stress of 2050 psi, and the tensile stress is less than the modulus of rupture of concrete of 503 psi. In this case, the uniform live load of 3033 lb/ft has been calculated as follows: Add the maximum allowable compressive stress of 2050 psi to the initial tensile stress at the top fibers of 225 psi to get 2275 psi. The moment that will produce a stress at the top fibers of 2275 psi is equal to

$$M = \sigma\left(\frac{bh^2}{6}\right)$$

$$= \frac{2.275}{6}(12)(24)^2 = 2620.8 \text{ K·in.} = 218.4 \text{ K·ft}$$

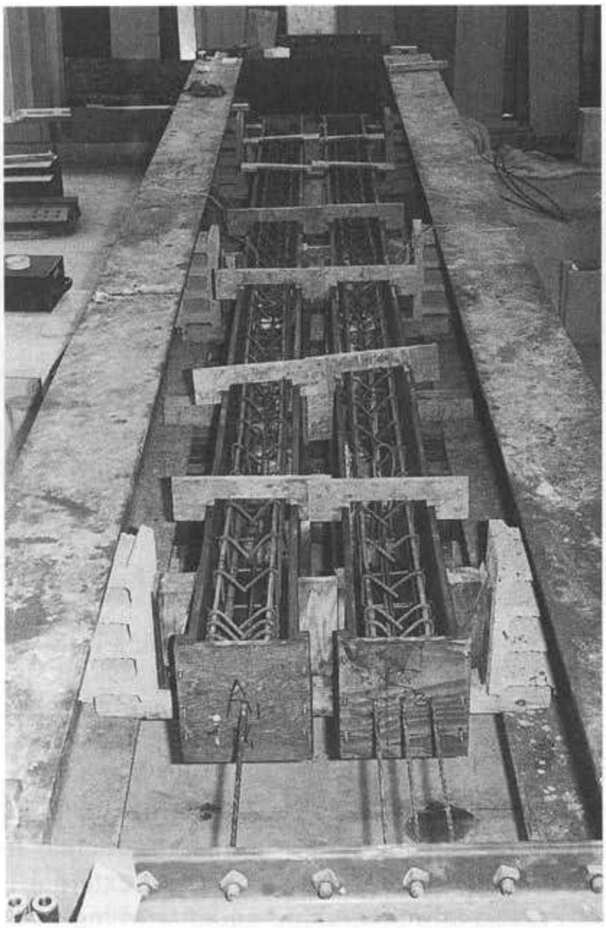
$$M = \frac{W_L L^2}{8} \quad \text{and} \quad W_L = \frac{8 \times 218.4}{(24)^2} = 3.033 \text{ K/ft}$$

Notes:

1. The entire concrete section is active in resisting the external loads.
2. The final tensile stress in the section is less than the modulus of rupture of concrete, which indicates that a crackless concrete section can be achieved under full load.
3. The allowable load on the beam has been increased appreciably due to the application of the prestressing force.
4. An increase in the eccentricity of the prestressing force will increase the allowable applied load, provided that the allowable stresses on the section are not exceeded.

19.1.2 Partial Prestressing

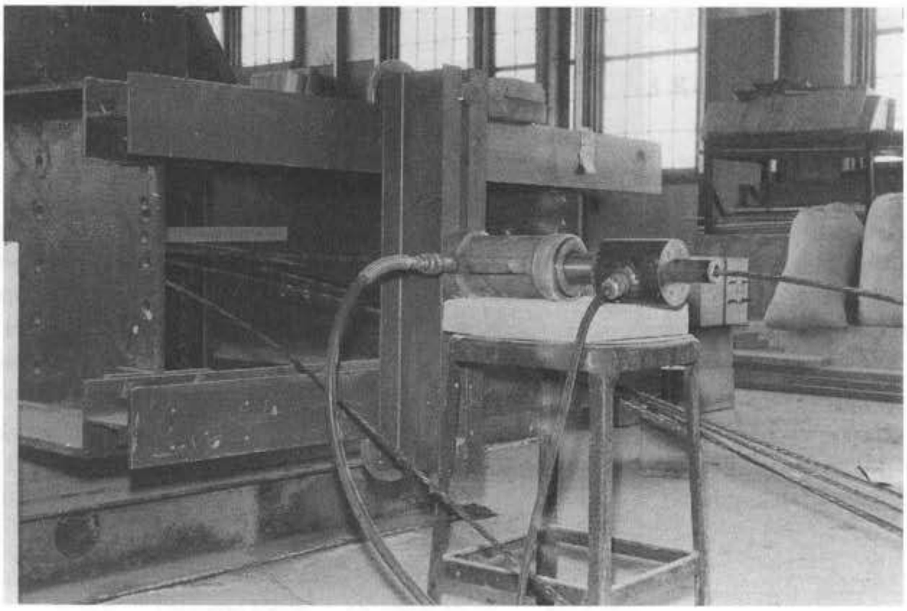
A partially prestressed concrete member can be defined as one in which (1) there have been introduced internal stresses to counteract part of the stresses resulting from external loadings, (2) tensile stresses are developed in the concrete under working loads, and (3) nonprestressed reinforcement may be added to increase the moment capacity of the member. That definition implies that there are two cases that could be considered as partially prestressed concrete:



Partially prestressed concrete beams.

1. A combination of prestressed and nonprestressed steel is used in the same section. The prestressed cables induce internal stresses designed to take only part of the ultimate capacity of the concrete section. The rest of the capacity is taken by nonprestressed steel placed along the same direction as the prestressed cables. The steel used as nonprestressed steel could be any common grade of carbon steel or high-tensile-strength steel of the same kind as the prestressing cables with ultimate strength of 250 ksi (1725 N/mm^2). The choice depends on two main factors: allowable deflection and allowable crack width. As for deflection, the ACI Code specifies a maximum ratio of span to depth of reinforced concrete members. With the smaller depth expected in partially prestressed concrete, and because a smaller steel percentage is used, excessive deflection under working loads must not be allowed. Cracks develop on the tension side of the concrete section or at the steel level because tensile stresses are allowed to occur under working loads. The maximum crack width that may be allowed is 0.016 in. (0.41 mm) for interior members and 0.013 in. (0.33 mm) for exterior members.
2. Internal stresses act on the member from prestressed steel only, but tensioned to a lower limit. In this case cracking develops earlier than in a fully prestressed member under similar loadings.

Partially prestressed concrete can be considered an intermediate form between reinforced and fully prestressed concrete. In reinforced concrete members, cracks develop under working loads; therefore, reinforcement is placed in the tension zone. In prestressed concrete members,



Prestressing jack with a load cell.

cracks do not usually develop under working loads. The compressive stresses due to prestressing may equal or exceed the tensile stresses due to external loadings. Therefore, a partially prestressed concrete member may be considered a reinforced concrete member in which internal stresses are introduced to counteract part of the stress from external loadings so that tensile stresses in the concrete do not exceed a limited value under working load. It reduces to reinforced concrete when no internal stresses act on the member. Full prestressing is an upper extreme of partial prestressing in which nonprestressed reinforcing steel reduces to 0.

Between a reinforced cracked member and a fully prestressed uncracked member, there exists a wide range of design in partial prestressing (Fig. 19.3). A proper choice of the degree of prestressing will produce a safe and economical structure.

