5. Confinement of bars by lateral ties. Adequate confinement by ties or stirrups prevents the spalling of concrete around bars.

7.2 DEVELOPMENT OF BOND STRESSES

7.2.1 Flexural Bond

Consider a length dx of a beam subjected to uniform loading. Let the moment produced on one side be M_1 and on the other side be M_2 with M_1 being greater than M_2. The moments will produce internal compression and tension forces, as shown in Fig. 7.1. Because M_1 is greater than M_2, T_1 is greater than T_2; consequently, C_1 is greater than C_2.

At any section, $T = M/jd$, where jd is the moment arm:

$$T_1 - T_2 = dT = \frac{dM}{jd}$$

but

$$T_1 = T_2 + u\Sigma O dx$$

where u is the average bond stress and ΣO is the sum of perimeters of bars in the section at the tension side. Therefore,

$$T_1 - T_2 = u\Sigma O dx = \frac{dM}{jd}$$

$$u = \frac{dM}{dx} \times \frac{1}{jd\Sigma O}$$

The rate of change of the moment with respect to x is the shear, or $dM/dx = V$. Therefore,

$$u = \frac{V}{jd\Sigma O} \tag{7.1}$$

The value u is the average bond stress; for practical calculations, j can be taken to be approximately equal to 0.87:

$$u = \frac{V}{0.87\, d\Sigma O}$$

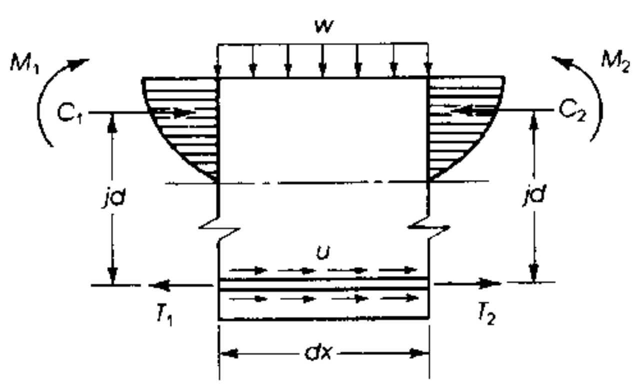


Figure 7.1 Flexural bond.

In the strength design method, the nominal bond strength is reduced by the capacity reduction factor, $\phi = 0.85$. Thus,

$$U_u = \frac{V_u}{\phi(0.87)d\Sigma O} \tag{7.2}$$

Based on the preceding analysis, the bond stress is developed along the surface of the reinforcing bar due to shear stresses and shear interlock.

7.2.2 Tests for Bond Efficiency

Tests to determine the bond stress capacity can be made using the pullout test (Fig. 7.2). This test evaluates the bond capacity of various types of bar surfaces relative to a specific embedded length. The distribution of tensile stresses will be uniform around the reinforcing bar at a specific section and varies along the anchorage length of the bar and at a radial distance from the surface of the bar (Fig. 7.2). However, this test does not represent the effective bond behavior in the surface of the bars in flexural members, because stresses vary along the depth of the concrete section. A second type of test can be performed on an embedded rod (Fig. 7.3). In these tests, the tensile force, P, is increased gradually and the number of cracks and their spacings and widths are recorded. The bond stresses vary along the bar length between the cracks. The strain in the steel bar is maximum at the cracked section and decreases toward the middle section between cracks.

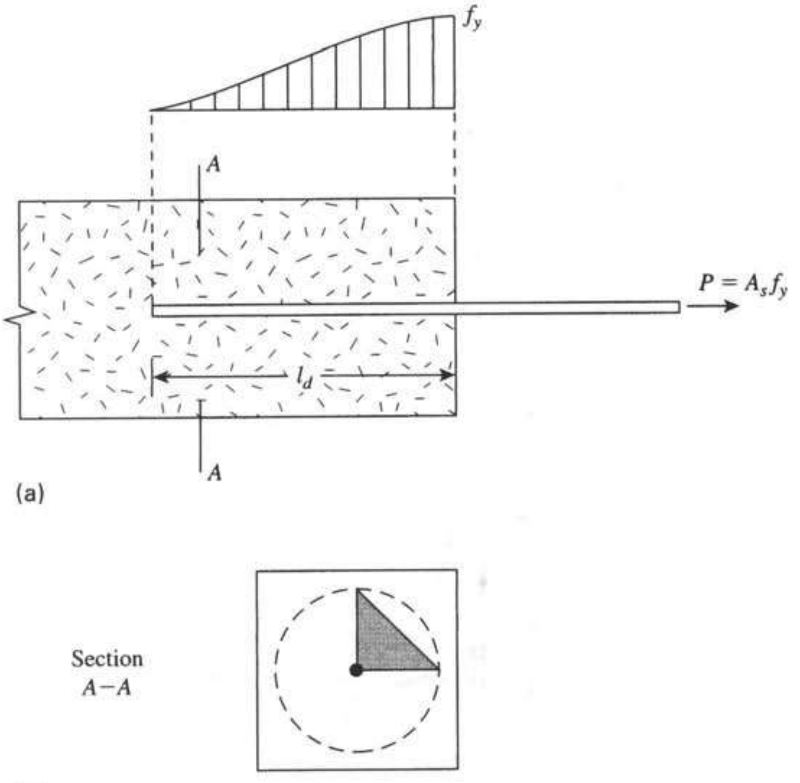


Figure 7.2 Bond stresses and development length. (*a*) Distribution of stress along l_d and (*b*) radial stress in concrete around the bar.

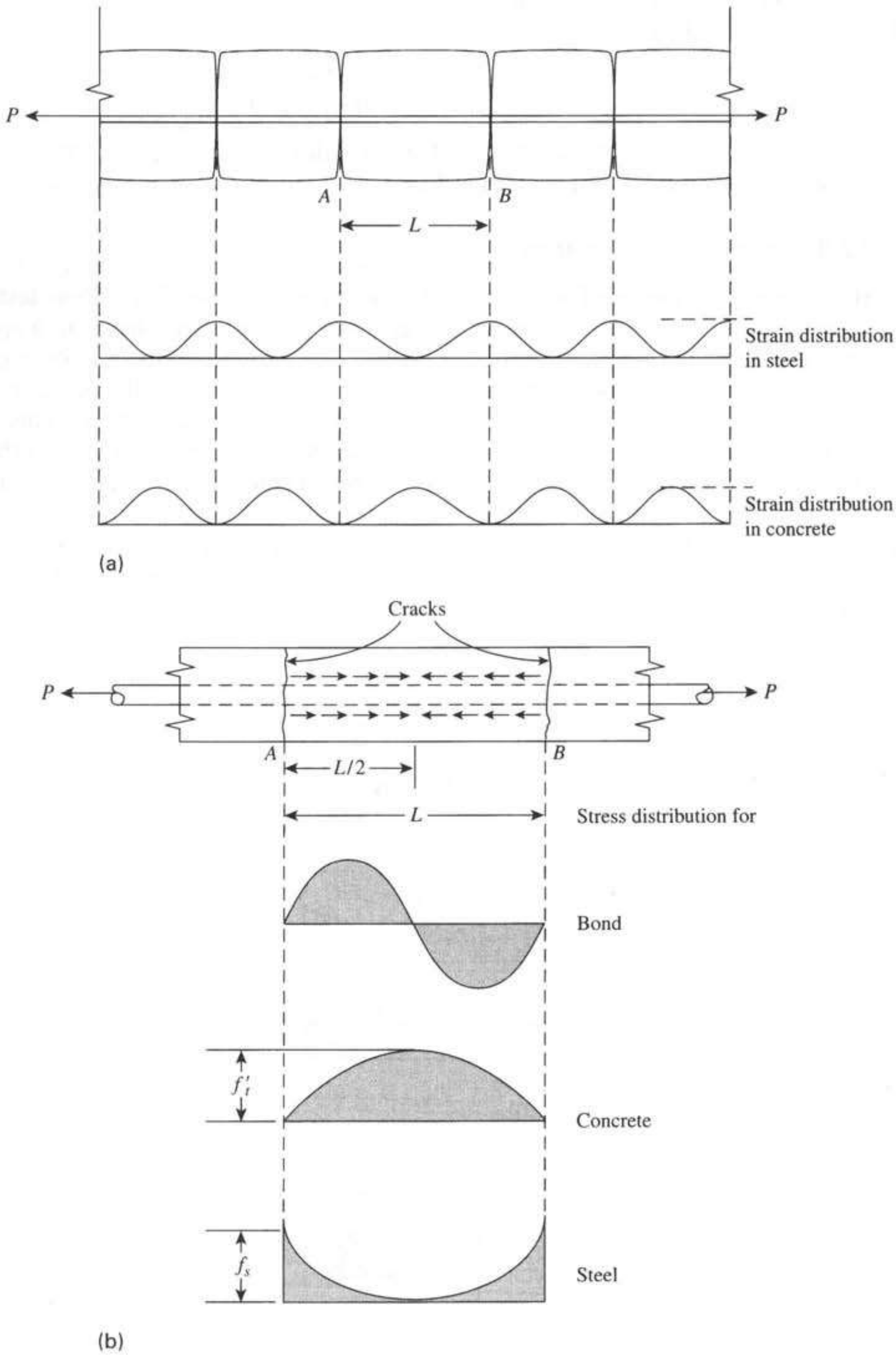


Figure 7.3 Bond mechanism in an embedded bar. Strain (*a*) and stress (*b*) distribution between cracks.

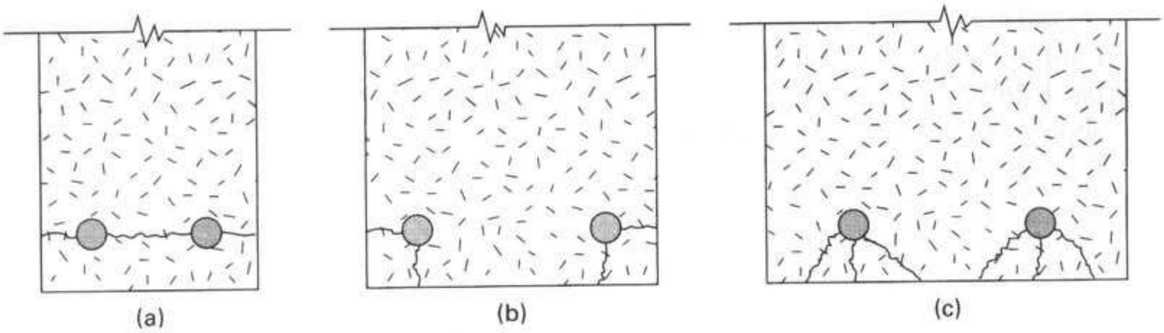


Figure 7.4 Examples of spalling of concrete cover. (*a*) High bottom cover, (*b*) wide spacing, and (*c*) small bottom cover.

Tests on flexural members are also performed to study the bond effectiveness along the surface of the tension bars. The analysis of bond stresses in the bars of these members was explained earlier, and they are represented by Eq. 7.2.

Based on this discussion, it is important to choose an appropriate length in each reinforcing bar to develop its full yield strength without a failure in the bond strength. This length is called the *development length*, l_d. If this length is not provided, the bond stresses in the tension zone of a beam become high enough to cause cracking and splitting in the concrete cover around the tension bars (Fig. 7.4). If the split continues to the end of the bar, the beam will eventually fail. Note that small spacings between tensile bars and a small concrete cover on the sides and bottom will reduce the bond capacity of the reinforcing bars (Fig. 7.4).

7.3 DEVELOPMENT LENGTH IN TENSION

7.3.1 Development Length, l_d

If a steel bar is embedded in concrete, as shown in Fig. 7.2, and is subjected to a tension force T, then this force will be resisted by the bond stress between the steel bar and the concrete. The maximum tension force is equal to $A_s f_y$, where A_s is the area of the steel bar. This force is resisted by another internal force of magnitude $U_u O l_d$, where U_u is the ultimate average bond stress, l_d is the embedded length of the bar, and O is the perimeter of the bar (πD). The two forces must be equal for equilibrium:

$$A_s f_y = U_u O l_d \quad \text{and} \quad l_d = \frac{A_s f_y}{U_u O}$$

For a combination of bars,

$$l_d = \frac{A_s f_y}{U_u \Sigma O} \tag{7.3}$$

The length l_d is the minimum permissible anchorage length and is called the development length.

$$l_d = \frac{\pi d_b^2 f_y}{4U_u(\pi d_b)} = \frac{d_b f_y}{4U_u} \tag{7.4}$$

where d_b = diameter of reinforcing bars.

This means that the development length is a function of the size and yield strength of the reinforcing bars in addition to the ultimate bond stress, which in turn is a function of $\sqrt{f'_c}$. The bar length l_d given in Eq. 7.4 is called the *development length*, l_d. The final development length should also include the other factors mentioned in Section 7.1. Equation 7.4 may be written as follows:

$$\frac{l_d}{d_b} = K\left(\frac{f_y}{\sqrt{f'_c}}\right) \tag{7.5}$$

where K is a general factor that can be obtained from tests to include factors such as the bar characteristics (bar size, spacing, epoxy coated or uncoated, location in concrete section, and bar splicing), amount of transverse reinforcement, and the provision of excess reinforcement compared to that required from design.

The ACI Code, Section 12.2.3, evaluated K as follows:

$$K = \left(\frac{3}{40\lambda}\right)\frac{\psi_t\psi_e\psi_s}{\dfrac{(c_b + K_{tr})}{d_b}} \tag{7.6}$$

and Eq. 7.5 becomes

$$\frac{l_d}{d_b} = \frac{3}{40\lambda}\frac{f_y}{\sqrt{f'_c}}\frac{\psi_t\psi_e\psi_s}{\left(\dfrac{c_b + K_{tr}}{d_b}\right)} \tag{7.7}$$

where

ψ_t = bar location
ψ_e = coating factor
ψ_s = bar-size factor
λ = lightweight aggregate concrete factor (ACI Code, Section 8.6.1)
= 1.0 normal-weight concrete
= shall not exceed 0.75 unless splitting tensile strength is specified, then
$\lambda = f_{ct}/(6.7\sqrt{f'_c}) \le 1$
c_b = spacing or cover dimension (in.), whichever is smaller
K_{tr} = transverse reinforcement index
= $40A_{tr}/sn$
n = number of bars or wires being developed along the plane of splitting
s = maximum spacing of transverse reinforcement within l_d, center to center (in.).
f_{yt} = yield strength of transverse reinforcement (psi)
A_{tr} = total sectional area of all transverse reinforcement within spacing s that crosses the potential plane of splitting through to the reinforcement being developed (in.^{2})

Notes:

1. $(c_b + K_{tr})/d_b$ shall not exceed 2.5 to safeguard against pullout-type failures.
2. The value of $\sqrt{f'_c}$ shall not exceed 100 psi (ACI Code, Section 12.1.2).
3. $K_{tr} = 0$ can be used as a design simplification (ACI Code, Section 12.2.3).