Figure 19.3 shows the load deflection curves of concrete beams containing different amounts and types of reinforcement. Curve a represents a reinforced concrete beam, which normally cracks at a small load W_{cr}. The cracking moment M_{cr} can be determined as follows:

$$M_{cr} = \frac{f_r I}{c}$$

where

f_r = the modulus of rupture of concrete $= 7.5\lambda\sqrt{f'_c}$

I = moment of inertia of the gross concrete section

c = distance from the neutral axis to the tensile extreme fibers

The cracking load can be determined from the cracking moment when the span and the type of loading are specified. For a simply supported beam subjected to a concentrated load at midspan, $W_{cr} = (4M_{cr})/L$.

Curves e and f represent underreinforced and overreinforced fully prestressed concrete beams, respectively. The overreinforced concrete beam fails by crushing of the concrete before the steel reaches its yield strength or proof stress. The beam has small deflection and undergoes brittle failure. The under-reinforced beam fails by the steel reaching its yield or ultimate strength.

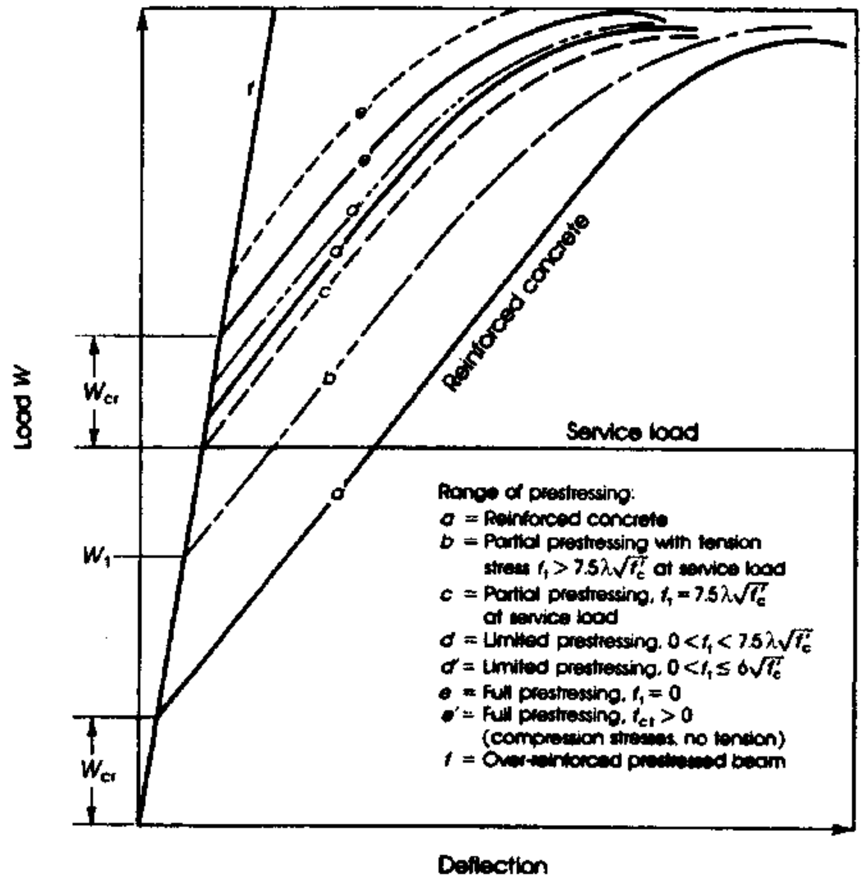


Figure 19.3 Load–deflection curves of concrete beams with different prestressing. The cracking load is W_{cr}.

It shows appreciable deflection and cracking due to elongation of the steel before the gradual crushing of the concrete and the collapse of the beam.

Between curves a and e is a wide range of concrete beams with varying amounts of reinforcement and subjected to varying amounts of prestress. The beam with little prestressing is closer to curve a, while the beam with a large prestress is closer to curve e. Depending upon the allowable concrete stress, deflection, and maximum crack width, a suitable combination of prestressed and nonprestressed reinforcement may be chosen for the required design.

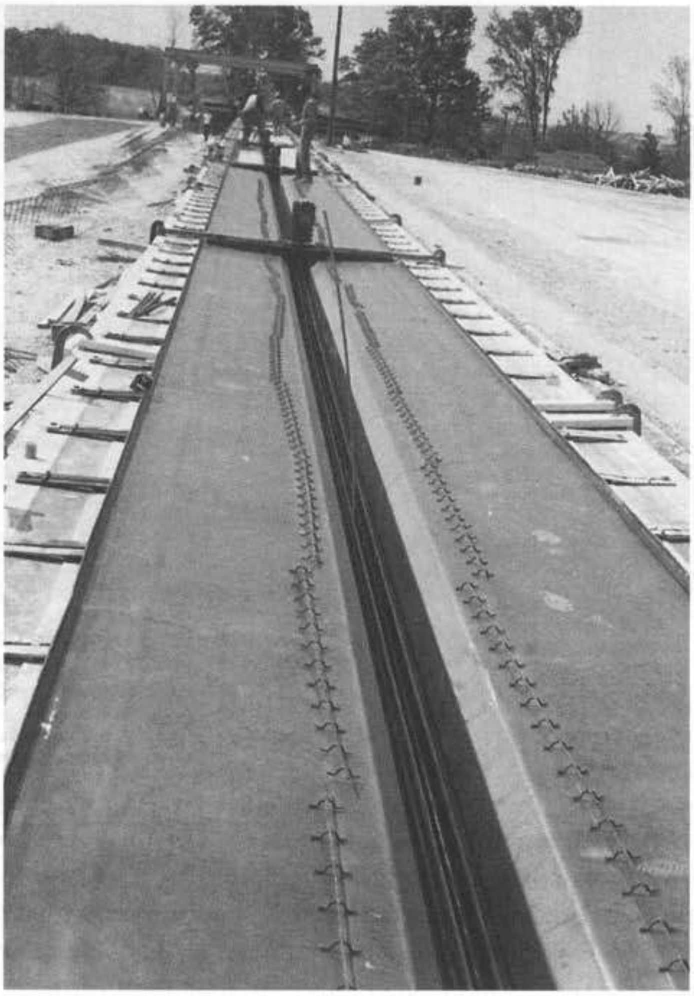
Curve b represents a beam that will crack under full working load. If only part of the live load L_1 occurs frequently on the structure, then W_1 represents the total dead load and that part of the live load L_1.

Curve c represents a beam that starts cracking at working load. The maximum tensile stress in the concrete $= 7.5\sqrt{f'_c}$.

Curve d represents a beam with limited prestress. The critical section of the beam will not crack under full working load, but it will have a maximum tensile stress of $0 < f_t < 7.5\sqrt{f'_c}$. The maximum tensile stress in concrete allowed by the current ACI Code is $6\sqrt{f'_c}$.

Curves e and e' represent fully prestressed concrete beams with no tensile stress under working loads. (See Fig. 19.4.)

The most important advantage of partial prestressing is the possibility of controlling camber. By reducing the prestressing force, the camber will be reduced and a saving in the amount of the prestressing steel, the amount of work in tensioning, and the number of end anchorages is realized.



Prestressing bed for T-beam sections.

Depending on the magnitude of the prestressing force, earlier cracking may occur in partially prestressed rather than in fully prestressed concrete members under service loads. Once cracks develop, the effective moment of inertia of the critical section is reduced and a greater deflection is expected. However, partial prestressing has been used with satisfactory results, and its practical application is increasing.