7.3.2 ACI Code Factors for Calculating l_d for Bars in Tension

1. ψ_t = bar location factor
   ψ_t = 1.3 for top bars defined as horizontal reinforcement, placed so that more than 12 in. of fresh concrete is below the development length, or splice
   ψ_t = 1.0 for all other reinforcement
2. ψ_e = coating factor
   ψ_e = 1.5 for epoxy-coated bars or wires with cover less than $3d_b$ or clear spacing less than $6d_b$
   ψ_e = 1.2 for all other epoxy coated bars or wires
   ψ_e = 1.0 for uncoated and zinc-coated (galvanized) reinforcement (However, the value of the product $\psi_t \psi_e$ should not exceed 1.7)
3. ψ_s = bar size factor
   ψ_s = 0.8 for no. 6 bars or smaller bars and deformed wires
   ψ_s = 1.0 for no. 7 bars and larger bars
4. λ = lightweight aggregate concrete factor
   λ = λ shall not exceed 0.75 unless f_{ct} is specified
   λ = 1.0 for normal-weight concrete
5. The ACI Code permits using $K_{tr} = 0$ even if transverse reinforcement is present. In this case,

$$\frac{l_d}{d_b} = \left(\frac{3}{40\lambda}\right)\left(\frac{f_y}{\sqrt{f'_c}}\right)\left(\frac{\psi_t \psi_e \psi_s}{(c_b/d_b)}\right) \tag{7.7a}$$

   The value of $\sqrt{f'_c}$ should not exceed 100 psi.
6. R_s is the reduction factor due to excess reinforcement. The ACI Code, Section 12.2.5, permits the reduction of l_d by the factor R_s when the reinforcement in a flexural member exceeds that required by analysis, except where anchorage or development for f_y is specifically required or the reinforcement is designed considering seismic effects.

$$R_s = \frac{A_s \text{ (required)}}{A_s \text{ (provided)}}$$

7. The development length, l_d, in all cases shall not be less than 12 in.

7.3.3 Simplified Expressions for l_d

The ACI Code, Section 12.2.2, permits the use of simplified expressions to calculate the ratio l_d/d_b. This is based on the fact that current practical construction cases utilize spacing and cover values along with confining reinforcement, such as stirrups and ties, that produce a value of $(c_b + K_{tr})/d_b \geq 1.5$. Moreover, tests indicated that the development length, l_d, can be reduced by 20% for no. 6 and smaller bars. Based on these assumptions and assuming $(c_b + K_{tr})/d_b = 1.5$, Eq. 7.7 can be reduced to the following expressions:

1. For no. 7 and larger bars,

$$\frac{l_d}{d_b} = \left(\frac{f_y}{\sqrt{f'_c}}\right)\frac{\psi_t \psi_e}{20\lambda} \tag{7.8}$$

   For no. 6 and smaller bars and deformed wires,

$$\frac{l_d}{d_b} = \left(\frac{f_y}{\sqrt{f'_c}}\right)\frac{\psi_t \psi_e}{25\lambda} \tag{7.9}$$

The ratio l_d/d_b in Eq. 7.9 represents 80% of that in Eq. 7.8. These equations are used when one of the following conditions is met:

a. Clear spacing of bars or wires being developed or spliced not less than d_b, clear cover not less than d_b, and stirrups or ties throughout l_d not less than the code minimum.

b. Clear spacing of bars or wires being developed or spliced not less than $2d_b$ and clear cover not less than d_b.

2. For all other cases, the value of l_d/d_b in Eqs. 7.8 and 7.9 must be multiplied by 1.5 to restore them to equivalence with Eq. 7.7.

These equations are relatively simple to use for the general conditions involved in practical design and construction. For example, in all structures with normal-weight concrete ($\psi_t = 1.0$), uncoated reinforcement ($\psi_e = 1.0$), no. 7 *or larger* bars ($\psi_s = 1.0$), Eq. 7.8 becomes

$$\frac{l_d}{d_b} = \frac{f_y}{(20\lambda\sqrt{f'_c})} \tag{7.10}$$

This equation is used when conditions a and b are met, whereas for all other cases, l_d/d_b is multiplied by 1.5, or

$$\frac{l_d}{d_b} = \frac{3f_y}{(40\lambda\sqrt{f'_c})} \tag{7.11}$$

Similarly, for the same conditions and for no. 6 *or smaller* bars, Eq. 7.9 becomes

$$\frac{l_d}{d_b} = \frac{f_y}{(25\lambda\sqrt{f'_c})} \tag{7.12}$$

This is used when conditions a and b are met; for all other cases, l_d/d_b is multiplied by 1.5, or

$$\frac{l_d}{d_b} = \frac{3f_y}{(50\lambda\sqrt{f'_c})} \tag{7.13}$$

It is quite common to use $f'_c = 4$ ksi and $f_y = 60$ ksi in the design and construction of reinforced concrete buildings. If these values are substituted in the preceding equations, and assuming normal-weight concrete ($\lambda = 1.0$) then

Equation 7.10 becomes	$l_d = 47.5d_b$	($\geq$ no. 7 bars).	(7.10a)
Equation 7.11 becomes	$l_d = 71.2d_b$	($\geq$ no. 7 bars).	(7.11a)
Equation 7.12 becomes	$l_d = 38d_b$	($\leq$ no. 6 bars).	(7.12a)
Equation 7.13 becomes	$l_d = 57d_b$	($\leq$ no. 6 bars).	(7.13a)

Other values of l_d/d_b ratios are shown in Table 7.1. Table 7.2 gives the development length, l_d, for different reinforcing bars (when $f_y = 60$ ksi and $f'_c = 3$ ksi and 4 ksi) for both cases, when conditions a and b are met and for all other cases.

7.4 DEVELOPMENT LENGTH IN COMPRESSION

The development length of deformed bars in compression is generally smaller than that required for tension bars, due to the fact that compression bars do not have the cracks that develop in tension concrete members that cause a reduction in the bond between bars and the surrounding

Table 7.1 Values of l_d/d_b for Various Values of f'_c and f_y (Tension Bars), ($\lambda = 1.0$)

	$f_y = 40$ ksi				$f_y = 60$ ksi			
	No. 6 Bars		≥ No. 7 Bars		No. 6 Bars		≥ No. 7 Bars	
f'_c (ksi)	Conditions met	Other cases	Conditions met	Other cases	Conditions met	Other cases	Conditions met	Other cases
3	29.3	43.9	36.6	54.8	43.9	65.8	54.8	82.2
4	25.3	38.0	31.7	47.5	38.0	57.0	47.5	71.2
5	22.7	34.0	28.3	42.5	34.0	51.0	42.5	63.7
6	20.7	31.0	25.9	38.8	31.0	46.5	38.8	58.1

Table 7.2 Development Length l_d (in.) for Tension Bars and $f_y = 60$ ksi ($\psi_t = \psi_e = \lambda = 1.0$)

		Development Length l_d (in.) — Tension Bars			
		$f'_c = 3$ksi		$f'_c = 4$ ksi	
Bar number	Bar diameter (in.)	Conditions met	Other cases	Conditions met	Other cases
3	0.375	17	25	15	21
4	0.500	22	33	19	29
5	0.625	28	41	24	36
6	0.750	33	50	29	43
7	0.875	48	72	42	63
8	1.000	55	83	48	72
9	1.128	62	93	54	81
10	1.270	70	105	61	92
11	1.410	78	116	68	102

concrete. The ACI Code, Section 12.3.2, gives the basic development length in compression for all bars as follows:

$$l_{dc} = \frac{0.02 d_b f_y}{\lambda \sqrt{f'_c}} \geq 0.0003 d_b f_y \tag{7.14}$$

which must not be less than 8 in. The development length, l_{dc}, may be reduced by multiplying l_{dc} by $R_s = (A_s \text{ required})/(A_s \text{ provided})$. For spirally reinforced concrete compression members with spirals of not less than $\frac{1}{4}$ in. diameter and a spacing of 4 in. or less, the value of l_{dc} in Eq. 7.14 may be multiplied by $R_{sl} = 0.75$. In general, $l_d = l_{dc} \times (R_s \text{ or } R_{sl}$, if applicable) ≥ 8 in. Tables 7.3 and 7.4 give the values of l_{dc}/d_b when $f_y = 60$ ksi.

Table 7.3 Values of l_d/d_b for Various Values of f'_c and f_y (Compression Bars), $\lambda = 1.0$, Minimum $l_{dc} =$ 8 in. $l_{dc}/d_b = 0.02 f_y / \lambda \sqrt{f'_c} \geq 0.0003 f_y$

f'_c(ksi)	3	4	5 or more
$f_y = 40$ ksi	15	13	12
$f_y = 60$ ksi	22	19	18

Table 7.4 Development Length, l_{dc} (in.), for Compression Bars ($f_y = 60$ ksi), $\lambda = 1.0$

Bar number	Bar diameter (in.)	Development Length, l_{dc} (in.) when f'_c = 3 (ksi)	4 (ksi)	5 (ksi) or more
3	0.375	9	8	8
4	0.500	11	10	9
5	0.625	14	12	12
6	0.750	17	15	14
7	0.875	20	17	16
8	1.000	22	19	18
9	1.128	25	22	21
10	1.270	28	25	23
11	1.410	31	27	26

7.5 SUMMARY FOR THE COMPUTATION OF l_d IN TENSION

Assuming normal construction practices, $(c_b + K_{tr})/d_b = 1.5$.

1. If one of the following two conditions is met:

a. Clear spacing of bars $\geq d_b$, clear cover $\geq d_b$, and bars are confined with stirrups not less than the code minimum.

b. Clear spacing of bars $\geq 2d_b$ and clear cover $\geq d_b$; then

$$\text{for no. 7 and larger bars, } \frac{l_d}{d_b} = \frac{\psi_t \psi_e f_y}{20\lambda\sqrt{f'_c}} \tag{7.8}$$

$$\text{for no. 6 or smaller bars, } \frac{l_d}{d_b} = \frac{\psi_t \psi_e f_y}{25\lambda\sqrt{f'_c}} \tag{7.9}$$

2. For all other cases, multiply these ratios by 1.5.

3. Note that $f'_c \leq 100$ psi and $\psi_t \psi_e \leq 1.7$; values of ψ_t, ψ_e, and λ are as explained earlier.

4. For bundled bars, either in tension or compression, l_d should be increased by 20% for three-bar bundles and by 33% for four-bar bundles. A unit of bundled bars is considered a single bar of a diameter and area equivalent to the total area of all bars in the bundle. This equivalent diameter is used to check spacings and concrete cover.

Example 7.1

Figure 7.5 shows the cross-section of a simply supported beam reinforced with four no. 8 bars that are confined with no. 3 stirrups spaced at 6 in. Determine the development length of the bars if the beam is made of normal-weight concrete, bars are not coated, $f'_c = 3$ ksi, and $f_y = 60$ ksi.

Solution

1. Check if conditions for spacing and concrete cover are met:

a. For no. 8 bars, $d_b = 1.0$ in.

b. Clear cover $= 2.5 - 0.5 = 2.0$ in. $> d_b$

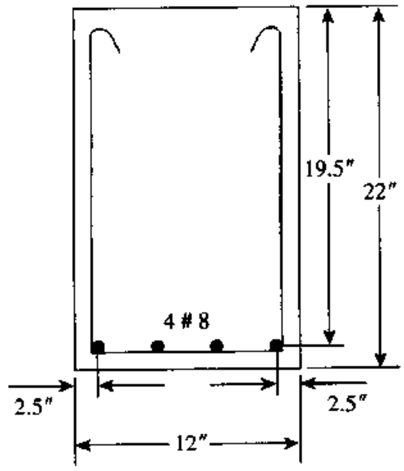


Figure 7.5 Example 7.1.

c. Clear spacing between bars $= \dfrac{12-5}{3} - 1.0 = 1.33$ in. $> d_b$

d. Bars are confined with no. 3 stirrup. The conditions are met. Then

$$\frac{l_d}{d_b} = \frac{\psi_t \psi_e f_y}{20\lambda\sqrt{f'_c}} \quad \text{(for bars > no. 7)} \tag{7.8}$$

2. Determine the multiplication factors: $\psi_t = 1.0$ (bottom bars), $\psi_e = 1.0$ (no coating), and $\lambda =$ 1.0 (normal-weight concrete). Also check that $\sqrt{f'_c} = 54.8$ psi < 100 psi.

$$\frac{l_d}{d_b} = \frac{60,000}{(20 \times 1 \times \sqrt{3000})} = 54.8$$

So, $l_d = 54.8(1.0) = 54.8$ in., say, 55 in. These values can be obtained directly from Tables 7.1 and 7.2. Note that if the general formula for l_d/d_b (Eq. 7.7) is used, assuming $K_{tr} = 0$, then

$$\frac{l_d}{d_b} = \left(\frac{3}{40\lambda}\right)\left(\frac{f_y}{\sqrt{f'_c}}\right)\left(\frac{\psi_t \psi_e}{c_b/d_b}\right) \tag{7.7}$$

In this example, $\psi_t = \psi_e = \lambda = 1$.
Also, c_b = smaller of distance from center of bar to the nearest concrete surface (c_1) or one-half the center-to-center of bars spacing (c_2).

$$c_1 = 2.5 \text{ in.} \quad c_2 = \frac{0.5(12-5)}{3} = 1.17 \text{ in. (controls)}$$

$f_y/(c_b + K_{tr})/d_b = 1.17/1.0 = 1.17 < 1.5$, so use $(c_b + K_{tr})/d_b = 1.5$. Consequently, $l_d/d_b =$ 60,000$(20\lambda\sqrt{f'_c})$ as in step 2, and $l_d = 55$ in.
Note : If the bars are not confined by stirrups, this value of l_d must be multiplied by 1.5 ($s =$ 1.33 in. $< 2d_b = 2.0$ in.).

Example 7.2

Repeat Example 7.1 if the beam is made of lightweight aggregate concrete, the bars are epoxy coated, and A_s required from analysis is 2.79 in.^{2}

Solution

1. Determine the multiplication factors: $\psi_t = 1.0$ (bottom bars), $\psi_e = 1.5$ (epoxy coated), $\lambda = 0.75$ (lightweight aggregate concrete), and $R_s = (A_s \text{ required})/(A_s \text{ provided}) = 2.79/3.14 = 0.89$. The value of ψ_e is 1.5, because the concrete cover is less than $3d_b = 3$ in. Check that $\psi_t \psi_e = 1.0(1.5) = 1.5 < 1.7$.
2. $$\frac{l_d}{d_b} = \frac{R_s \psi_t \psi_e f_y}{20\lambda\sqrt{f'_c}} \quad \text{(for bars > no. 7)}$$
$$= \frac{0.89(1.0)(1.5)(60,000)}{((20)(0.75)\sqrt{3000})} = 73.1 \text{ in.,} \quad \text{say, 74 in.}$$
3. The development length l_d can be obtained from Table 7.2 ($l_d = 55$ in. for no. 8 bars) and then divided by the factor 0.75.

Example 7.3

A reinforced concrete column is reinforced with eight no. 10 bars, which should extend to the footing. Determine the development length needed for the bars to extend down in the footing. Use normal-weight concrete with $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

The development length in compression is

$$l_{dc} = \frac{0.02 d_b f_y}{\lambda\sqrt{f'_c}} \geq 0.0003 d_b f_y$$

$$l_{dc} = \frac{0.02(1.27)(60,000)}{(1)\sqrt{4000}} = 24.1 \text{ in.} \quad \text{(controls)}$$

The minimum l_{dc} is 0.0003(1.27)(60,000) = 22.86 in., but it cannot be less than 8 in. Because there are no other multiplication factors, then $l_d = 24.1$ in., or 25 in. (The same value is shown in Table 7.4.)

7.6 CRITICAL SECTIONS IN FLEXURAL MEMBERS

The critical sections for development of reinforcement in flexural members are

- At points of maximum stress
- At points where tension bars within the span are terminated or bent
- At the face of the support
- At points of inflection at which moment changes signs

The critical sections for a typical uniformly loaded continuous beam are shown in Fig. 7.6. The sections and the relative development lengths are explained as follows:

1. Three sections are critical for the negative moment reinforcement: Section 1 is at the face of the support, where the negative moment as well as stress are at maximum values. Two development lengths, x_1 and x_2, must be checked.

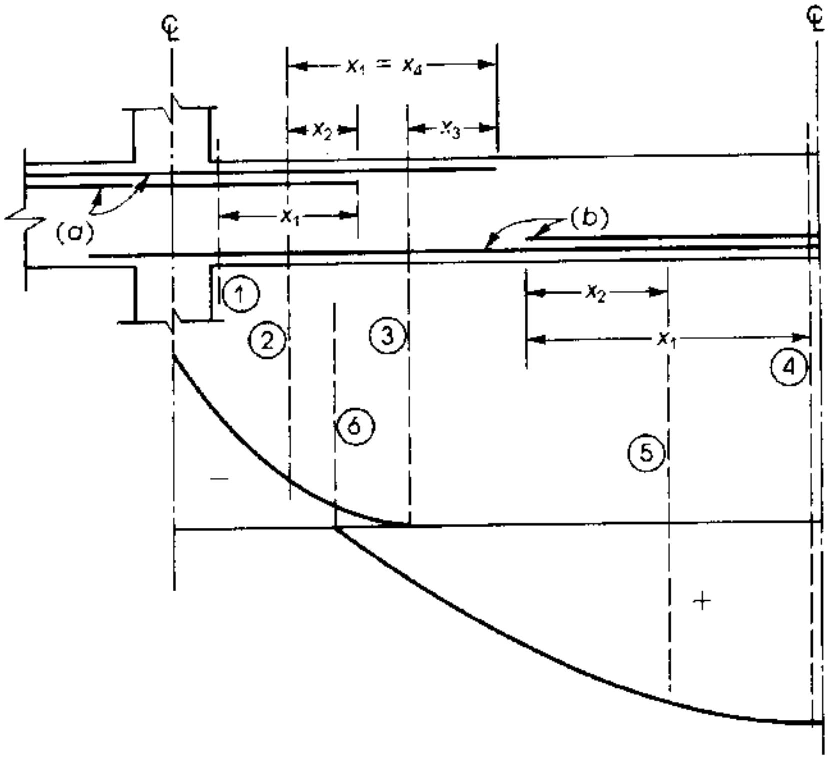


Figure 7.6 Critical sections (circled numbers) and development lengths ($X_1 - X_4$).

Section 2 is the section where part of the negative reinforcement bars can be terminated. To develop full tensile force, the bars should extend a distance x_2 before they can be terminated. Once part of the bars are terminated, the remaining bars develop maximum stress.

Section 3 is at the point of inflection. The bars shall extend a distance x_3 beyond section 3:x_3 must be equal to or greater than the effective depth, d, 12 bar diameters, or $\frac{1}{16}$ clear span, whichever is greater. At least one-third of the total reinforcement provided for negative moment at the support shall be extended a distance x_3 beyond the point of inflection, according to the ACI Code, Section 12.12.3.

2. Three sections are critical for positive moment reinforcement: Section 4 is that of maximum positive moment and maximum stresses. Two development lengths, x_1 and x_2, have to be checked. The length x_1 is the development length l_d specified by the ACI Code, Section 12.11, as mentioned later. The length x_2 is equal to or greater than d or 12 bar diameters.

   Section 7.5 is where part of the positive reinforcement bars may be terminated. To develop full tensile force, the bars should extend a distance x_2. The remaining bars will have a maximum stress due to the termination of part of the bars. At the face of support, section 7.1, at least one-fourth of the positive moment reinforcement in continuous members shall extend along the same face of the member into the support, according to the ACI Code, Section 12.11.1. For simple members, at least one-third of the reinforcement shall extend into the support.

   At points of inflection, section 7.6, limits are according to Section 12.11.3 of the ACI Code.

Example 7.4

A continuous beam has the bar details shown in Fig. 7.7. The bending moments for maximum positive and negative moments are also shown. We must check the development lengths at all critical sections. Given: $f_c' = 3$ ksi normal-weight concrete, $f_y = 40$ ksi, $b = 12$ in., $d = 18$ in., and span $L = 24$ ft.

Solution

The critical sections are (1) at the face of the support for tension and compression reinforcement (section 1), (7.2) at points where tension bars are terminated within the span (sections 2 and 5), (3) at point of inflection (sections 3 and 6), and (4) at midspan (section 4).

1. Development lengths for negative-moment reinforcement, from Fig. 7.7, are as follows: Three no. 9 bars are terminated at a distance $x_1 = 4.5$ ft from the face of the support, whereas the other three bars extend to a distance of 6 ft 0 in. (72 in.) from the face of the support.

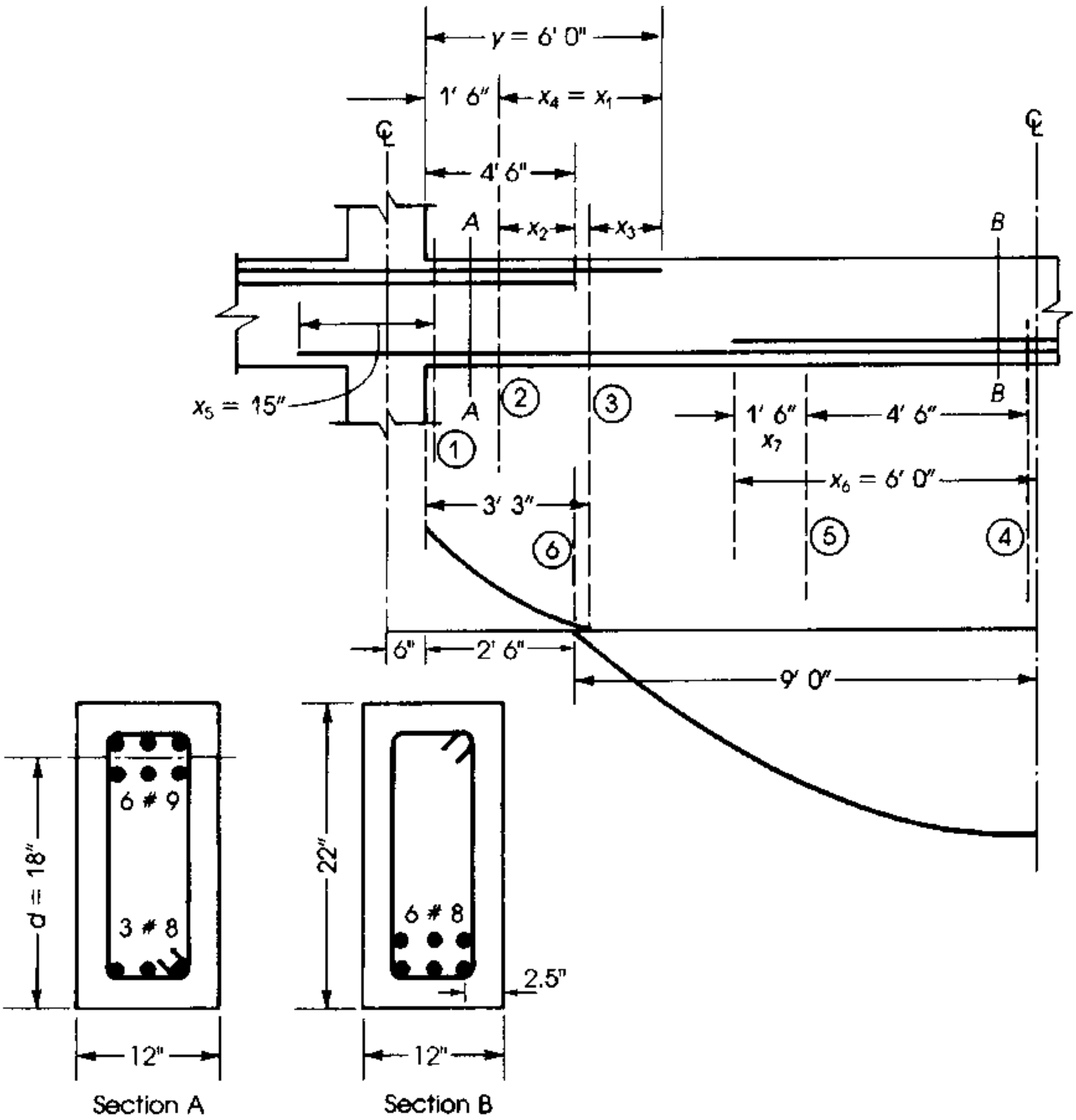


Figure 7.7 Example 7.4: Development length of a continuous beam.

a. The development length of no. 9 tension bars is $36.3d_b$ (Table 7.1) if conditions of spacing and cover are met.

$$\text{For no.9bars, } d_b = 1.128 \text{ in.}$$

$$\text{Cover} = 2.5 - \frac{1.128}{2} = 1.94 \text{ in.} > d_b$$

$$\text{Clear spacing} = \frac{12 - 5}{2} - 1.128 = 2.37 \text{ in.} > 2d_b$$

Then conditions are met, and $l_d = 36.6(1.128) = 41.3$ in. For top bars, $x_1 = l_d = 1.3(41.3) = 54$ in. $= 4.50$ ft $= x_1 > 12$ in. (minimum).

b. The development length x_2 shall extend beyond the point where three no. 9 bars are not needed, either $d = 18$ in. or $12d_b = 13.6$ in., whichever is greater. Thus, $x_2 = 18$ in. The required development length is $x_4 = 4.50$ ft, similar to x_1. Total length required is $y = x_1 + 1.5\text{ ft} = 6.0$ ft.

c. Beyond the point of inflection (section 3), three no. 9 bars extend a length $x_3 = y - 39 = 72 - 39 = 33$ in. The ACI Code requires that at least one-third of the bars should extend beyond the inflection point. Three no. 9 bars are provided, which are adequate. The required development length of x_3 is the greatest of $d = 18$ in., $12d_b = 13.6$ in., or $L/16 = 24 \times \frac{12}{16}$ in. $= 18$ in., which is less than x_3 provided.

2. Compressive reinforcement at the face of the support (section 7.1) (no. 8 bars): The development length x_5 is equal to

$$l_{dc} = \frac{0.02 d_b f_y}{\lambda\sqrt{f_c'}} = \frac{0.02 \times 1 \times 40,000}{1 \times \sqrt{3000}} = 14.6 \text{ in.}$$

So, we can use 15 in.

$$\text{Minimum } l_{dc} = 0.0003 d_b f_y = 0.0003 \times 1 \times 40,000 = 12 \text{ in.}$$

but it cannot be less than 8 in. The length 15 in. controls. For no. 8 bars, $d_b = 1$ in.; l_{dc} provided $= 15$ in., which is greater than that required.

3. Development length for positive moment reinforcement: Three no. 8 bars extend 6 ft beyond the centerline, and the other bars extend to the support. The development length x_6 from the centerline is $l_d = 36.6d_b = 37$ in. (Table 7.1), but it cannot be less than 12 in. That is, x_6 provided is 6 ft $= 72$ in. > 37 in.

The length x_7 is equal to d or $12d_b$, that is, 18 in. or $12 \times 1 = 12$ in. The provided value is 18 in., which is adequate.

The actual position of the termination of bars within the span can be determined by the moment-resistance diagram, as will be explained later.

7.7 STANDARD HOOKS (ACI CODE, SECTIONS 12.5 AND 7.1)

A *hook* is used at the end of a bar when its straight embedment length is less than the necessary development length, l_d. Thus the full capacity of the bar can be maintained in the shortest distance of embedment. The minimum diameter of bend, measured on the inside of the main bar of a standard hook D_b, is as follows (Fig. 7.8) [[9]]:

- For no. 3 to no. 8 bars (10–25 mm), $D_b = 6d_b$.
- For no. 9 to no. 11 bars (28, 32, and 36 mm), $D_b = 8d_b$.
- For no. 14 and no. 18 bars (43 and 58 mm), $D_b = 10d_b$.

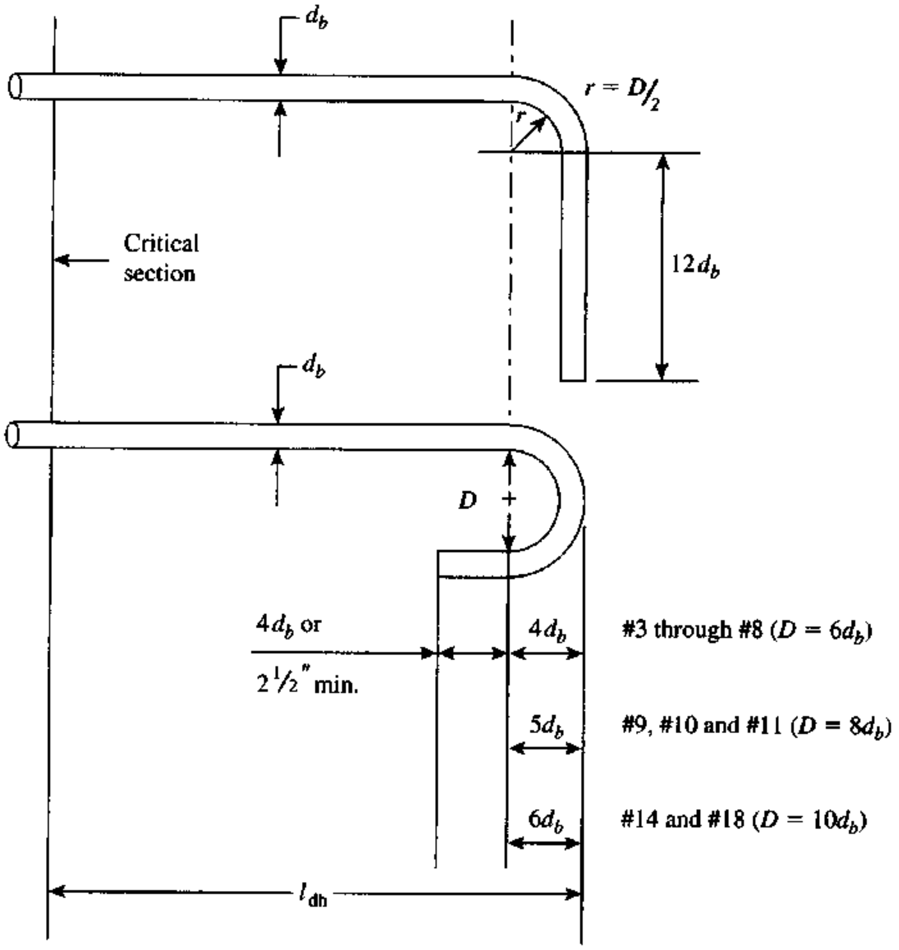


Figure 7.8 Hooked-bar details for the development of standard hooks [9]. Courtesy of ACI.

The ACI Code, Section 12.5.2, specifies a development length l_{dh} for hooked bar as follows:

$$l_{dh} = \left(\frac{0.02\psi_e f_y}{\lambda\sqrt{f_c'}}\right) \text{(Modification Factor) } d_b \tag{7.15}$$

where

$\psi_e = 1.2$ for epoxy-coated bars
$\lambda = 0.75$ for lightweight aggregate concrete unless f_{ct} is specified then $\lambda = f_{ct}/(6.7(\sqrt{f_c'}) \leq 1$
ψ_e and $\lambda = 1.0$ for all other cases

For grade 60 hooked bar ($f_y = 60$ ksi) with $\psi_e = \lambda = 1$, l_{dh} becomes:

$$l_{dh} = \frac{1200d_b}{\sqrt{f_c'}} \text{(Modification Factor) } d_b \tag{7.15a}$$

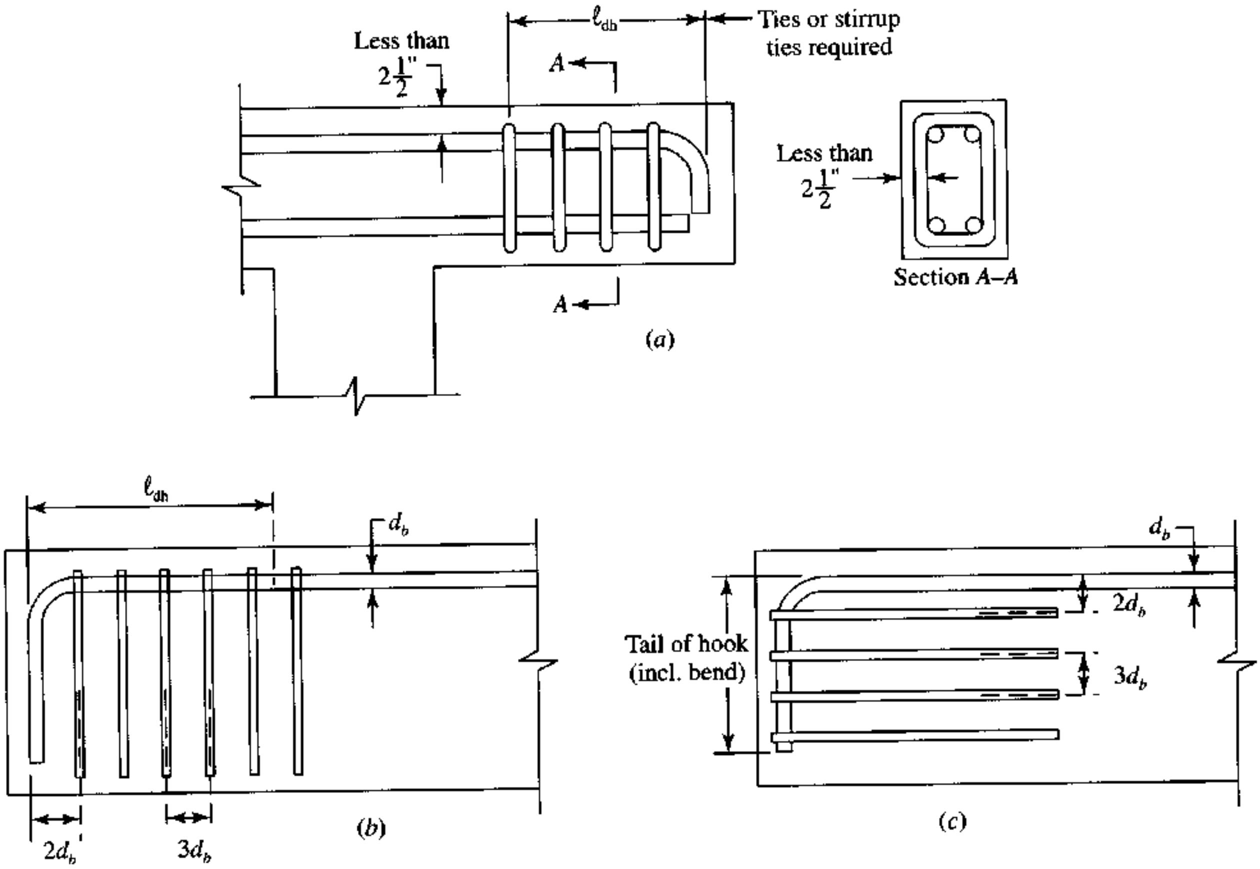


Figure 7.9 (*a*) Concrete cover limitations, (*b* and *c*) stirrups or ties placed perpendicular or parallel to the bar being developed [9]. Courtesy of ACI.