19.1.3 Classification of Prestressed Concrete Flexural Members

The ACI Code, Section 18.3.3, divided prestressed concrete members into three classes based on the computed extreme tensile fiber stress, f_t, in the tension zone at service load as follows:

1. Class U (uncracked section), with $f_t \leq 7.5\sqrt{f_c'}$. In this uncracked concrete section, the gross section properties are used to check deflection at service load. No cracks will develop in this section and no skin reinforcement is needed.
2. Class T (section in the transition zone), with $7.5\sqrt{f_c'} < f_t \leq 12\sqrt{f_c'}$. This type of sections has a tensile stress in concrete higher than the modulus of rupture of concrete, $f_r = 7.5\sqrt{f_c'}$ producing a case between uncracked and cracked sections. In this case, the gross section

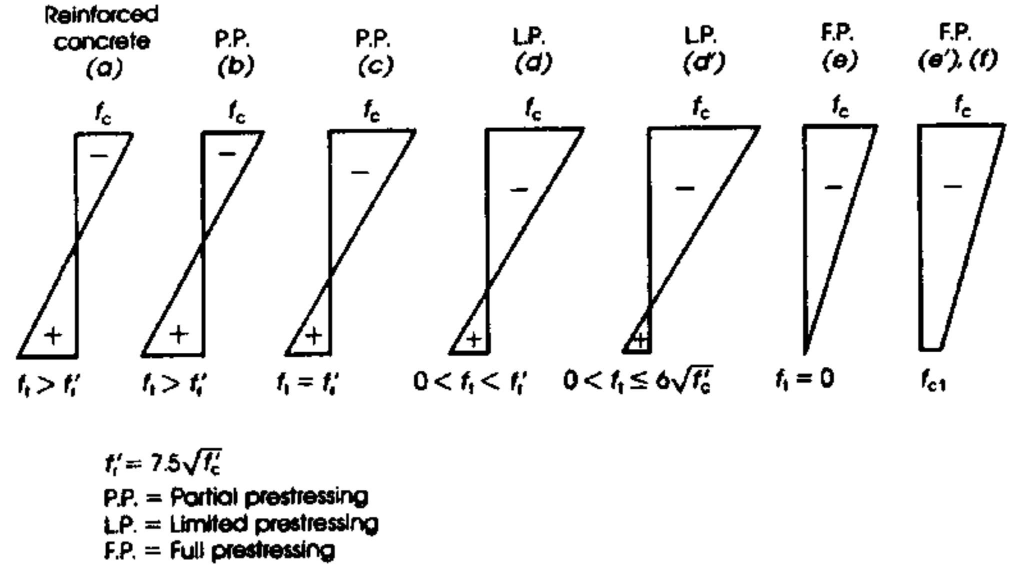


Figure 19.4 Distribution of stresses in beams with varying amounts of prestressed and nonprestressed reinforcement.

properties are used to check stresses, while the cracked section bilinear section is used to calculate deflection. No skin reinforcement is needed in the tension zone.

3. Class C (Cracked section), with $f_t > 12\sqrt{f_c'}$. The tensile stress in the section exceeds 1.6 times the modulus of rupture. Therefore, cracks will develop as in the case of partially prestressed concrete members. In this case a cracked section properties should be used to check stresses, cracking, and deflection. Crack control provisions and skin reinforcement should be used as explained in Section 6.7 for reinforced concrete members with the effective depth of $d > 36$ in.

19.2 MATERIALS AND SERVICEABILITY REQUIREMENTS

19.2.1 Concrete

The physical properties of concrete were discussed in Chapter 2. Although reinforced concrete members are frequently made of concrete with a compressive strength of 3 to 5 ksi (21 to 35 MPa), prestressed concrete members are made of higher strength material, usually from 4 to 8 ksi (28 to 56 MPa). High-strength concrete may be adopted for precast, prestressed concrete members where components are prepared under optimum control of mixing concrete, placing, vibrating, and curing.

The allowable stresses in concrete according to the ACI Code, Section 18.4, are as follows.

1. Stresses after prestress transfer and before prestress losses:
   a. Maximum compressive stress of $0.6 f_{ci}'$
   b. Maximum compressive stress at ends of simply supported $0.7 f_{ci}'$
   c. Maximum tensile stress (experts as permitted below in d) of $3\sqrt{f_{ci}'}$
   d. Maximum tensile stress at the ends of simply supported members of $6\sqrt{f_{ci}'}$ where f_{ci}' is the strength of concrete at transfer

If the maximum tensile stresses are exceeded in c or d, then reinforcement must be provided in the tensile zone to resist the total tensile force in concrete (based on uncracked gross section).

2. Stresses at service loads after all losses (for class U and class T) are as follows: Maximum compressive stress of $0.45f'_c$ due to prestresses plus sustained loads and of $0.6\sqrt{f'_c}$ due to prestress plus total load.
3. These stresses may be exceeded if it is shown by tests or analysis that performance is satisfactory.

19.2.2 Prestressing Steel

The most common type of steel tendons used in prestressed concrete are strands (or cables) made with several wires, usually seven or 19. Wires and bars are also used. The strands and wires are manufactured according to ASTM Standard A421 for uncoated stress-relieved wires and A416 for uncoated seven-wire stress-relieved strands. Properties of prestressing steel are given in Table 19.1.

Table 19.1 Properties of Prestressing Steel, Nominal Diameters, Areas, and Weights

Type	Diameter (in.)	Area (in.^{2})	Weight (lb/ft)	Diameter (mm)	Area (mm^2)	Mass (kg/m)
Seven-wire strand (grade 250)	$\frac{1}{4}$ (0.250)	0.036	0.12	6.350	23.2	0.179
	$\frac{5}{16}$ (0.313)	0.058	0.20	7.950	37.4	0.298
	$\frac{3}{8}$ (0.375)	0.080	0.27	9.525	51.6	0.402
	$\frac{7}{16}$ (0.438)	0.108	0.37	11.125	69.7	0.551
	$\frac{1}{2}$ (0.500)	0.144	0.49	12.700	92.9	0.729
	(0.600)	0.216	0.74	15.240	139.4	1.101
Seven-wire strand (grade 270)	$\frac{3}{8}$ (0.375)	0.085	0.29	9.525	54.8	0.432
	$\frac{7}{16}$ (0.438)	0.115	0.40	11.125	74.2	0.595
	$\frac{1}{2}$ (0.500)	0.153	0.53	12.700	98.7	0.789
	(0.600)	0.215	0.74	15.250	138.7	1.101
Prestressing wire grades (250)	0.192	0.029	0.10	4.877	18.7	0.146
(250)	0.196	0.030	0.10	4.978	19.4	0.149
(240)	0.250	0.049	0.17	6.350	31.6	0.253
(235)	0.276	0.060	0.20	7.010	38.7	0.298
Prestressing bars (smooth)	$\frac{3}{4}$ (0.750)	0.44	1.50	19.050	283.9	2.232
(grade 145 or 160)	$\frac{7}{8}$ (0.875)	0.60	2.04	22.225	387.1	3.036
	1 (1.000)	0.78	2.67	25.400	503.2	3.973
	$1\frac{1}{8}$ (1.125)	0.99	3.38	28.575	638.7	5.030
	$1\frac{1}{4}$ (1.250)	1.23	4.17	31.750	793.5	6.206
	$1\frac{3}{8}$ (1.385)	1.48	5.05	34.925	954.8	7.515
Prestressing bars (deformed)	$\frac{5}{8}$ (0.625)	0.28	0.98	15.875	180.6	1.458
(grade 150–160)	$\frac{3}{4}$ (0.750)	0.42	1.49	19.050	271.0	2.218
	1 (1.000)	0.85	3.01	25.400	548.4	4.480
	$1\frac{1}{4}$ (1.250)	1.25	4.39	31.750	806.5	6.535
	$1\frac{3}{8}$ (1.385)	1.58	5.56	34.925	1006	8.274

Seven-wires prestressing strands (shipped in coils as shown).