Based on different conditions, the development length, l_{dh}, must be multiplied by one of the following factors:

1. For 90° hooks of no. 11 or smaller bars are used and the hook is enclosed vertically or horizontally within stirrups or ties spaced not greater than three times the diameter of the hooked bar, the basic development length is multiplied by 0.8.
2. When no. 11 or smaller bars are used and the side concrete cover, normal to the plane of the hook, is not less than 2.5 in., the development length is multiplied by 0.7. The same factor applies for a 90° hook when the concrete cover on bar extension beyond the hook is not less than 2 in.
3. For 180° hooks of no. 11 or smaller bars that are enclosed with ties or stirrups perpendicular to the bar and spaced not greater than $3d_b$, the development length is multiplied by 0.8.
4. When a bar anchorage is not required, the basic development length for the reinforcement in excess of that required is multiplied by the ratio

$$\frac{A_s \text{ (required)}}{A_s \text{ (provided)}}$$

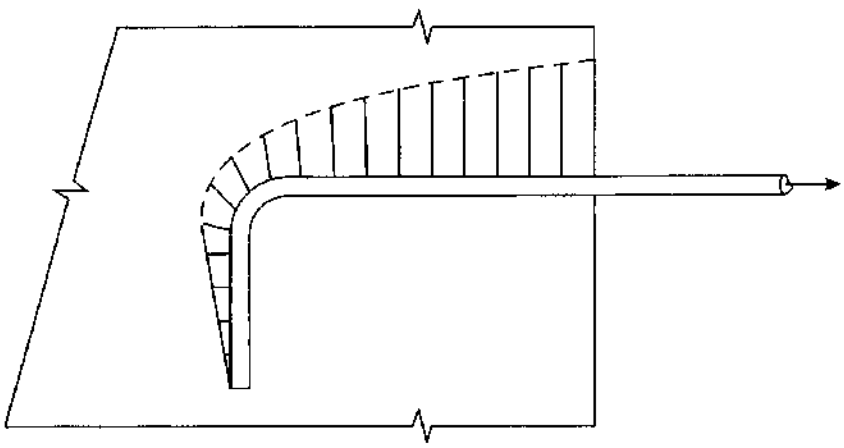


Figure 7.10 Stress distribution in 90° hooked bar.

5. When standard hooks with less than a 2.5-in. concrete cover on the side and top or bottom are used at a discontinuous end of a member, the hooks shall be enclosed by ties or stirrups spaced at no greater than $3d_b$. Moreover, the factor 0.8 given in item 1 shall not be used.

The development length, l_{dh}, of a standard hook for deformed bars in tension must not be less than $8d_b$ or 6 in., whichever is greater. Note that hooks are not effective for reinforcing bars in compression and may be *ignored* (ACI Code, Section 12.5).

Details of standard 90° and 180° hooks are shown in Fig. 7.8 [9]. The dimensions given are needed to protect members against splitting and spalling of concrete cover. Figure 7.9*a* shows details of hooks at a discontinuous end with a concrete cover less than 2.5 in. that may produce concrete spalling [9]. The use of closed stirrups is necessary for proper design. Figures 7.9*b* and *c* show placement of stirrups or ties perpendicular and parallel to the bar being developed, spaced along the development length. Figure 7.10 shows the stress distribution along a 90° hooked bar under a tension force p.

The development length required for deformed welded wire fabric is covered in Section 12.7 in the ACI Code. The basic development length (measured from the critical section) with at least one cross wire within the development length and not less than 2 in. shall be the greater of $(f_y - 35{,}000)/f_y$ (units in psi) or $5d_b/S_w$ but should not be taken greater than 1.0, where S_w = spacing of wire to be developed or spliced (in.).

Example 7.5

Compute the development length required for the top no. 8 bars of the cantilever beam shown in Fig. 7.11 that extend into the column support if the bars are

a. Straight
b. Have a 90° hook at the end
c. Have a 180° hook at the end

The bars are confined by no. 3 stirrups spaced at 6 in. and have a clear cover = 1.5 in. and clear spacings = 2.0 in. Use $f'_c = 4$ ksi normal-weight concrete and $f_y = 60$ ksi.

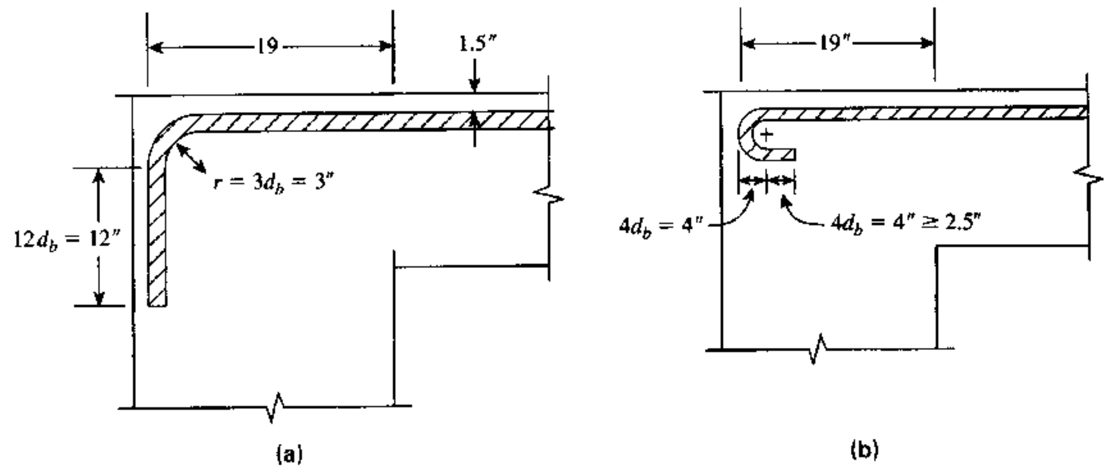


Figure 7.11 Example 7.5.

Solution

a. Straight bars: For no. 8 bars, $d_b = 1.0$ in. Because clear spacing $= 2d_b$ and clear cover is greater than d_b with bars confined by stirrups, then conditions *a* and *b* are met. Equation 7.10 can be used to calculate the basic l_d or you can get it directly from Table 7.2: $l_d = 48$ in. For top bars, $\psi_e = 1.3$ and final $l_d = 1.3(48) = 63$ in.

b. Bars with 90° hook: For no. 8 bars, $d_b = 1.0$ in. development length for $f_y = 60$ ksi $l_{dh} = 1200d_b/\sqrt{f_c'} = 1200(1.0)/\sqrt{4000} = 19$ in. Because no other modifications apply, then $l_{dh} = 19$ in. $> 8d_b = 8$ in. or 6 in. Other details are shown in Fig. 7.11. The factor $\psi_e = 1.3$ for top bars does not apply to hooks.

c. Bars with 180° hook: $l_{dh} = 19$ in., as calculated before. No other modifications apply; then $l_{dh} = 19$ in. $> 8d_b = 8$ in. Other details are shown in Fig. 7.11.

7.8 SPLICES OF REINFORCEMENT

7.8.1 General

Steel bars that are used as reinforcement in structural members are fabricated in lengths of 20, 40, and 60 ft (6, 12, and 18 m), depending on the bar diameter, transportation facilities, and other reasons. Bars are usually tailored according to the reinforcement details of the structural members. When some bars are short, it is necessary to splice them by lapping the bars a sufficient distance to transfer stress through the bond from one bar to the other.

Splices may be made by lapping or welding or with mechanical devices that provide positive connection between bars. Lap splices should not be used for bars larger than no. 11 (36 mm). For noncontact lap splices in flexural members, bars should not be spaced transversely farther apart than one-fifth the required length or 6 in. (150 mm). An approved welded splice is one in which the bars are butted and welded to develop in tension at least 125% of the specified yield strength of the bar. The ACI Code, Section 12.14, also specifies that full positive mechanical connections must develop in tension or compression at least 125% of the specified yield strength of the bar.

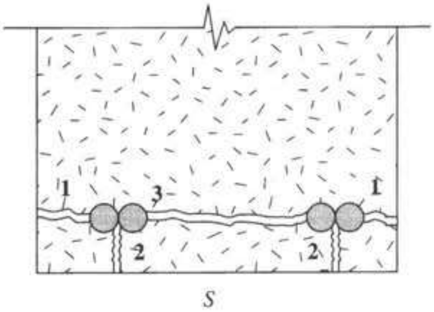


Figure 7.12 Lap splice failure due to the development of one or more cracks.

Splices should not be made at or near sections of maximum moments or stresses. Also, it is recommended that no bars should be spliced at the same location to avoid a weakness in the concrete section and to avoid the congestion of bars at the same location, which may cause difficulty in placing the concrete around the bars.

The stresses developed at the end of a typical lap splice are equal to 0, whereas the lap length, l_d, embedded in concrete is needed to develop the full stress in the bar, f_y. Therefore, a minimum lap splice of l_d is needed to develop a continuity in the spliced tension or compression bars. If adequate splice length is not provided, splitting and spalling occurs in the concrete shell (Fig. 7.12).

Splices in tension and compression are covered by Sections 12.15 and 12.16 of the ACI Code.

7.8.2 Lap Splices in Tension, l_{st}

Depending upon the percentage of bars spliced on the same location and the level of stress in the bars or deformed wires, the ACI Code introduces two classes of splices (with a minimum length of 12 in.):

1. Class A splices: These splices have a minimum length $l_{st} = l_d$ and are used when (a) one-half or less of the total reinforcement is spliced within the required lap length; and (b) the area of reinforcement provided is at least twice that required by analysis over the entire length of the splice. The length l_d is the development length of the bar, as calculated earlier.
2. Class B splices: These splices have a minimum length $l_{st} = 1.3l_d$ and are used for all other cases that are different from the aforementioned conditions. For example, class B splices are required when all bars or deformed wires are spliced at the same location with any ratio of (A_s provided)/(A_s required). Splicing all the bars in one location should be avoided when possible.
3. l_d in class A and B splice is calculated without the 12 in. minimum requirement and without the modification factor of (A_s required)/(A_s provided).

7.8.3 Lap Splice in Compression, l_{sc}

The splice lap length of the reinforcing bars in compression, l_{sc}, should be equal to or greater than the development length of the bar in compression, l_d (including the modifiers), calculated earlier (Eq. 7.14).Moreover, the lap length shall satisfy the following (ACI Code, Section 12.16.1):

Table 7.5 Lap-Splice Length in Compression, l_{sc}(in.), ($f'_c \geq$ 3ksi and Minimum l_{sc} = 12 in.)

Bar number	Bar diameter (in.)	f_y (ksi)		
		40	60	80
3	0.375	12	12	18
4	0.500	12	15	24
5	0.625	13	19	30
6	0.750	15	23	36
7	0.875	18	27	42
8	1.000	20	30	48
9	1.128	23	34	55
10	1.270	26	39	61
11	1.410	29	43	68

$$l_{sc} \geq (0.0005 f_y d_b) \quad (\text{for } f_y \leq 60{,}000 \text{ psi}) \tag{7.16}$$

$$l_{sc} = (0.0009 f_y - 24) d_b \quad (\text{for } f_y > 60{,}000 \text{ psi}) \tag{7.17}$$

For all cases, the lap length must not be less than 12 in. Table 7.5 gives the lap-splice length for various f_y values. If the concrete strength, f'_c, is less than 3000 psi, the lap length, l_{sc}, must be increased by one-third.

In spirally reinforced columns, lap-splice length within a spiral may be multiplied by 0.75 but may not be less than 12 in. In tied columns, with ties within the splice length having a minimum effective area of $0.0015hs$, lap splice may be multiplied by 0.83 but may not be less than 12 in., where h = overall thickness of column and s = spacing of ties (in.).

Example 7.6

Calculate the lap-splice length for six no. 8 tension bottom bars (in two rows) with clear spacing = 2.5 in. and clear cover = 1.5 in. for the following cases:

a. When three bars are spliced and (A_s provided)/(A_s required) > 2
b. When four bars are spliced and (A_s provided)/(A_s required) < 2
c. When all bars are spliced at the same location. Given: f'_c = 5 ksi and f_y = 60 ksi.

Solution

a. For no. 8 bars, d_b = 1.0 in., and $\psi_t = \psi_e = \lambda = 1.0$.: check first for $\sqrt{5000} = 70.7$ psi < 100 psi, and then calculate l_d from Equation 7.8 or Table 7.1, $l_d = 42.5 d_b$, conditions for clear spacings and cover are met. $l_d = 42.5(1.0) = 42.5$ in., or 43 in. For (A_s provided)/(A_s required) > 2, class A splice applies, $l_{st} = 1.0 l_d$ = 43 in. > 12 in. (minimum). Bars spliced are less than half the total number.
b. l_d = 43 in., as calculated before. Because (A_s provided)/(A_s required) is less than 2, class B splice applies, $l_{st} = 1.3 l_d = 1.3(42.5) = 55.25$ in., say, 56 in., which is greater than 12 in.
c. Class B splice applies and l_{st} = 56 in. > 12 in.

Example 7.7

Calculate the lap-splice length for a no. 10 compression bar in a tied column when $f_c' = 5$ ksi and when (a) $f_y = 60$ ksi and (b) $f_y = 80$ ksi.

Solution

a. For no. 10 bars, $d_b = 1.27$ in., and the development length from Table 7.4 or 7.3 is 23 in. Because no modifiers apply, $l_{sc} = 23$ in. > 12 in. Check that $l_{sc} \geq 0.0005 d_b f_y = 0.0005(1.27)(60{,}000) = 38.1$ in. Therefore, $l_{sc} = 39$ in. controls.

b. The basic l_d is 23 in., as calculated before. Check that $l_{sc} \geq (0.0009 f_y - 24) d_b = [0.0009(80{,}000) - 24](1.27) = 61$ in. Therefore, $l_{sc} = 61$ in. controls.

7.9 MOMENT-RESISTANCE DIAGRAM (BAR CUTOFF POINTS)

The moment capacity of a beam is a function of its effective depth, d, width, b, and the steel area for given strengths of concrete and steel. For a given beam, with constant width and depth, the amount of reinforcement can be varied according to the variation of the bending moment along the span. It is a common practice to cut off the steel bars where they are no longer needed to resist the flexural stresses. In some other cases, as in continuous beams, positive-moment steel bars may be bent up, usually at 45°, to provide tensile reinforcement for the negative moments over the supports.

The factored moment capacity of an under-reinforced concrete beam at any section is

$$M_u = \phi A_s f_y \left(d - \frac{a}{2}\right) \tag{7.18}$$

The lever arm ($d - a/2$) varies for sections along the span as the amount of reinforcement varies; however, the variation in the lever arm along the beam length is small and is never less than the value obtained at the section of maximum bending moment. Thus, it may be assumed that the moment capacity of any section is proportional to the tensile force or the area of the steel reinforcement, assuming proper anchorage lengths are provided.

To determine the position of the cutoff or bent points, the moment diagram due to external loading is drawn first. A moment-resistance diagram is also drawn on the same graph, indicating points where some of the steel bars are no longer required. The factored moment resistance of one bar, M_{ub}, is

$$M_{ub} = \phi A_{sb} f_y \left(d - \frac{a}{2}\right) \tag{7.19}$$

where

$$a = \frac{A_s f_y}{0.85 f_c' b}$$

A_{sb} = area of one bar

The intersection of the moment-resistance lines with the external bending moment diagram indicates the theoretical points where each bar can be terminated. To illustrate this discussion, Fig. 7.13 shows a uniformly loaded simple beam, its cross-section, and the bending moment diagram. The bending moment curve is a parabola with a maximum moment at midspan of

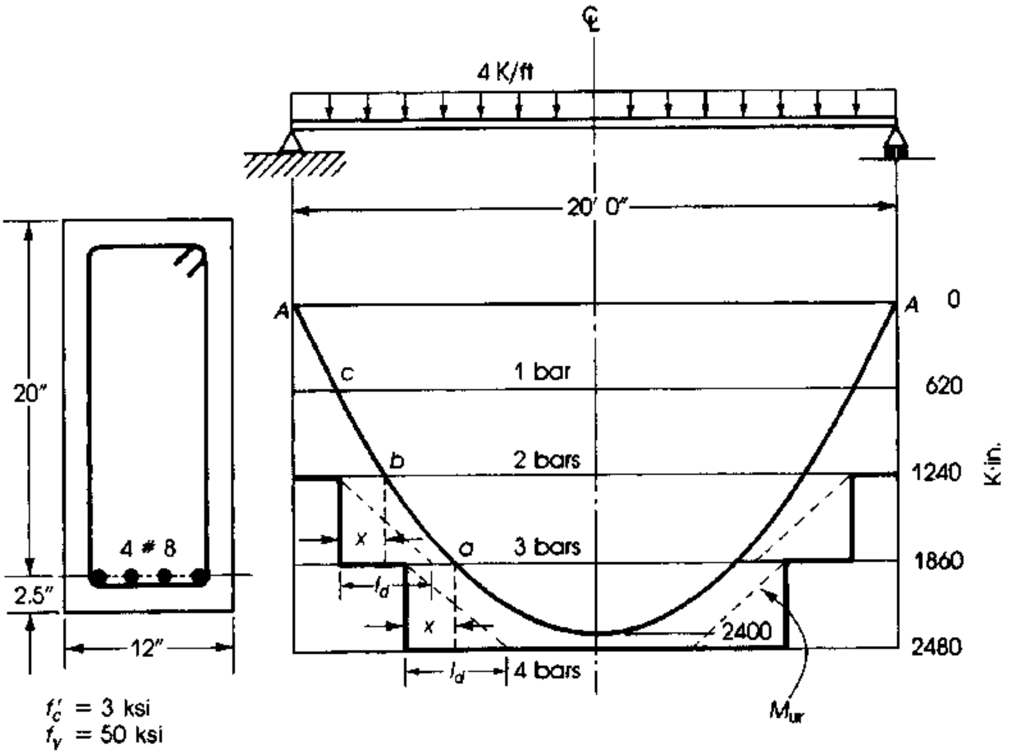


Figure 7.13 Moment-resistance diagram.

2400 K·in. Because the beam is reinforced with four no. 8 bars, the factored moment resistance of one bar is

$$M_{ub} = \phi A_{sb} f_y \left(d - \frac{a}{2}\right)$$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{4 \times 0.79 \times 50}{0.85 \times 3 \times 12} = 5.2 \text{ in.}$$

$$M_{ub} = 0.9 \times 0.79 \times 50 \left(20 - \frac{5.2}{2}\right) = 620 \text{ K·in.}$$

The factored moment resistance of four bars is thus 2480 K·in., which is greater than the external moment of 2400 K·in. If the moment diagram is drawn to scale on the base line A-A, it can be seen that one bar can be terminated at point a, a second bar at point b, the third bar at point c, and the fourth bar at the support end a. These points are the theoretical positions for the termination of the bars. However, it is necessary to develop part of the strength of the bar by bond, as explained earlier. The ACI Code specifies that every bar should be continued at least a distance equal to the effective depth, d, of the beam or 12 bar diameters, whichever is greater, beyond the theoretical points a, b, and c. The Code (Section 12.11.1) also specifies that at least one-third of the positive moment reinforcement must be continued to the support for simple beams. Therefore, for the example discussed here, two bars must extend into the support, and the moment-resistance diagram, M_{ur}, shown in Fig. 7.13, must enclose the external bending moment diagram at all points. Full load capacity of each bar is attained at a distance l_d from its end.

For continuous beams, the bars are bent at the required points and used to resist the negative moments at the supports. At least one-third of the total reinforcement provided for the negative moment at the support must be extended beyond the inflection points a distance not less than the effective depth, 12 bar diameters, or $\frac{1}{6}$ the clear span, whichever is greatest (ACI Code, Section 12.12.3).

Bent bars are also used to resist part of the shear stresses in beams. The moment–resistance diagram for a typical continuous beam is shown in Fig. 7.14.

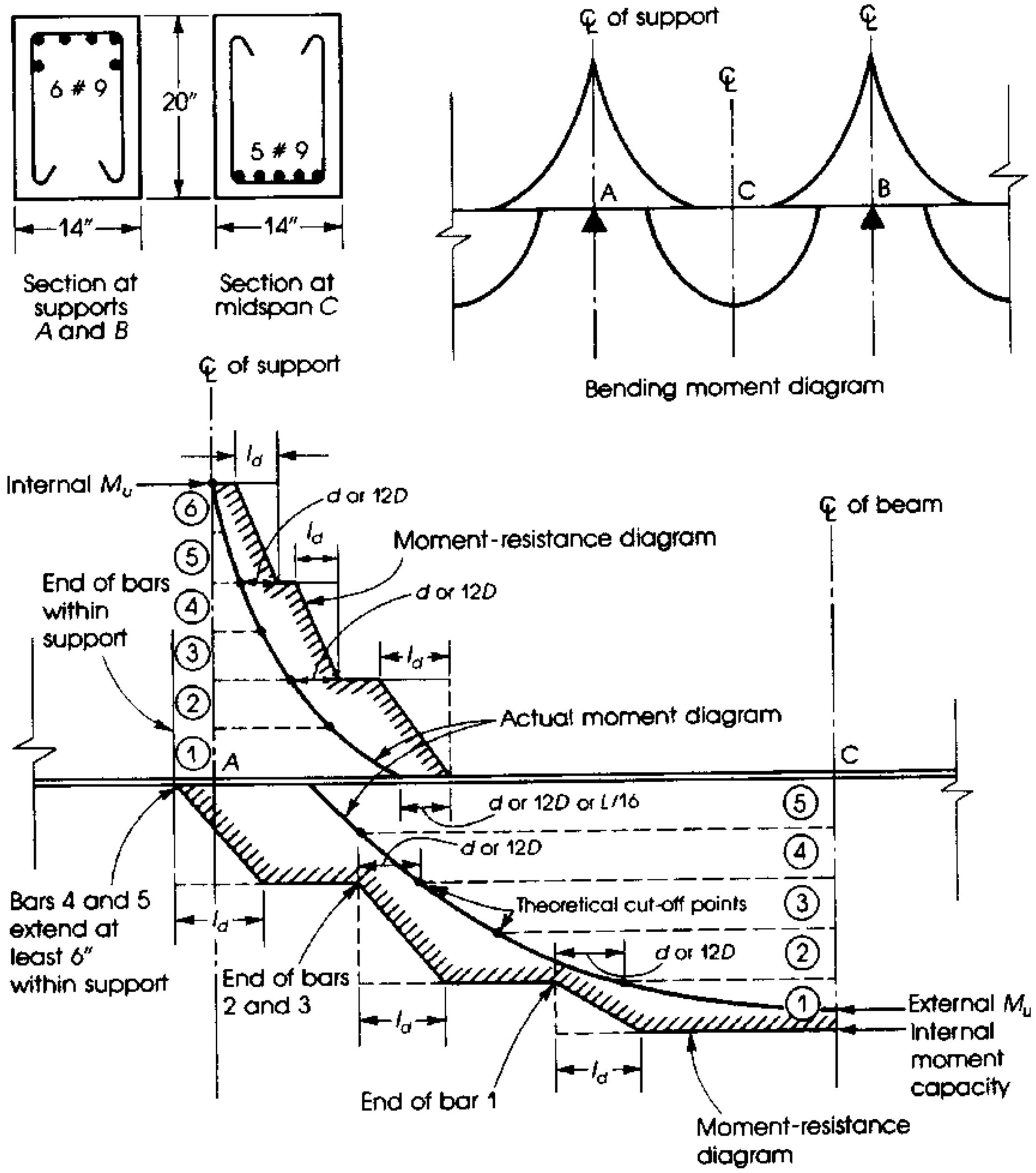


Figure 7.14 Sections and bending moment diagram (*top*) and moment–resistance diagram (*bottom*) of a continuous beam. Bar diameter is signified by *D*.

Example 7.8

For the simply supported beam shown in Fig. 7.15, design the beam for the given factored loads and draw the moment—resistance diagram. Also, show where the reinforcing bars can be terminated. Use $b = 10$ in., a steel ratio of 0.018, $f'_c = 3$ ksi, and $f_y = 40$ ksi.

Solution

For $\rho = 0.018$, $R_u = 556$ psi and $M_u = R_u bd^2$. $M_u = 132.5$ K·ft. Now $132.5(12) = 0.556(10)d^2$, so $d = 17$ in.; let $h = 20$ in. $A_s = 0.018(10)(17) = 3.06\,\text{in.}^2$; use four no. 8 bars ($A_s = 3.14\,\text{in.}^2$). Actual $d = 20 - 2.5 = 17.5$ in.

$$M_{ur} = \phi A_s f_y \left(d - \frac{a}{2}\right) \text{ and } a = \frac{3.14(40)}{0.85(3)(10)} = 4.93 \text{ in.}$$

$$M_{ur} \text{ (for one bar)} = 0.9(0.79)(40)\left(17.5 - \frac{4.93}{2}\right)$$

$$= 427.7\text{K·in.} = 35.64 \text{ K·ft}$$

$$M_{ur} \text{ (for all four bars)} = 1710.8 \text{ K·in.} = 142.6 \text{ K·ft}$$

For the calculation of 'a', the four no. 8 bars were utilized rather than calculating the 'a' for the extended two bars. This assumption will slightly increase the length of the bars beyond the cutoff point.

Details of the moment–resistance diagram are shown in Fig. 7.15. Note that the bars can be bent or terminated at a distance of 17.5, say, 18 in. (or 12 bar diameters, whichever is greater), beyond the points where (theoretically) the bars are not needed. The development length, l_d, for no. 8 bars is $36.6d_b = 37$ in. (Table 7.1). The cutoff points of the first and second bars are at points A and B, but the actual points are at A' and B', 18 in. beyond A and B. From A', a length $l_d = 37$ in. backward is shown to establish the moment–resistance diagram (the dashed line). The end of the last two bars extending to the support will depend on how far they extend inside the support, say, at C'. Normally, bars are terminated within the span at A' and B' as bent bars are not commonly used to resist shear.

SUMMARY

Sections 7.1–7.2

Bond is influenced mainly by the roughness of the steel surface area, the concrete mix, shrinkage, and the cover of concrete. In general,

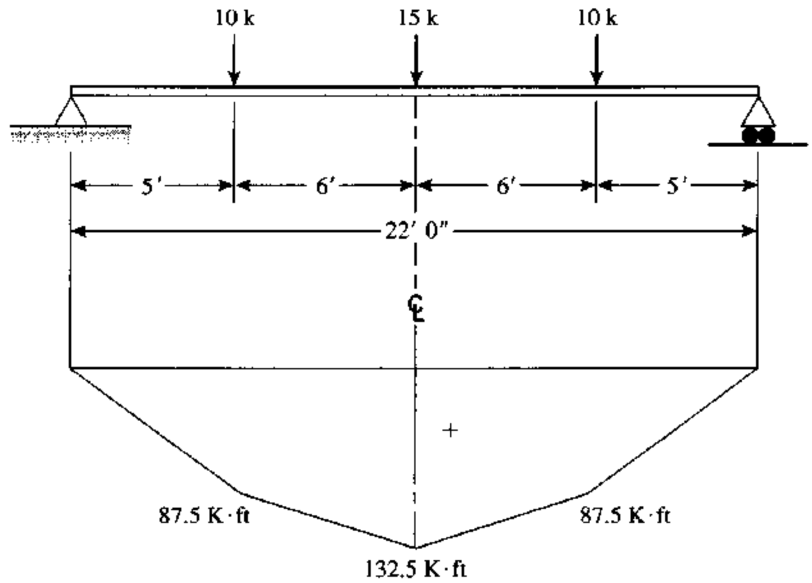
$$l_d = \frac{A_s f_y}{U_u \Sigma O} \tag{7.3}$$

Sections 7.3 and 7.5

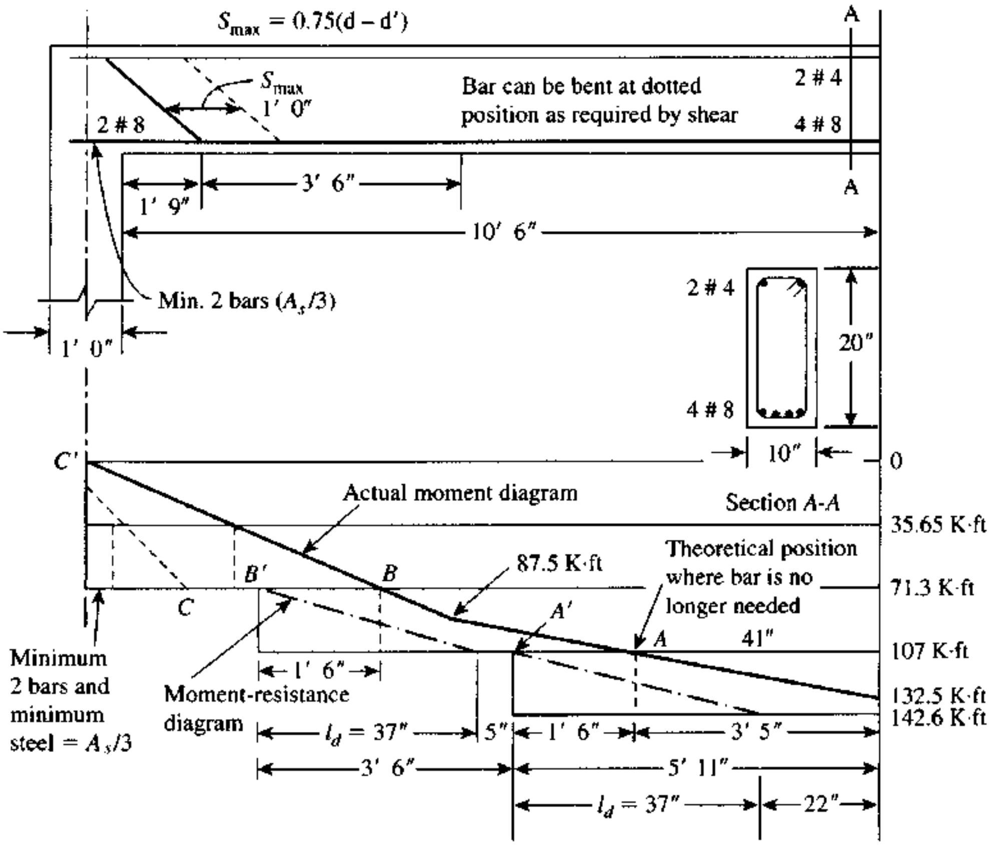
1. The general formula for the development length of deformed bars or wire shall be

$$\frac{l_d}{d_b} = \left(\frac{3}{40}\right)\left(\frac{f_y}{\lambda\sqrt{f'_c}}\right)\frac{\psi_t\psi_e\psi_s}{(c_b + k_{tr})/d_b} \tag{7.7}$$

As design simplification, K_{tr} may be assumed to be zero. Other values of l_d/d_b are given in Tables 7.1 and 7.2. ψ_t, ψ_e, ψ_s, and λ are multipliers defined in Section 7.3.1.