Prestressing steel used in prestressed concrete must be of high-strength quality, usually of ultimate strength, f_{pu}, of 250 ksi to 270 ksi (1730–1860 MPa). High-strength steel is necessary to permit high elongation and to maintain a permanent sufficient prestress in the concrete after the inelastic shortening of the concrete.

The allowable stresses in prestressing steel according to the ACI Code, Section 18.5, are as follows:

1. Maximum stress due to tendon jacking force must not exceed the smaller of $0.8 f_{pu}$ or $0.94 f_{py}$. (The smaller value must not exceed that stress recommended by the manufacturer of tendons or anchorages.)
2. Maximum stress in pretensioned tendons immediately after transfer must not exceed the smaller of $0.74 f_{pu}$ or $0.82 f_{py}$.
3. Maximum stress in posttensioning tendons after tendon is anchored is $0.70 f_{pu}$.

19.2.3 Reinforcing Steel

Nonprestressed reinforcing steel is commonly used in prestressed concrete structural members, mainly in the prestressed, precast concrete construction. The reinforcing steel is used as shear reinforcement, as supplementary reinforcement for transporting and handling the precast elements, and in combination with the prestressing steel in partially prestressed concrete members. The types and allowable stresses of reinforcing bars were discussed in Chapters 2 and 5.

19.3 LOSS OF PRESTRESS

19.3.1 Lump-Sum Losses

Following the transfer of the prestressing force from the jack to the concrete member, a continuous loss in the prestressing force occurs; the total loss of prestress is the reduction in the prestressing force during the lifespan of the structure. The amount of loss in tendon stress varies between 15% and 30% of the initial stress, because it depends on many factors. For most normal-weight concrete structures constructed by standard methods, the tendon stress loss due to elastic shortening, shrinkage, creep, and relaxation of steel is about 35 ksi (241 MPa) for pretensioned members and 25 ksi (172 MPa) for posttensioned members. Friction and anchorage slip are not included.

Two current recommendations for estimating the total loss in prestressed concrete members are presented by AASHTO and the Posttensioning Institute (PTI). AASHTO [23] recommends a total loss (excluding friction loss) of 45 ksi (310 MPa) for pretensioned strands and 33 ksi (228 MPa) for posttensioned strands and wires when a concrete strength of $f_c' = 5$ ksi is used. The PTI [24] recommends a total lump-sum prestress loss for posttensioned members of 35 ksi (241 MPa) for beams and 30 ksi (207 MPa) for slabs (excluding friction loss). These values can be used unless a better estimate of the prestress loss by each individual source is made, as is explained shortly.

In general, the sources of prestress loss are

- Elastic shortening of concrete
- Shrinkage of concrete
- Creep of concrete
- Relaxation of steel tendons
- Friction
- Anchorage set

19.3.2 Loss Due to Elastic Shortening of Concrete

In pretensioned members, estimating loss proceeds as follows. Consider a pretensioned concrete member of constant section and stressed uniformly along its centroidal axis by a force F_o. After the transfer of the prestressing force, the concrete beam and the prestressing tendon shorten by an equal amount, because of the bond between the two materials. Consequently, the starting prestressing force F_o drops to F_i and the loss in the prestressing force is $F_o - F_i$. Also, the strain in the concrete, ε_c, must be equal to the change in the tendon strain, $\Delta\varepsilon_s$. Therefore, $\varepsilon_c = \Delta\varepsilon_s$, or $(f_c/E_c) = (\Delta f_s/E_s)$, and the stress loss due to the elastic shortening is

$$\Delta f_s = \frac{E_s}{E_c} \times f_c = n f_c = \frac{nF_i}{A_c} \approx \frac{nF_o}{A_c} \tag{19.1}$$

where

A_c = the area of the concrete section
$n = E_s/E_c$ = modular ratio
f_c = the stress in the concrete at the centroid of the prestressing steel

Multiply the stress by the area of the prestressing steel, A_{sp}, to get the total force; then the elastic loss is

$$ES = F_o - F_i = \Delta f_s A_{sp} = (nf_c)A_{sp} \approx \left(\frac{nF_o}{A_c}\right) A_{sp} \tag{19.2}$$

$$F_i = F_o - (nf_c)A_{sp} \tag{19.3}$$

For practical design, the loss in the prestressing force, Δf_s per unit A_{sp}, may be taken to be approximately nF_o/A_c. If the force F_o acts at an eccentricity e, then the elastic loss due to the presence of F_o and the applied dead load at transfer is

$$ES = -(nf_c)A_{sp} \text{ (due to prestress)} + (nf_c)A_{sp} \quad \text{(dead load)}$$

$$ES = F_o - F_i = -\left(\frac{F_i}{A} + \frac{F_i e^2}{I}\right) nA_{sp} + \left(\frac{M_D e}{I}\right) nA_{sp} \tag{19.4}$$

An approximate value of $F_i = (0.63 f_{pu})A_{sp}$ may be used in the above equation.

$$F_o + f_c(\text{D.L.})nA_{sp} = F_i\left[1 + nA_{sp}\left(\frac{1}{A} + \frac{e^2}{I}\right)\right] \tag{19.5}$$

$$F_i = \frac{F_o + (nA_{sp})f_c(\text{D.L.})}{1 + (nA_{sp})\left(\frac{1}{A} + \frac{e^2}{I}\right)}$$

For posttensioned members where the tendons or individual strands are not stressed simultaneously, the loss of the prestress can be taken as half the value ES for pretensioned members.

Also, it is practical to consider the elastic shortening loss in slabs equal to one-quarter of the equivalent pretensioned value, because stretching of one tendon will have little effect on the stressing of the other tendons.

19.3.3 Loss Due to Shrinkage

The loss of prestress due to shrinkage is time dependent. It may be estimated as follows:

$$SH = \Delta f_s \text{ (shrinkage)} = \varepsilon_{sh} E_s \tag{19.6}$$

where $E_s = 29 \times 10^6$ psi and ε_{sh} = shrinkage strain in concrete.

The average strain due to shrinkage may be assumed to have the following values: for pretensioned members, $\varepsilon_{sh_1} = 0.0003$; for posttensioned members, $\varepsilon_{sh_2} = 0.0002$. If posttensioning is carried out within 5 to 7 days after concreting, the shrinkage strain can be taken to be $0.8\varepsilon_{sh_1}$. If posttensioning is carried out between 1 and 2 weeks, $\varepsilon_{sh} = 0.7\varepsilon_{sh_1}$ can be used, and if it occurs more than 2 weeks later, $\varepsilon_{sh} = \varepsilon_{sh2}$ can be adopted. Shrinkage loss, SH, can also be estimated as follows [28]:

$$SH = 8.2 \times 10^{-6} K_{sh} E_s \left(1 - \frac{0.06V}{S}\right)(100 - RH)$$

where V/S = volume-to-surface ratio and RH = average relative humidity. K_{sh} is 1.0 for pretensioned members and is 0.8, 0.73, 0.64, and 0.58 for posttensioned members if posttensioning is carried out after 5, 10, 20, and 30 days, respectively.

19.3.4 Loss Due to Creep of Concrete

Creep is a time-dependent deformation that occurs in concrete under sustained loads. The developed deformation causes a loss of prestress from 5% to 7% of the applied force.