(a)



(b)

Figure 7.15 Example 7.8: Details of reinforcing bars and the moment-resistance diagram.

2. Simplified expressions are used when conditions for concrete cover and spacings are met. For no. 7 and larger bars,

$$\frac{l_d}{d_b} = \left(\frac{f_y}{\sqrt{f'_c}}\right)\left(\frac{\psi_t\psi_e}{20\lambda}\right) = Q \tag{7.8}$$

For no. 6 and smaller bars,

$$\frac{l_d}{d_b} = 0.8Q \tag{7.9}$$

3. For all other cases, multiply the previous Q by 1.5.
4. Minimum length is 12 in.

Section 7.4

Development length in compression for all bars is

$$l_d = \frac{0.02 d_b f_y}{\lambda\sqrt{f'_c}} \geq 0.0003 d_b f_y \geq 8 \text{ in.} \tag{7.14}$$

For specific values, refer to Tables 7.3 and 7.4.

Section 7.6

The critical sections for the development of reinforcement in flexural members are

- At points of maximum stress
- At points where tension bars are terminated within the span
- At the face of the support
- At points of inflection

Section 7.7

The minimum diameter of bends in standard hooks is

- For no. 3 to no. 8 bars, $6d_b$
- For no. 9 to no. 11 bars, $8d_b$

The development length l_{dh} of a standard hook is

$$l_{dh} = \left(\frac{0.02\psi f_y}{\lambda\sqrt{f'_c}}\right)(\text{Modification factor})\, d_b \tag{7.15}$$

Section 7.8

1. For splices in tension, the minimum lap-splice length is 12 in. If (a) one-half or less of the total reinforcement is spliced within the required lap length and (b) the area of reinforcement provided is at least twice that required by analysis over the entire length of the splice, then $l_{st} = 1.0 l_d$ = class A splice.

2. For all other cases, class B has to be used when $l_{st} = 1.3l_d$.
3. For splices in compression, the lap length should be equal to or greater than l_{dc} in compression, but it also should satisfy the following: $l_{sc} \geq 0.0005\ f_y d_b$ (for $f_y \leq 60{,}000\,\text{psi}$).

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PROBLEMS

7.1 For each assigned problem, calculate the development length required for the following tension bars. All bars are bottom bars in normal-weight concrete unless specified otherwise in the notes.

No.	Bar no.	f'_c (ksi)	f_y (ksi)	Clear cover (in.)	Clear spacing (in.)	Notes
a	5	3	60	2.0	2.25	
b	6	4	60	2.0	2.50	Lightweight aggregate concrete
c	7	5	60	2.0	2.13	Epoxy coated
d	8	3	40	2.5	2.30	Top bars, lightweight aggregate concrete
e	9	4	60	1.5	1.5	
f	10	5	60	2.0	2.5	No. 3 stirrups at 6 in.
g	11	5	60	3.0	3.0	
h	9	3	40	2.0	1.5	Epoxy coated
i	8	4	60	2.0	1.75	(A_s provided)/(A_s required) = 1.5
j	6	4	60	1.5	1.65	Top bars, epoxy coated and no. stirrup at 4 in.

7.2 For each assigned problem, calculate the development length required for the following bars in compression.

No.	Bar no.	f'_c (ksi)	f_y (ksi)	Notes
a	8	3	60	
b	9	4	60	
c	10	4	40	
d	11	5	60	(A_s required)/(A_s provided) = 0.8
e	7	6	60	(A_s required)/(A_s provided) = 0.9
f	9	5	60	Column with spiral no. 3 at 2 in.

7.3 Compute the development length required for the top no. 9 bars of a cantilever beam that extend into the column support if the bars are

a. Straight

b. Have a 90° hook at the end

c. Have a 180° hook at the end

The bars are confined with no. 3 stirrups spaced at 5 in. and have a clear cover of 2.0 in. Use $f'_c = 4$ ksi and $f_y = 60$ ksi. (Clear spacing = 2.5 in.)

7.4 Repeat Problem 7.3 when no. 7 bars are used.

7.5 Repeat Problem 7.3 when $f'_c = 3$ ksi and $f_y = 40$ ksi.

7.6 Repeat Problem 7.3 when no. 10 bars are used.

7.7 Calculate the lap-splice length for no. 9 tension bottom bars with clear spacing of 2.0 in. and clear cover of 2.0 in. for the following cases:

a. When 50% of the reinforcement is spliced and (A_s provided)/(A_s required) = 2

b. When 75% of the reinforcement is spliced and (A_s provided)/(A_s required) = 1.5

c. When all bars are spliced at one location and (A_s provided)/(A_s required) = 2

d. When all bars are spliced at one location and (A_s provided)/(A_s required) = 1.3. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

7.8 Repeat Problem 7.7 using $f'_c = 3$ ksi and $f_y = 60$ ksi.

7.9 Calculate the lap splice length for no. 9 bars in compression when $f'_c = 5$ ksi and $f_y = 60$ ksi.

7.10 Repeat Problem 7.9 when no. 11 bars are used.

7.11 Repeat Problem 7.9 when $f_y = 80$ ksi.

7.12 Repeat Problem 7.9 when $f'_c = 4$ ksi and $f_y = 60$ ksi.

7.13 A continuous beam has the typical steel reinforcement details shown in Fig. 7.16. The sections at midspan and at the face of the support of a typical interior span are also shown. Check the development lengths of the reinforcing bars at all critical sections. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

7.14 Design the beam shown in Fig. 7.17 using ρ_{max}. Draw the moment—resistance diagram and indicate where the reinforcing bars can be terminated. The beam carries a uniform dead load, including self-weight of 1.5 K/ft, and a live load of 2.2 K/ft. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and $b = 12$ in.

7.15 Design the beam shown in Fig. 7.18 using a steel ratio $\rho = 1/2\ \rho_b$. Draw the moment—resistance diagram and indicate the cutoff points. Use $f'_c = 3$ ksi, $f_y = 60$ ksi, and $b = 12$ in.

7.16 Design the section at support B of the beam shown in Fig. 7.19, ρ_{max}. Adopting the same dimensions of the section at B for the entire beam ABC, determine the reinforcement required for part AB and draw the moment–resistance diagram for the beam ABC. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and $b = 12$ in.

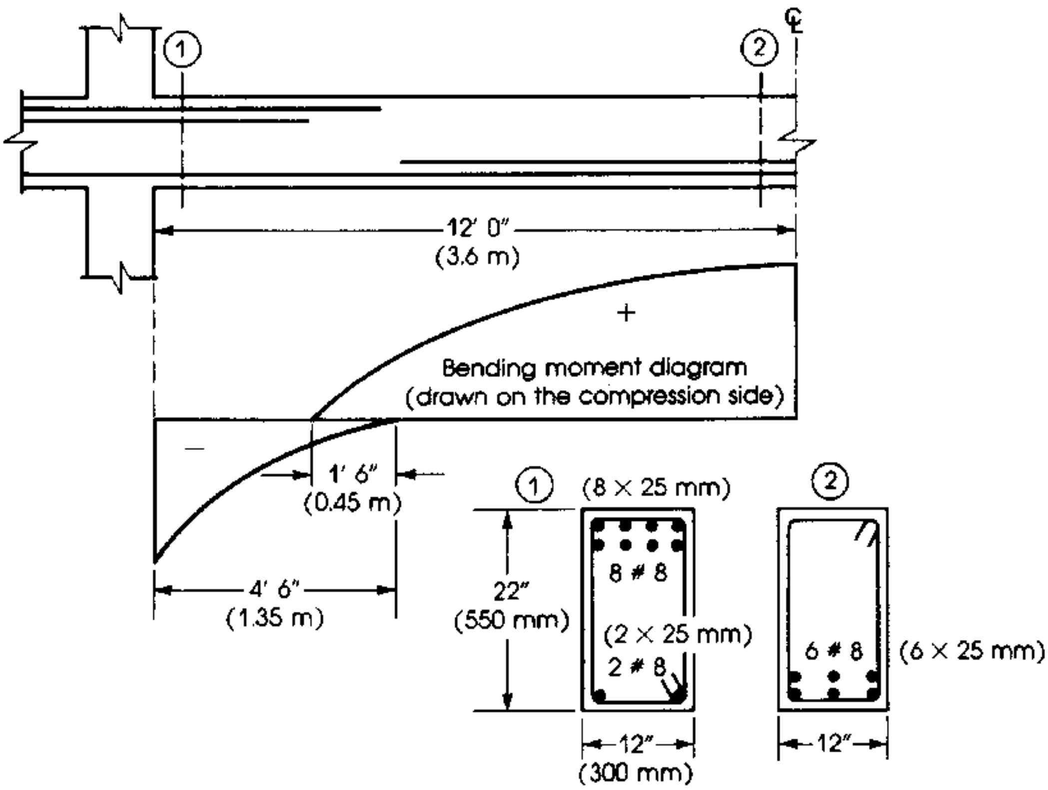


Figure 7.16 Problem 7.13.

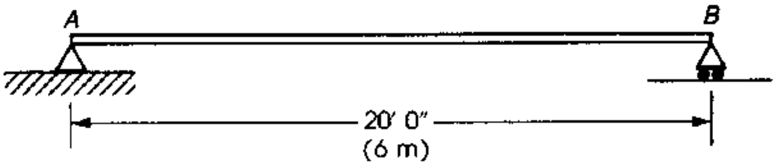


Figure 7.17 Problem 7.14: Dead load = 1.5 K/ft (22.5 kN/m), live load = 2.2 K/ft (33 kN/m).

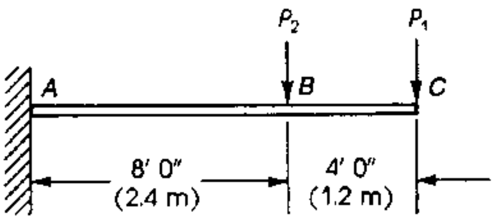


Figure 7.18 Problem 7.15: Dead load = 2 K/ft (30 kN/m), live load (concentrated loads only) is P_1 = 10 K (45 kN), P_2 = 16 K (72 kN).

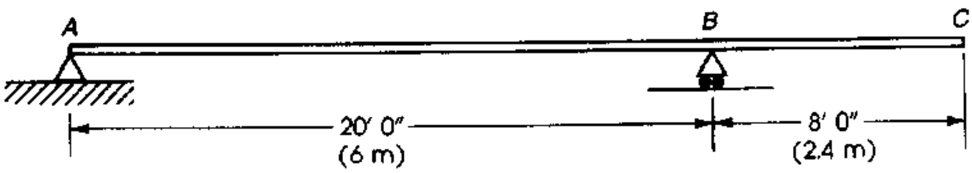


Figure 7.19 Problem 7.16: Dead load = 6 K/ft (90 kN/m), live load = 4 K/ft (60 kN/m).

CHAPTER 8

SHEAR AND DIAGONAL TENSION


Office building, Chicago, Illinois.

8.1 INTRODUCTION

When a simple beam is loaded as shown in Fig. 8.1, bending moments and shear forces develop along the beam. To carry the loads safely, the beam must be designed for both types of forces. Flexural design is considered first to establish the dimensions of the beam section and the main reinforcement needed, as explained in the previous chapters.

The beam is then designed for shear. If shear reinforcement is not provided, shear failure may occur. *Shear failure* is characterized by small deflections and lack of ductility, giving little or no warning before failure. On the other hand, flexural failure is characterized by a gradual increase in deflection and cracking, thus giving warning before total failure. This is due to the ACI Code limitation on flexural reinforcement. The design for shear must ensure that shear failure does not occur before flexural failure.

8.2 SHEAR STRESSES IN CONCRETE BEAMS

The general formula for the shear stress in a homogeneous beam is

$$v = \frac{VQ}{Ib} \tag{8.1}$$

where

V = total shear at the section considered

Q = statical moment about the neutral axis of that portion of cross-section lying between a line through the point in question parallel to the neutral axis and nearest face, upper or lower, of the beam

I = moment of inertia of cross-section about the neutral axis

b = width of beam at the given point

The distribution of bending and shear stresses according to elastic theory for a homogeneous rectangular beam is as shown in Fig. 8.2. The bending stresses are calculated from the flexural

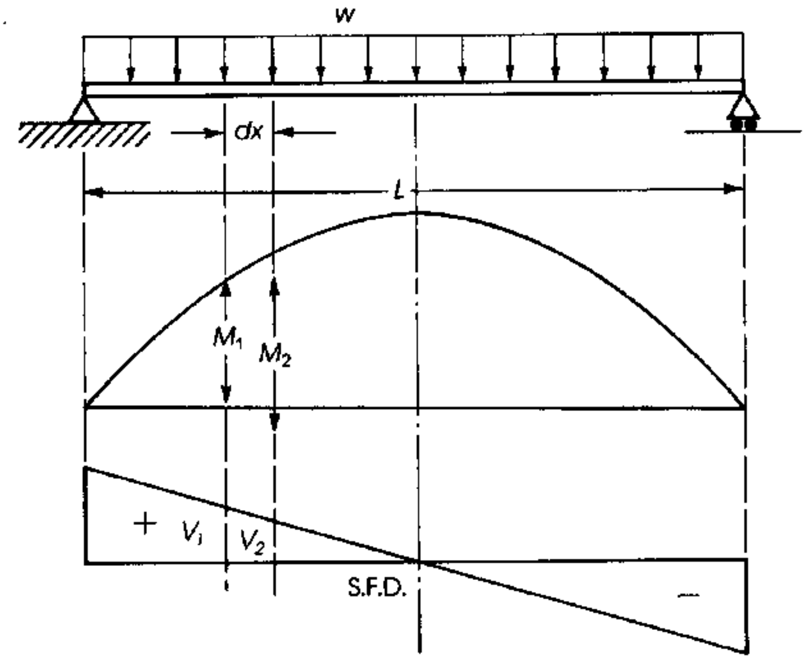


Figure 8.1 Bending moment and shearing force diagrams for a simple beam.

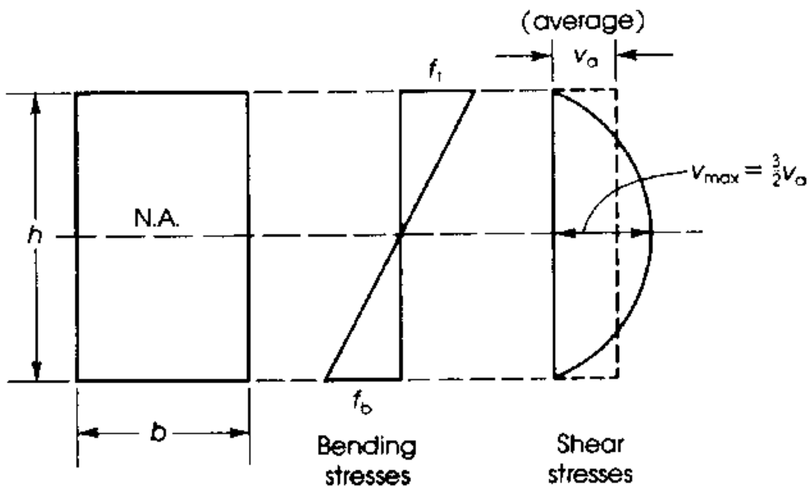


Figure 8.2 Bending and shear stresses in a homogeneous beam, according to elastic theory.

formula $f = Mc/I$, whereas the shear stress at any point is calculated by the shear formula of Eq. 8.1. The maximum shear stress is at the neutral axis and is equal to $1.5v_a$ (average shear), where $v_a = V/bh$. The shear stress curve is parabolic.

For a singly reinforced concrete beam, the distribution of shear stress above the neutral axis is a parabolic curve. Below the neutral axis, the maximum shear stress is maintained down to the level of the tension steel, because there is no change in the tensile force down to this point and the concrete in tension is neglected. The shear stress below the tension steel is 0 (Fig. 8.3). For doubly reinforced and T-sections, the distribution of shear stresses is as shown in Fig. 8.3.

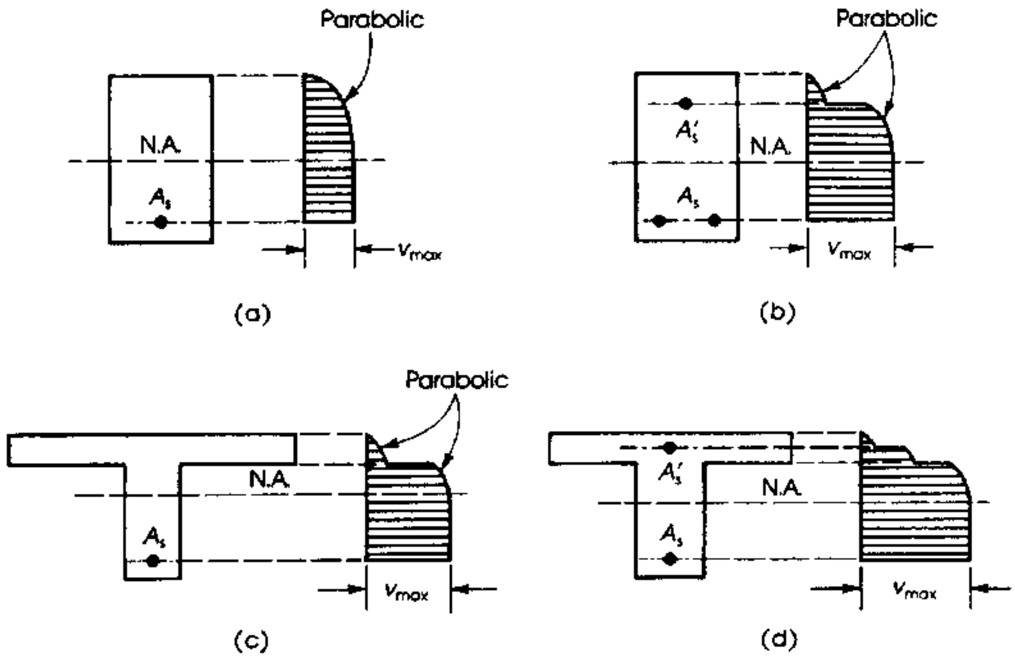


Figure 8.3 Distribution of shear stresses in reinforced concrete beams: (*a*) singly reinforced, (*b*) doubly reinforced, (*c*) T-section, (*d*) T-section with compression steel.

It can be observed that almost all the shear force is resisted by the web, whereas the flange resists a very small percentage; in most practical problems, the shear capacity of the flange is neglected.

Referring to Fig. 8.1 and taking any portion of the beam dx, the bending moments at both ends of the element, M_1 and M_2 are not equal. Because $M_2 > M_1$ and to maintain the equilibrium of the beam portion dx, the compression force C_2 must be greater than C_1 (Fig. 8.4). Consequently, a shear stress v develops along any horizontal section a-a_1 or b-b_1 (Fig. 8.4*a*). The normal and shear stresses on a small element at levels a-a_1 and b-b_1 are shown in Fig. 8.4*b*. Notice that the normal stress at the level of the neutral axis b-b_1 is 0, whereas the shear stress is maximum. The horizontal shear stress is equal to the vertical shear stress, as shown in Fig. 8.4*b*. When the normal stress f is 0 or low, a case of pure shear may occur. In this case, the maximum tensile stress f_t, acts at 45° (Fig. 8.4*c*).

The tensile stresses are equivalent to the principal stresses, as shown in Fig. 8.4*d*. Such principal stresses are traditionally called *diagonal tension stresses*. When the diagonal tension stresses reach the tensile strength of concrete, a diagonal crack develops. This brief analysis explains the concept of diagonal tension and diagonal cracking. The actual behavior is more complex, and it is affected by other factors, as explained later. For the combined action of shear and normal stresses at any point in a beam, the maximum and minimum diagonal tension (principal stresses) f_p are given by the equation

$$f_p = \frac{f}{2} \pm \sqrt{\left(\frac{f}{2}\right)^2 + v^2} \tag{8.2}$$

where

f = intensity of normal stress due to bending

v = shear stress

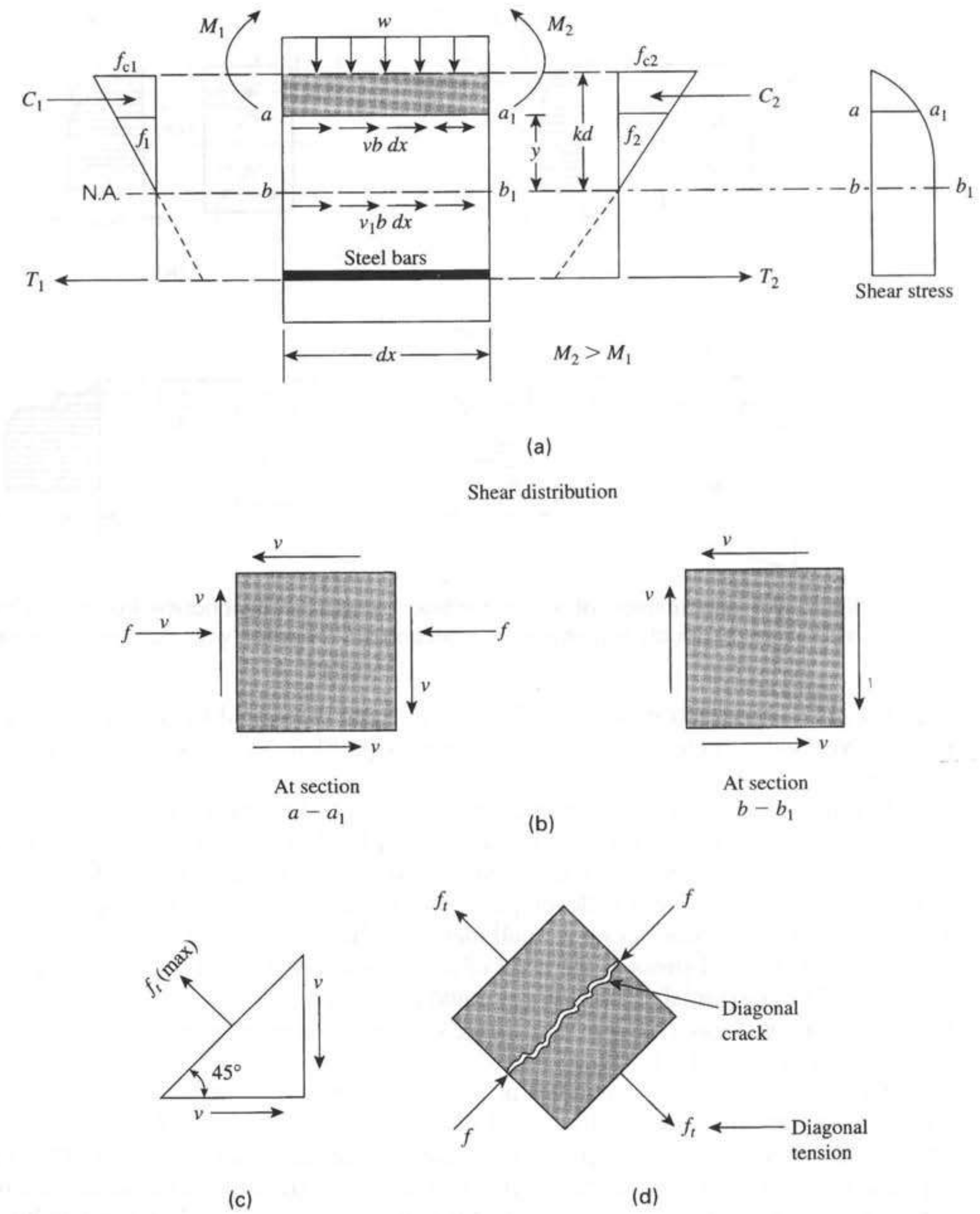


Figure 8.4 (*a*) Forces and stresses along the depth of the section, (*b*) normal and shear stresses, (*c*) pure shear, and (*d*) diagonal tension.

The shear failure in a concrete beam is most likely to occur where shear forces are maximum, generally near the supports of the member. The first evidence of impending failure is the formation of diagonal cracks.

8.3 BEHAVIOR OF BEAMS WITHOUT SHEAR REINFORCEMENT

Concrete is weak in tension, and the beam will collapse if proper reinforcement is not provided. The tensile stresses develop in beams due to axial tension, bending, shear, torsion, or a combination of these forces. The location of cracks in the concrete beam depends on the direction of principal stresses. For the combined action of normal stresses and shear stresses, maximum diagonal tension may occur at about a distance d from the face of the support.

The behavior of reinforced concrete beams with and without shear reinforcement tested under increasing load was discussed in Section 3.3. In the tested beams, vertical flexural cracks developed at the section of maximum bending moment when the tensile stresses in concrete exceeded the modulus of rupture of concrete, or $f_r = 7.5\lambda\sqrt{f'_c}$. Inclined cracks in the web developed at a later stage at a location very close to the support.

An inclined crack occurring in a beam that was previously uncracked is generally referred to as a *web-shear crack*. If the inclined crack starts at the top of an existing flexural crack and propagates into the beam, the crack is referred to as *flexural-shear crack* (Fig. 8.5). Web-shear cracks occur in beams with thin webs in regions with high shear and low moment. They are relatively uncommon cracks and may occur near the inflection points of continuous beams or adjacent to the supports of simple beams.

Flexural-shear cracks are the most common type found in reinforced concrete beams. A flexural crack extends vertically into the beam; then the inclined crack forms, starting from the top of the beam when shear stresses develop in that region. In regions of high shear stresses, beams must be reinforced by stirrups or by bent bars to produce ductile beams that do not rupture at a failure. To avoid a shear failure before a bending failure, a greater factor of safety must be provided against a shear failure. The ACI Code specifies a capacity reduction factor, ϕ, of 0.75 for shear.

Shear resistance in reinforced concrete members is developed by a combination of the following mechanisms [2] (Fig. 8.5):

- Shear resistance of the uncracked concrete, V_z [3]
- Interface shear transfer, V_a, due to aggregate interlock tangentially along the rough surfaces of the crack [3]
- Arch action [4]
- Dowel action, V_d, due to the resistance of the longitudinal bars to the transverse shearing force [5]

In addition to these forces, shear reinforcement increases the shear resistance V_s, by which depends on the diameter and spacings of stirrups used in the concrete member. If shear reinforcement is not provided in a rectangular beam, the proportions of the shear resisted by the various mechanisms are 20% to 40% by V_z 35% to 50% by V_a and 15% to 25% by V_d [6].

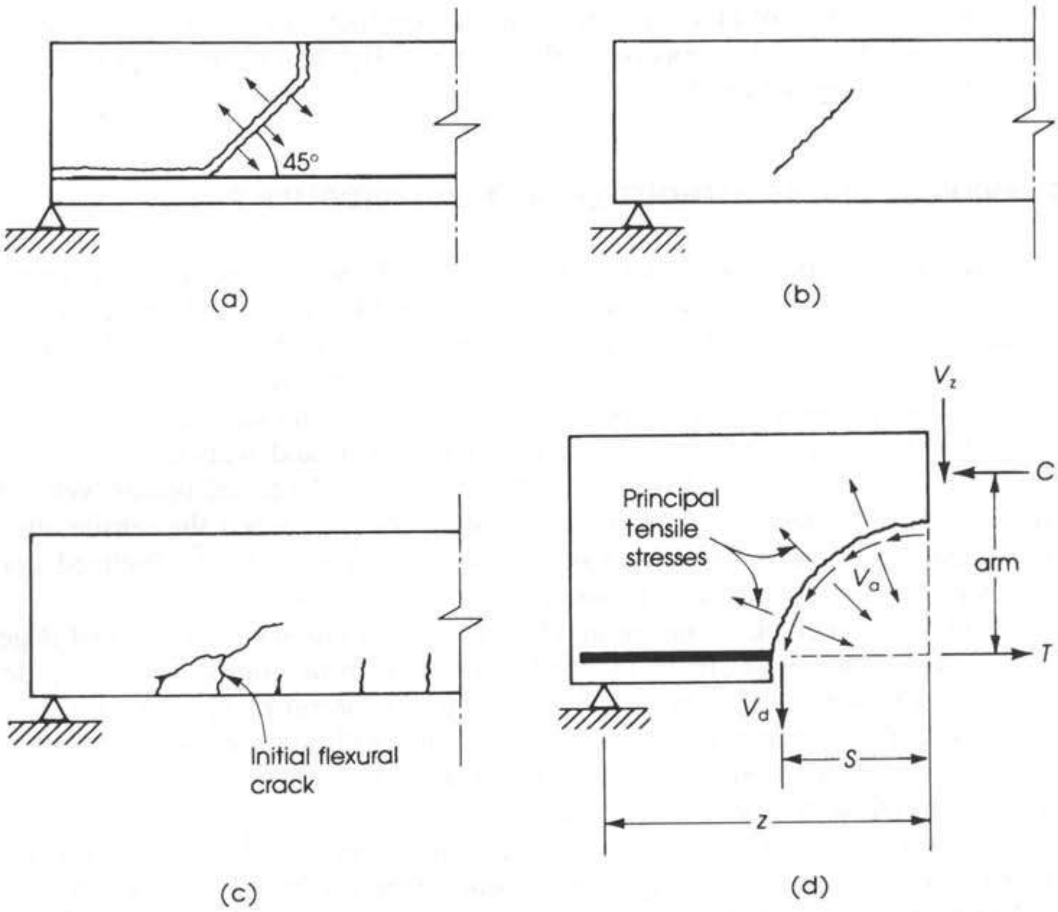


Figure 8.5 Shear failure: (*a*) general form, (*b*) web-shear crack, (*c*) flexural-shear crack, (*d*) analysis of forces involved in shear. V_a is interface shear, V_z is shear resistance, and V_d is dowel force.

8.4 MOMENT EFFECT ON SHEAR STRENGTH

In simply supported beams under uniformly distributed load, the midspan section is subjected to a large bending moment and zero or small shear, whereas sections near the ends are subjected to large shear and small bending moments (Fig. 8.1). The shear and moment values are both high near the intermediate supports of a continuous beam. At a location of large shear force and small bending moment, there will be little flexural cracking, and an average stress v is equal to V/bd. The diagonal tensile stresses are inclined at about 45° (Fig. 8.4*c*). Diagonal cracks can be expected when the diagonal tensile stress in the vicinity of the neutral axis reaches or exceeds the tensile strength of concrete. In general, the factored shear strength varies between $3.5\sqrt{f_c'}$ and $5\sqrt{f_c'}$. After completing a large number of beam tests on shear and diagonal tension [1], it was found that in regions with large shear and small moment, diagonal tension cracks were formed at an average shear force of

$$V_c = 3.5\sqrt{f_c'}b_w d \tag{8.3}$$

where b_w is the width of the web in a T-section or the width of a rectangular section and d is the effective depth of the beam.

In locations where shear forces and bending moments are high, flexural cracks are formed first. At a later stage, some cracks bend in a diagonal direction when the diagonal tension stress at the upper end of such cracks exceeds the tensile strength of concrete. Given the presence of large moments on a beam, for which adequate reinforcement is provided, the nominal shear force at which diagonal tension cracks develop is given by

$$V_c = 1.9\lambda\sqrt{f_c'}b_w d \tag{8.4}$$

This value is a little more than half the value in Eq. 8.3 when bending moment is very small. This means that large bending moments reduce the value of shear stress for which cracking occurs. The following equation has been suggested to predict the nominal shear stress at which a diagonal crack is expected [1]:

$$v_c = \frac{V}{b_w d} = \left(1.9\lambda\sqrt{f_c'} + 2500\rho\frac{Vd}{M}\right) \leq 3.5\lambda\sqrt{f_c'} \tag{8.5}$$

1. ACI Code, Section 11.2.2.1, adopted this equation for the nominal shear force to be resisted by concrete for members subjected to shear and flexure only by:

$$V_c = \left[1.9\lambda\sqrt{f_c'} + 2500\rho_w\frac{V_u d}{M_u}\right] b_w d \leq 3.5\lambda\sqrt{f_c'}b_w d \tag{8.6}$$

where $\rho_w = A_s/b_w$, d and b_w are the web width in a T-section or the width of a rectangular section, and V_u and M_u are the factored shearing force and bending moment occurring simultaneously on the considered section.