The creep strain varies with the magnitude of the initial stress in the concrete, the relative humidity, and time. The loss in stress due to creep can be expressed as follows:

$$CR = \Delta f_s \text{ (creep)} = C_c(nf_c) = C_c(\varepsilon_{cr} E_s) \tag{19.7}$$

where

$$C_c = \text{creep coefficient} = \frac{\text{creep strain, } \varepsilon_{cp}}{\text{initial elastic strain, } \varepsilon_i}$$

The value of C_c may be taken as follows 22.

Concrete strength	$f'_c \le 4$ **ksi**		$f'_c > 4$ **ksi**	
Relative humidity	%	50%	100%	50%
C_c	1–2	2–4	0.7–1.5	1.5–3

Linear interpolation can be made between these values. Considering that half the creep takes place in the first 134 months of the first 6 months after transfer and under normal humidity conditions, the creep strain can be assumed for practical design as follows:

1. For pretensioned members, $\varepsilon_{cr} = 48 \times 10^{-5}$ stress in concrete (ksi).
2. For posttensioned members, $\varepsilon_{cr} = 36 \times 10^{-5} \times$ stress in concrete (ksi). This value is used when posttensioning is made within 2 to 3 weeks. For earlier posttensioning, an intermediate value may be used.

These values apply when the strength of concrete at transfer is $f'_{ci} \geq 4$ ksi. When $f'_{ci} <$ 4 ksi, the creep strain should increase in the ratio of (4/actual strength).

$$\text{Total loss of prestress due to creep} = \varepsilon_{cr} E_s \tag{19.8}$$

19.3.5 Loss Due to Relaxation of Steel

Relaxation of steel causes a time-dependent loss in the initial prestressing force, similar to creep in concrete. The loss due to relaxation varies for different types of steel; its magnitude is usually furnished by the steel manufacturers. The loss is generally assumed to be 3% of the initial steel stress for posttensioned members and 2% to 3% for pretensioned members. If test information is not available, the loss percentages for relaxation at 1000 h can be assumed as follows:

1. In low-relaxation strands, when the initial prestress is $0.7f_{pu}$ and $0.8f_{pu}$, relaxation (RE) is 2.5% and 3.5%, respectively.
2. In stress-relieved strands or wire, when the initial prestress is $0.7f_{pu}$ or $0.8f_{pu}$, relaxation (RE) is 8% and 12%, respectively.

19.3.6 Loss Due to Friction

With pretensioned steel, friction loss occurs when wires or strands are deflected through a diaphragm. This loss is usually small and can be neglected. When the strands are deflected to follow a concordant profile, the friction loss may be considerable. In such cases, accurate load measuring devices are commonly used to determine the force in the tendon.

With posttensioned steel, the effect of friction is considerable because of two main factors: the curvature of the tendon and the lack of alignment (wobble) of the duct. The curvature effect may be visualized if a belt around a fixed cylinder is tensioned on one end with a force P_2; then the force, P_1, at the other end to initiate slippage in the direction of P_1 is

$$P_1 = P_2 e^{\mu \alpha_{px}} \tag{19.9}$$

where μ = the coefficient of static angular friction and α_{px} = the angle between P_1 and P_2. It is a general practice to treat the wobbling effect similarly:

$$P_x = P_s e^{-(\mu\alpha + Kl_x)}$$

$$P_{px} = P_{px} e^{+(Kl_{px} + \mu_p \alpha_{px})}$$

$$P_{px} = P_{pj} e^{-(Kl_{px} + \mu_p \alpha_{px})} \tag{19.10}$$

where

P_{pj} = the prestressing tendon force at any point x
P_{px} = the prestressing tendon force at the jacking end
μ_p = curvature friction coefficient

α_{px} = total angular change of prestressing tendon profile, in radians, from tendon jacking end to any point x

$= \dfrac{\text{length of curve}}{\text{radius of curvature}}$

K = wobble friction coefficient per foot of the prestressing tendon

As an approximation, the ACI Code gives the following expression:

$$P_{px} = P_{pj}(1 + Kl_{px} + \mu_p\alpha_{px})^{-1} \quad \text{(ACI Code, Eq. 18.2)} \tag{19.11}$$

provided that $(\mu_p\alpha_{px} + Kl_x) \le 0.30$.

The frictional coefficients α and K depend on the type of prestressing strands or wires, type of duct, and the surface conditions. Some approximate values for μ and K are given in the ACI Code, Section R.18.6.2, and in Table 19.2.

Friction loss in the jack is variable and depends on many factors, including the length of travel of the arm over a given load range. The use of accurate load cells to measure directly the force in the tendon is recommended. The use of pressure gauges may lead to inaccuracies unless they are calibrated against a known force in the tendon.

The friction loss in the anchorage is dependent mainly upon the type of anchorage and the amount of deviation of the tendon as it passes through the anchorage. This loss is usually small and may be neglected. Guidance in particular cases should be obtained from the manufacturers.

19.3.7 Loss Due to Anchor Set

When the force in a tendon is transferred from the jack to the anchorage unit, a small inward movement of the tendon takes place due to the seating of the gripping device or wedges. The slippage causes a shortening of the tendon, which results in a loss in the prestressing force. The magnitude of slippage varies between 0.1 and 0.25 in. (2.5 and 6 mm) and is usually specified by the manufacturer. The loss due to the anchor set may be calculated as follows:

$$\Delta f_s = \Delta\varepsilon E_s = \frac{\Delta L}{L} \times E_s \tag{19.12}$$

where

$\Delta\varepsilon$ = magnitude of the anchor slippage

$E_s = 29 \times 10^6$ psi

L = length of the tendon

Table 19.2 Friction Coefficients for Posttensioned Tendons

Type of Tendon	Wobble Coefficient K Per Foot ($\times 10^{-3}$)	Curvature Coefficient μ
Tendon in flexible metal sheathing (grouted)		
Wire tendons	1.0–1.5	0.15–0.25
Seven-wire strand	0.5–2.0	0.15–0.25
High-strength bars	0.1–0.6	0.08–0.30
Pregreased unbonded tendon		
Wire tendons and seven-wire strand	0.3–2.0	0.05–0.15
Mastic-coated unbonded tendons		
Wire tendons and seven-wire strand	1.0–2.0	0.05–0.15

Because the loss in stress is inversely proportional to the length of the tendon (or approximately half the length of the tendon if it is stressed from both ends simultaneously), the percentage loss in steel stress decreases as the length of the tendon increases. If the tendon is elongated by $\Delta\varepsilon$ at transfer, the loss in prestress due to slippage is neglected.

Example 19.2

A 36-ft-span pretensioned simply supported beam has a rectangular cross-section with $b = 18$ in. and $h = 32$ in. Calculate the elastic loss and all time-dependent losses. Given: prestressing force at transfer is $F_i = 435$ K, area of prestressing steel is $A_{PS} = 3.0$ in.^{2}, $f_c' = 5$ ksi, $E_c = 5000$ ksi, $E_s = 29{,}000$ ksi, profile of tendon is parabolic, eccentricity at midspan $= 6.0$ in., and eccentricity at ends $= 0$.

Solution

1. Elastic shortening: Stress due to the prestrepssing force at transfer is

$$\frac{F_i}{A_{ps}} = \frac{435}{3} = 145 \text{ ksi}$$

$$\text{Strain in prestressing steel} = \frac{f_s}{E_s} = \frac{145}{29{,}000} = 0.005$$

Using Eq. 19.1,

$$n = \frac{E_s}{E_c} = \frac{29{,}000}{5000} = 5.8 \quad \text{or} \quad 6$$

$$\Delta f_s = \frac{nF_i}{A_c} = \frac{6 \times 435}{32 \times 18} = 4.5 \text{ ksi}$$

Considering the variation in the eccentricity along the beam,

$$\text{Strain at end of section} = \frac{F_i}{A_c E_c} = \frac{435}{(18 \times 32) \times 5000} = 0.151 \times 10^{-3}$$

$$\text{Strain at midspan} = \frac{F_i}{A_c E_c} + \frac{F_i e^2}{I E_c}$$

$$I = \frac{bh^3}{12} = \frac{18(32)^3}{12} = 49{,}152 \text{ in.}^4$$

$$\text{Strain} = 0.151 \times 10^{-3} + \frac{435(6)^2}{49{,}152(5000)} = 0.215 \times 10^{-3}$$

$$\text{Average strain} = \frac{1}{2}(0.151 + 0.215) \times 10^{-3} = 0.183 \times 10^{-3}$$

$$\text{Prestress loss} = \text{strain} \times E_s = 0.183 \times 10^{-3} \times 29{,}000 = 5.3 \text{ ksi}$$

$$\text{Percent loss} = \frac{5.3}{145} = 3.66\%$$

2. Loss due to shrinkage:

$$\text{Shrinkage strain} = 0.0003$$

$$\Delta f_s = \varepsilon_{sh} E_s = 0.0003 \times 29{,}000 = 8.7 \text{ ksi}$$

$$\text{Percent loss} = \frac{8.7}{145} = 6\%$$

3. Loss due to creep of concrete: Assuming $C_c = 2.0$, then $\Delta f_s = C_c(\varepsilon_{cr} E_s)$

$$\text{Elastic strain} = \frac{F_i}{A_c E_c} = 0.151 \times 10^{-3}$$

$$\Delta f_s = 2(0.151 \times 10^{-3} \times 29{,}000) = 8.8 \text{ ksi}$$

$$\text{Percent loss} = \frac{8.8}{145} = 6.1\%$$

Or, approximately, $\varepsilon_{cr} = 48 \times 10^{-5} \times$ stress in the concrete (ksi):

$$\varepsilon_{cr} = 48 \times 10^{-5} \left(\frac{435}{32 \times 18}\right) = 36 \times 10^{-5}$$

$$\Delta f_s = \varepsilon_{cr} E_s = 36 \times 10^{-5} \times 29{,}000 = 10.4 \text{ ksi}$$

$$\text{Percent loss} = \frac{10.4}{145} = 7.2\%$$

This is a conservative value, and the same ratio is obtained if $C_c = 2.38$ is adopted in the preceding calculations.

4. Loss due to relaxation of steel: For low-relaxation strands, the loss is assumed to be 2.5%.

$$\Delta f_s = 0.025 \times 145 = 3.6 \text{ ksi}$$

5. Assume the losses due to bending, friction of cable spacers, and the end block of the pretensioning system are 2%.

$$\Delta f_s = 0.02 \times 145 = 2.9 \text{ ksi}$$

6. Loss due to friction in tendon is 0.
7. Total losses are as follows.

Elastic shortening loss	5.3 ksi	3.6%
Shrinkage loss	8.7 ksi	6.0%
Creep of concrete loss	8.8 ksi	6.1%
Relaxation of steel loss	3.6 ksi	2.5%
Other losses	2.9 ksi	2.0%
Total losses	29.3 ksi	20.2%

$$\text{Effective prestress} = 145 - 24 = 121 \text{ ksi}$$

$$\text{Effective prestressing force } F = 121 \times 3\text{in.}^2 = 363 \text{ ksi}$$

$$F = (1 - 0.166)F_i = 0.834F_i$$

For $F = \eta F_i$, $\eta = 0.834$.

Example 19.3

Calculate all losses of a 120-ft-span posttensioned beam that has an I-section with the following details. Area of concrete section (A_c) = 760 in.^{2}; moment of inertia (I_g) = 1.64×10^5 in.^{4}; prestressing force at transfer (F_i) = 1110 K; area of prestressing steel (A_{ps}) = 7.5 in.^{2}; $f'_c = 5$ ksi, $E_c = 5000$ ksi, and $E_s = 29{,}000$ ksi; profile of tendon is parabolic; eccentricity at midspan = 20 in; and eccentricity at ends = 0.

Solution

1. Loss due to elastic shortening:

$$\text{Steel stress at transfer} = \frac{F_i}{A_{ps}} = \frac{1110}{7.5} = 148 \text{ ksi}$$

$$\text{Stress in concrete at end section} = \frac{1110}{760} = 1.46 \text{ ksi}$$

$$\text{Stress in concrete at midspan} = \frac{F_i}{A_c} + \frac{F_i e^2}{I} - \frac{M_D e}{I}$$

$$\text{Weight of beam} = \frac{760}{144} \times 150 = 790 \text{ lb/ft}$$

$$M_D = 0.79\frac{(120)^2}{8} = 1422 \text{ K·ft}$$

$$\text{Stress at midspan} = \frac{1110}{760} + \frac{1110(20)^2}{164{,}000} - \frac{(1422 \times 12)(20)}{164{,}000}$$

$$= 1.46 + 2.71 - 2.08 = 2.09 \text{ ksi}$$

$$\text{Average stress} = \frac{1.46 + 2.09}{2} = 1.78 \text{ ksi}$$

$$\text{Average strain} = \frac{1.78}{E_c} = \frac{1.78}{5000} = 0.356 \times 10^{-3}$$

Elastic loss is $\Delta f_s = \varepsilon_c E_s = 0.356 \times 103 \times 29{,}000 = 10.3$ ksi, assuming that the tendons are tensioned two at a time. The first pair will have the greatest loss, whereas the last pair will have 0 loss. Therefore, average $\Delta f_s = 10.3/2 = 5.15$ ksi.

$$\text{Percent loss} = \frac{5.15}{148} = 3.5\%$$

2. Loss due to shrinkage of concrete:

$$\Delta f_s \text{ (shrinkage)} = 0.0002E_s = 0.0002 \times 29{,}000 = 5.8 \text{ ksi}$$

$$\text{Percent loss} = \frac{5.8}{148} = 3.9\%$$

3. Loss due to creep of concrete: Assume $C_c = 1.5$.

$$\text{Elastic strain} = \frac{F_i}{A_c E_c} = \frac{1110}{760 \times 5000} = 0.92 \times 10^{-3}$$

$$\Delta f_s \text{ (creep)} = C_c(\varepsilon_{cr} E_s)$$

$$= 1.5(0.292 \times 10^{-3} \times 29{,}000) = 12.7 \text{ ksi}$$

$$\text{Percent loss} = \frac{12.7}{148} = 8.6\%$$

4. Loss due to relaxation of steel: For low-relaxation strands, the loss is 2.5%.

$$\Delta f_s = 0.025 \times 148 = 3.7 \text{ ksi}$$

5. Slip in anchorage: For tensioning from one end only, assume a slippage of 0.15 in. The length of the cable is $120 \times 12 = 1440$ in.

$$\Delta f_s = \frac{\Delta L}{L} \times E_s = \frac{0.15}{1440} \times 29{,}000 = 3 \text{ ksi} \qquad (19.12)$$

To allow for anchorage slip, set the tensioned force to 148 + 3 = 151 ksi on the pressure gauge to leave a net stress of 148 ksi in the tendons.

6. Loss due to friction: The equation of parabolic profile is

$$e_x = \frac{4e}{L^2}(Lx - x^2)$$

where e_x = the eccentricity at a distance x measured from the support and e = eccentricity at midspan.

$$\frac{d(e_x)}{dx} = \frac{4e}{L^2}(L - 2x)$$

is the slope of the tendon at any point. At the support, $x = 0$ and the slope

$$\frac{d(e_x)}{dx} = \frac{4e}{L} = \frac{4 \times 20}{120 \times 12} = 0.056$$

The slope at midspan is 0; therefore, $\alpha_{px} = 0.056$. Using flexible metallic sheath, $\mu_p = 0.5$ and $K = 0.001$. At midspan, $x = 60$ ft. Check if $(\mu_p \alpha_{px} + K l_x) \leq 0.30$:

$$\mu_p \alpha_{px} + K l_x = 0.5 \times 0.056 + 0.001 \times 60 = 0.0088 < 0.3$$

$$P_{px} = P_{pj}(1 + K l_{px} + \mu_p \alpha_{px})$$

$$= P_x(1 + 0.088) = 1.088 P_x$$

$$= 1.088 \times 148 = 161 \text{ K} \quad \text{(force at jacking end)} \qquad (19.11)$$

$$\Delta f_s = 161 - 148 = 13 \text{ ksi}$$

$$\text{Percent loss} = \frac{13}{148} = 8.8\%$$

7. Total losses:

Elastic shortening loss	5.2 ksi	3.5%
Shrinkage loss	5.8 ksi	3.9%
Creep of concrete loss	12.7 ksi	8.6%
Relaxation of steel loss	3.7 ksi	2.5%
Friction losses	13.0 ksi	8.8%
Total losses	40.4 ksi	27.3%

$$\text{Effective prestress} = 148 - 35.2 = 112.8 \text{ ksi}$$

$$\text{Effective prestressing force}(F) = (1 - 0.238)F_i = 0.762 F_i$$

$$F = 0.762 \times 1110 = 846 \text{ K}$$

For $F = \eta F_i$, $\eta = 0.762$.

19.4 ANALYSIS OF FLEXURAL MEMBERS

19.4.1 Stresses Due to Loaded and Unloaded Conditions

In the analysis of prestressed concrete beams, two extreme loadings are generally critical. The first occurs at transfer, when the beam is subjected to the prestressing force, F_i, and the weight of the beam or the applied dead load at the time of transfer of the prestressing force. No live load or additional dead loads are considered. In this unloaded condition, the stresses at the top and bottom fibers of the critical section must not exceed the allowable stresses at transfers, f_{ci} and f_{ti}, for the compressive and tensile stresses in concrete, respectively.

The second case of loading occurs when the beam is subjected to the prestressing force after all losses F and all dead and live loads. In this loaded condition, the stresses at the top and bottom fibers of the critical section must not exceed the allowable stresses, f_c and f_t, for the compressive and tensile stresses in concrete, respectively.

These conditions can be expressed mathematically as follows.

1. For the unloaded condition (at transfer):
   - At top fibers,

$$\alpha_{ti} = -\frac{F_i}{A} + \frac{(F_i e)y_t}{I} - \frac{M_D y_t}{I} \leq f_{ti} \tag{19.14}$$

   - At bottom fibers,

$$\alpha_{bi} = -\frac{F_i}{A} - \frac{(F_i e)y_b}{I} + \frac{M_D y_b}{I} \geq -f_{ci} \tag{19.15}$$

2. For the loaded condition (all loads are applied after all losses):
   - At top fibers,

$$\sigma_t = -\frac{F}{A} + \frac{(Fe)y_t}{I} - \frac{M_D y_t}{I} - \frac{M_L y_t}{I} \geq -f_c \tag{19.16}$$

   - At bottom fibers,

$$\sigma_b = -\frac{F}{A} - \frac{(Fe)y_b}{I} + \frac{M_d y_b}{I} + \frac{M_L y_b}{I} \leq f_t \tag{19.16}$$

where

F_i and F = the prestressing force at transfer and after all losses
f_{ti} and f_t = allowable tensile stress in concrete at transfer and after all losses
f_{ci} and f_c = allowable compressive stress in concrete at transfer and after all losses
M_D and M_L = moments due to dead load and live load
y_t and y_b = distances from the neutral axis to the top and bottom fibers

In this analysis, it is assumed that the materials behave elastically within the working range of stresses applied.

19.4.2 Kern Limits

If the prestressing force is applied at the centroid of the cross-section, uniform stresses will develop. If the prestressing force is applied at an eccentricity, e below the centroid such that the stress at the top fibers is equal to 0, that prestressing force is considered acting at the lower Kern point (Fig. 19.5). In this case e is denoted by K_b, and the stress distribution is triangular, with maximum compressive stress at the extreme bottom fibers. The stress at the top fibers is

$$\begin{aligned} \sigma_t &= -\frac{F_i}{A} + \frac{(F_i e)y_t}{I} = 0 \\ e &= K_b = \text{lower Kern} = \frac{I}{Ay_t} \end{aligned} \tag{19.17}$$

Similarly, if the prestressing force is applied at an eccentricity e' above the centroid such that the stress at the bottom fibers is equal to 0, that prestressing force is considered acting at the upper Kern point (Fig. 19.5). In this case the eccentricity it e' is denoted by K_t, and the stress distribution is triangular, with maximum compressive stress at the extreme top fibers. The stress at the bottom fibers is

$$\begin{aligned} \sigma_b &= -\frac{F_i}{A} + \frac{(F_i e')y_b}{I} = 0 \\ e' &= K_t = \text{upper Kern} = \frac{I}{Ay_b} \end{aligned} \tag{19.18}$$

The Kern limits of a rectangular section are shown in Fig. 19.5.

19.4.3 Limiting Values of Eccentricity

The four stress equations, Eqs. 19.13 through 19.16,can be written as a function of the eccentricity e for the various loading conditions. For example, Eq. 19.13 can be rewritten as follows:

$$\sigma_{\text{ti}} = -\frac{F_i}{A} + \frac{(F_i e)y_t}{I} - \frac{M_D y_t}{I} \leq f_{\text{ti}}$$

$$\frac{(f_i e)y_t}{I} \leq f_{\text{ti}} + \frac{F_i}{A} + \frac{M_D y_t}{I}$$

$$e \leq \frac{I}{F_i y_t}\left(\frac{F_i}{A} + \frac{M_D y_t}{I} + f_{\text{ti}}\right) \tag{19.19}$$

If the lower Kern limit $K_b = I/Ay_t$ is used, then

$$e \leq K_b + \frac{M_D}{F_i} + \frac{f_{\text{ti}} A K_b}{F_i} \tag{19.20}$$

This value of e represents the maximum eccentricity based on the top fibers, unloaded condition.

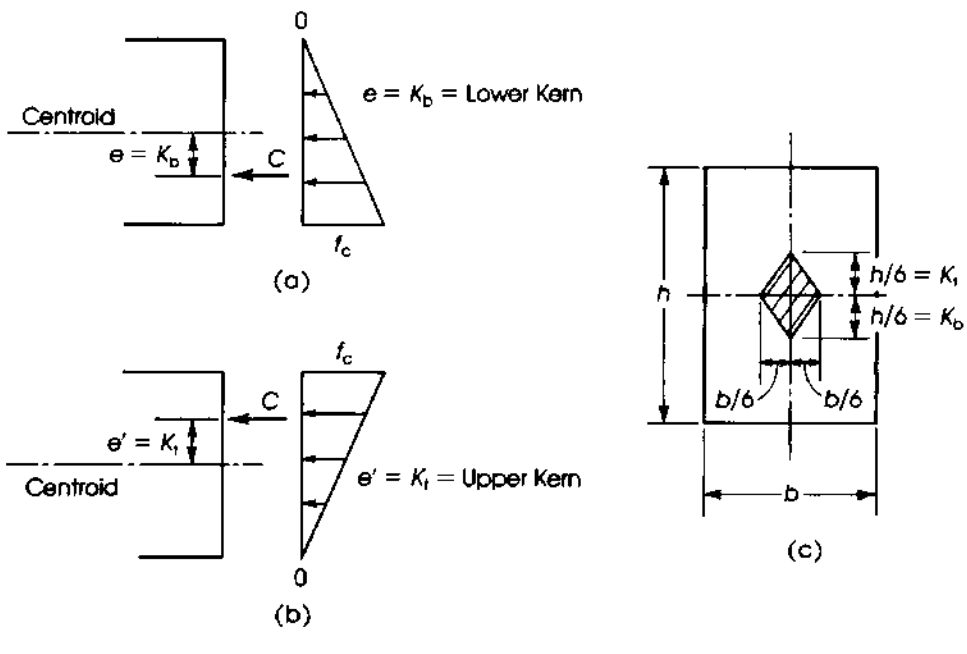


Figure 19.5 Kern points: (*a*) lower, (*b*) upper, and (*c*) central.

Similarly, from Eq. 19.14,

$$e \leq \frac{I}{F_i y_b}\left(-\frac{F_i}{A} + \frac{M_D y_b}{I} + f_{ci}\right) \tag{19.21}$$

$$e \leq -K_t + \frac{M_D}{F_i} + \frac{f_{ci} A K_t}{F_i} \tag{19.22}$$

This value of e represents the maximum eccentricity based on the bottom fibers, unloaded condition. The two maximum values of e should be calculated from the preceding equations and the smaller value used.

From Eq. 19.15,

$$e \geq \frac{I}{F y_t}\left(\frac{F}{A} + \frac{M_T y_t}{I} - f_c\right) \tag{19.23}$$

$$e \geq K_b + \frac{M_T}{F} - \frac{f_c A K_b}{F} \tag{19.24}$$

where M_T = moment due to dead and live loads = $(M_D + M_L)$. This value of e represents the minimum eccentricity based on the top fibers, loaded condition. From Eq. 19.17,

$$e \geq \frac{I}{F y_b}\left(-\frac{F}{A} + \frac{M_T y_b}{I} - f_t\right) \tag{19.25}$$

$$e \geq K_t + \frac{M_T}{F} - \frac{f_t A K_t}{F} \tag{19.26}$$

This value of e represents the minimum eccentricity based on the bottom fibers, loaded condition. The two minimum values of e should be calculated from the preceding equations and the larger of the two minimum eccentricities used.

19.4.4 Limiting Values of the Prestressing Force at Transfer F_i

Considering that $F = \eta F_i$, where η represents the ratio of the net prestressing force after all losses, and for the different cases of loading, Eqs. 19.20, 19.22, 19.24, and 19.26 can be rewritten as follows:

$$(e - K_b)F_i \leq M_D + f_{ti} A K_b \tag{19.27}$$

$$(e + K_t)F_i \leq M_D + f_{ci} A K_t \tag{19.28}$$

$$(e - K_b)F_i \geq \frac{M_D}{\eta} + \frac{M_L}{\eta} - \frac{1}{\eta}(f_c A K_t) \tag{19.29}$$

$$(e + K_t)F_i \geq \frac{M_D}{\eta} + \frac{M_L}{\eta} - \frac{1}{\eta}(f_t A K_t) \tag{19.30}$$

Subtract Eq. 19.29 from Eq. 19.32 to get

$$F_i(K_b + K_t) \geq M_D\left(\frac{1}{\eta} - 1\right) + \frac{M_L}{\eta} - \frac{f_t A K_t}{\eta} - f_{ti} A K_t$$

or

$$F_i \geq \frac{1}{(K_b + K_t)}\left[\left(\frac{1}{\eta} - 1\right) M_D + \frac{M_L}{\eta} - \left(\frac{f_t A K_t}{\eta}\right) - (f_{ti} A K_b)\right] \tag{19.31}$$

This value of F_i represents the *minimum* value of the prestressing force at transfer without exceeding the allowable stresses under the loaded and unloaded conditions. Subtract Eq. 19.29 from Eq. 19.28 toget

$$F_i \le \frac{1}{(K_b + K_t)}\left[\left(1 - \frac{1}{\eta}\right)M_D - \frac{M_L}{\eta} + \left(\frac{f_c A K_b}{\eta}\right) + (f_{ci} A K_t)\right] \tag{19.32}$$

This value of F_i represents the *maximum* value of the prestressing force at transfer without exceeding the allowable stresses under the loaded and unloaded conditions. Subtracting Eq. 19.31 from Eq. 19.32,

$$\left(1 - \frac{1}{\eta}\right)2M_D - \frac{2M_L}{\eta} + \left(f_{ti} + \frac{f_c}{\eta}\right)AK_b + \left(f_{ci} + \frac{f_t}{\eta}\right)AK_t \ge 0 \tag{19.33}$$

This equation indicates that (maximum F_i) − (minimum F_i) ≥ 0. If this equation is checked for any given section and proved to be satisfactory, then the section is adequate.

Example 19.4

A pretensioned simply supported beam of the section shown in Fig. 19.6 is to be used on a span of 48 ft. The beam made with normal-weight concrete must carry a dead load of 900 lb/ft (excluding its own weight), which will be applied at a later stage, and a live load of 1100 lb/ft. Assuming that prestressing steel is made of 20 tendons that are 716 in. in diameter, with $E_s = 29 \times 10^6$ psi, $F_o =$ 175 ksi, and ultimate strength $f_{pu} = 250$ ksi, it is required to do the following:

1. Determine the location of the upper and lower limits of the tendon profile (centroid of the prestressing steel) for the section at midspan and for three other sections between the midspan section and the beam end.
2. Locate the tendon to satisfy these limits by harping some of the tendons at one-third points of the span. Check the limiting values of the prestressing force at transfer.
3. Revise the prestress losses, taking into consideration the chosen profile of the tendons and the variation of the eccentricity, *e*.
   Use f_{ci} (at transfer) = 4 ksi, $f'_c = 5$ ksi, $E_c = 4000$ ksi, and $E_{ci} = 3600$ ksi.

Solution

1. Determine the properties of the section:

$$\text{Area} = 18 \times 6 + 24 \times 6 + 12 \times 10 = 372 \text{ in.}^2$$

Determine the centroid of the section by taking moments about the base line.

$$y_b = \frac{1}{372}(120 \times 5 + 144 \times 22 + 108 \times 37) = 20.8 \text{ in.}$$

$$y_t = 40 - 20.8 = 19.2 \text{ in.}$$

Calculate the gross moment of inertia, I_g:

$$I_g = \left[\frac{18(6)^3}{12} + 108(16.2)^2\right] + \left[\frac{6(24)^3}{12} + 144(1.2)^2\right] + \left[\frac{12(10)^3}{12} + 120(15.8)^2\right]$$

$$= 66{,}862 \text{ in.}^4$$

$$K_b = \frac{I}{Ay_t} = \frac{66{,}862}{372 \times 19.2} = 9.4 \text{ in.}$$

$$K_t = \frac{I}{Ay_b} = \frac{66{,}862}{372 \times 20} = 8.6 \text{ in.}$$