The value of $V_u d/M_u$ must not exceed 1.0 in Eq. 8.6. If M_u is large in Eq. 8.6, the second term becomes small and v_c approaches $1.9\lambda\sqrt{f_c'}$. If M_u is small, the second term becomes large and the upper limit of $3.5\lambda\sqrt{f_c'}$ controls. As an alternative to Eq. 8.6, the ACI Code, Section 11.2.1.1, permits evaluating the shear strength of concrete as follows:

$$V_c = 2\lambda\sqrt{f_c'}b_w d \tag{8.7}$$

$$V_c = 0.17\lambda\sqrt{f_c'}b_w d \qquad \text{(SI)}$$

2. For members subjected to significant axial compression force N_u,

$$V_c = \left(1.9\lambda\sqrt{f_c'} + 2500\rho_w\frac{V_u d}{M_m}\right) b_w d \tag{8.8}$$

$$M_m = M_u - N_u\left(\frac{4h - d}{8}\right)$$

where

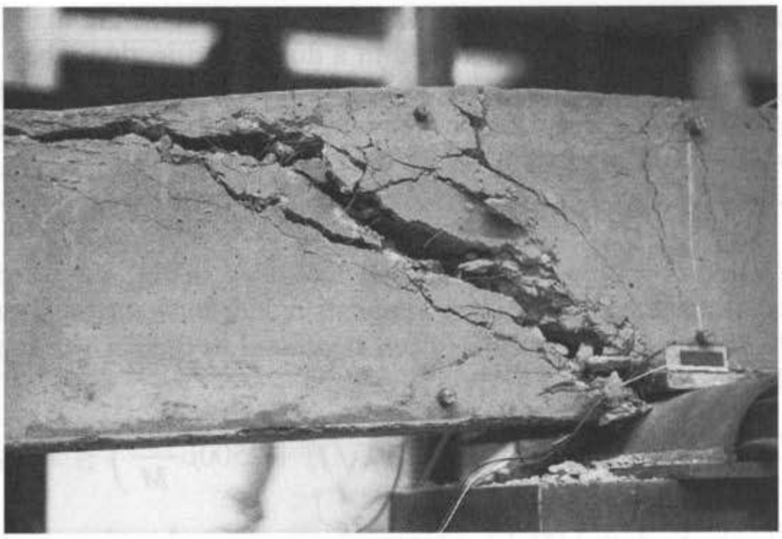
$$\rho_w = \frac{A_s}{b_w d}$$

h = overall depth

$V_u d/M_u$ may be greater than 1.0, but V_c must not exceed

$$V_c = 3.5\lambda\sqrt{f_c'}b_w d\sqrt{1 + \frac{N_u}{500A_g}} \tag{8.9}$$

where A_g is the gross area in.^{2}



Shear failure near a middle support.

Alternatively, V_c may be computed by

$$V_c = b_w d \left(2 + 0.001 \frac{N_u}{A_g}\right) \lambda \sqrt{f'_c} \tag{8.10}$$

3. In the case of members subjected to significant axial tensile force N_u,

$$V_c = b_w d \left(2 + 0.004 \frac{N_u}{A_g}\right) \lambda \sqrt{f'_c} \tag{8.11}$$

where N_u is to be taken as negative for tension and N_u/A_g is in psi.
If V_c is negative, V_c should be taken equal to zero.

8.5 BEAMS WITH SHEAR REINFORCEMENT

Different types of shear reinforcement may be used:

1. Stirrups, which can be placed either perpendicular to the longitudinal reinforcement or inclined, usually making a 45° angle and welded to the main longitudinal reinforcement. Vertical stirrups, using no. 3 or no. 4 U-shaped bars, are the most commonly used shear reinforcement in beams (Fig. 8.6*a*).
2. Bent bars, which are part of the longitudinal reinforcement, bent up (where they are no longer needed) at an angle of 30° to 60°, usually at 45°.
3. Combinations of stirrups and bent bars.
4. Welded wire fabric with wires perpendicular to the axis of the member.
5. Spirals, circular ties, or hoops in circular sections, as columns.

The shear strength of a reinforced concrete beam is increased by the use of shear reinforcement. Prior to the formation of diagonal tension cracks, shear reinforcement contributes very little to the shear resistance. After diagonal cracks have developed, shear reinforcement augments the shear resistance of a beam, and a redistribution of internal forces occurs at the cracked section.

When the amount of shear reinforcement provided is small, failure due to yielding of web steel may be expected, but if the amount of shear reinforcement is too high, a shear-compression failure may be expected, which should be avoided.

Concrete, stirrups, and bent bars act together to resist transverse shear. The concrete, by virtue of its high compressive strength, acts as the diagonal compression member of a lattice girder system, where the stirrups act as vertical tension members. The diagonal compression force is such that its vertical component is equal to the tension force in the stirrup. Bent-up reinforcement acts also as tension members in a truss, as shown in Fig. 8.6.

In general, the contribution of shear reinforcement to the shear strength of a reinforced concrete beam can be described as follows [2]:

1. It resists part of the shear, V_s.
2. It increases the magnitude of the interface shear, V_a (Fig. 8.5), by resisting the growth of the inclined crack.

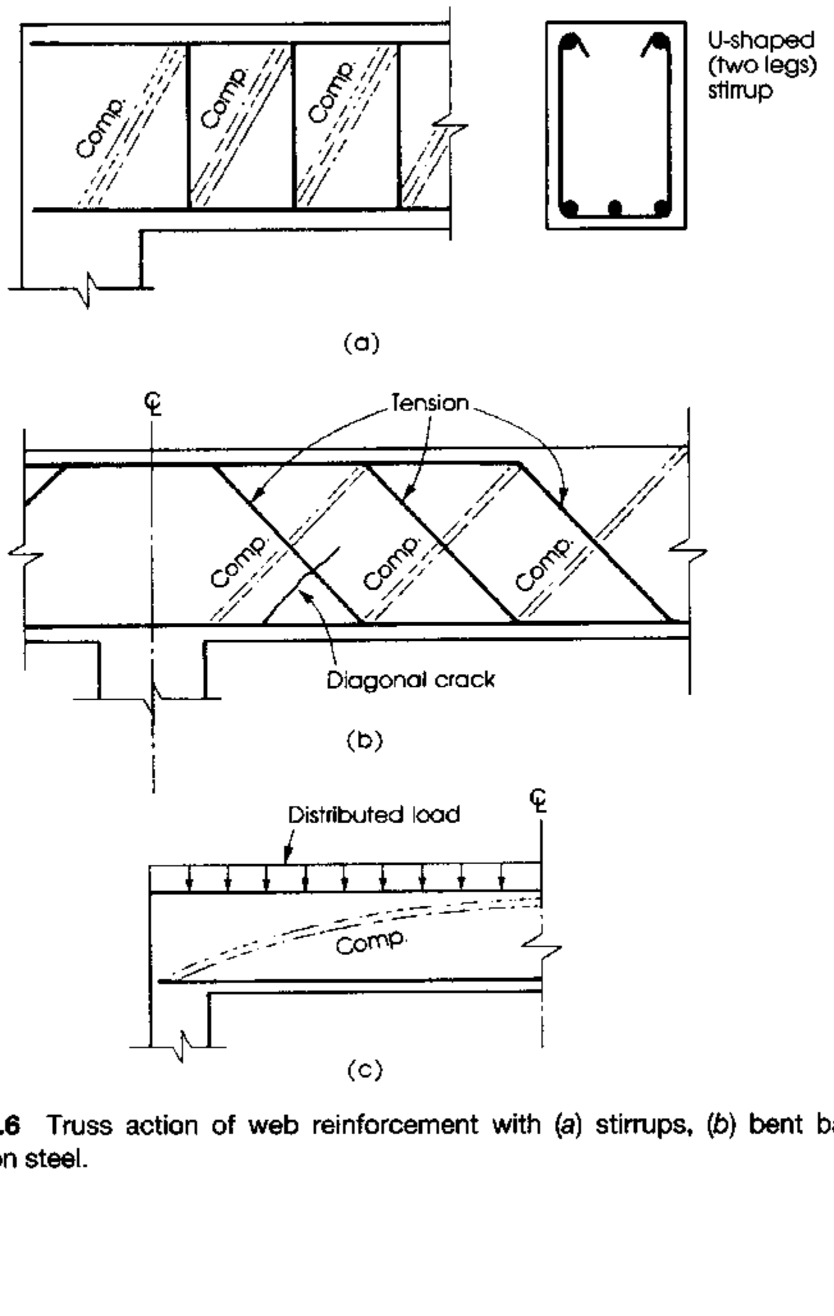


Figure 8.6 Truss action of web reinforcement with (*a*) stirrups, (*b*) bent bars, and (*c*) tension steel.

3. It increases the dowel force, V_d (Fig. 8.5), in the longitudinal bars.
4. The confining action of the stirrups on the compression concrete may increase its strength.
5. The confining action of stirrups on the concrete increases the rotation capacity of plastic hinges that develop in indeterminate structures at ultimate load and increases the length over which yielding takes place [7].

The total nominal shear strength of beams with shear reinforcement V_n is due partly to the shear strength attributed to the concrete V_c and partly to the shear strength contributed by the shear reinforcement V_s:

$$V_n = V_c + V_s \tag{8.12}$$

The shear force V_u produced by factored loads must be less than or equal to the total nominal shear strength V_n or

$$V_u \le \phi V_n = \phi(V_c + V_s) \tag{8.13}$$

where $V_u = 1.2V_D + 1.6V_L$ and $\phi = 0.75$.

An expression for V_s may be developed from the truss analogy (Fig. 8.7). For a 45° crack and a series of inclined stirrups or bent bars, the vertical shear force V_s resisted by shear reinforcement is equal to the sum of the vertical components of the tensile forces developed in the inclined bars. Therefore,

$$V_s = nA_v f_{yt} \sin\alpha \tag{8.14}$$

where A_v is the area of shear reinforcement with a spacing s, and f_{yt} is the yield strength of shear reinforcement, ns is defined as the distance aa_1a_2:

$$d = a_1a_4 = aa_1 \tan 45^\circ \qquad \text{(from triangle } aa_1a_4\text{)}$$

$$d = a_1a_4 = a_1a_2 \tan\alpha \qquad \text{(from triangle } a_1a_2a_4\text{)}$$

$$\begin{aligned} ns &= aa_1 + a_1a_2 \\ &= d(\cot 45^\circ + \cot\alpha) = d(1 + \cot\alpha) \\ n &= \frac{d}{S}(1 + \cot\alpha) \end{aligned}$$

Substituting this value in Eq. 8.14 gives

$$V_s = \frac{A_v f_{yt} d}{S} \sin\alpha(1 + \cot\alpha) = \frac{A_v f_{yt} d}{S}(\sin\alpha + \cos\alpha) \tag{8.15}$$

For the case of vertical stirrups, $\alpha = 90^\circ$ and

$$V_s = \frac{A_v f_{yt} d}{S} \quad \text{or } s = \frac{A_v f_{yt} d}{V_s} \tag{8.16}$$

In the case of T-sections, b is replaced by the width of web b_w in all shear equations. When $\alpha = 45^\circ$, Eq. 8.15 becomes

$$V_s = 1.4\left(\frac{A_v f_{yt} d}{S}\right) \text{or } s = \frac{1.4\ A_v f_{yt} d}{V_s} \tag{8.17}$$

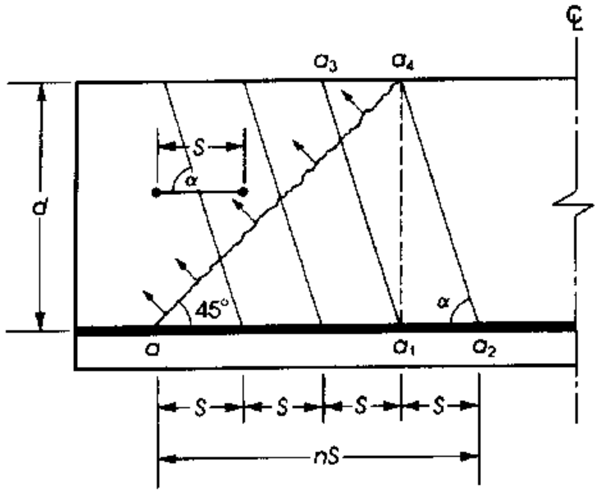


Figure 8.7 Factors in inclined shear reinforcement.

For a single bent bar or group of parallel bars in one position, the shearing force resisted by steel is

$$V_s = A_v f_{yt} \sin\alpha \text{ or } A_v = \frac{V_s}{f_{yt}\sin\alpha} \tag{8.18}$$

For $\alpha = 45°$,

$$A_v = 1.4\left(\frac{V_s}{f_{yt}}\right) \tag{8.19}$$

For circular sections, mainly in columns, V_s shall be computed from Eq. 8.16 using $d = 0.8 \times$ diameter and A_v = two times the area of the bar in a circular tie, hoop, or spiral.

8.6 ACI CODE SHEAR DESIGN REQUIREMENTS

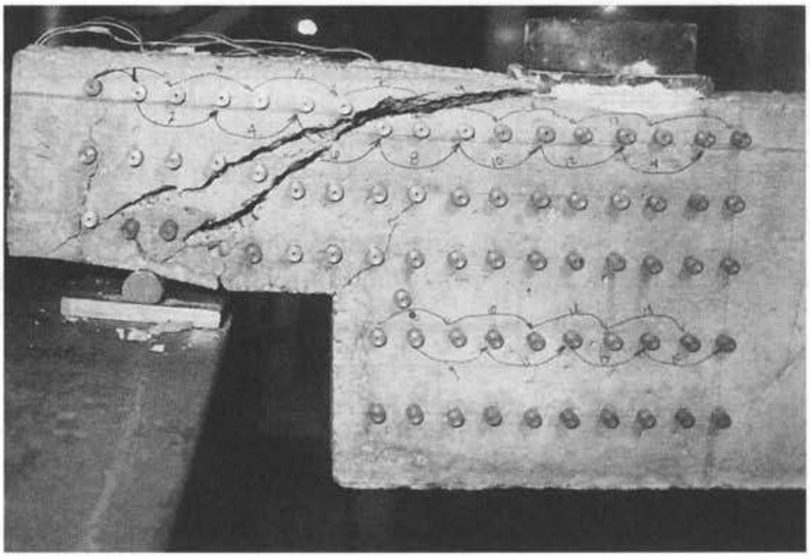
8.6.1 Critical Section for Nominal Shear Strength Calculation

The ACI Code, Section 11.1.3, permits taking the critical section for nominal shear strength calculation at a distance d from the face of the support. This recommendation is based on the fact that the first inclined crack is likely to form within the shear span of the beam at some distance d away from the support. The distance d is also based on experimental work and appeared in the testing of the beams discussed in Chapter 3. This critical section is permitted on the condition that the support reaction introduces compression into the end region, loads are applied at or near the top of the member, and no concentrated load occurs between the face of the support and the location of the critical section.

The Code also specifies that shear reinforcement must be provided between the face of the support and the distance d, using the same reinforcement adopted for the critical section.

8.6.2 Minimum Area of Shear Reinforcement

The presence of shear reinforcement in a concrete beam restrains the growth of inclined cracking. Moreover, ductility is increased, and a warning of failure is provided. If shear reinforcement is



Shear failure in dapped-end beam.

not provided, brittle failure will occur without warning. Accordingly, a minimum area of shear reinforcement is specified by the Code. The ACI Code, Section 11.4.6 requires all stirrups to have a minimum shear reinforcement area, A_v, equal to

$$A_v = 0.75\sqrt{f'_c}\left(\frac{b_w s}{f_{yt}}\right) \geq \frac{50 b_w s}{f_{yt}} \tag{8.20}$$

where b_w is the width of the web and s is the spacing of the stirrups. The minimum amount of shear reinforcement is required whenever V_u exceeds $\phi V_c/2$, except in

- Slabs and footings
- Concrete floor joist construction
- Beams where the total depth does not exceed 10 in., 2.5 times the flange thickness for T-shaped flanged sections, or one-half the web width, whichever is greatest.

If $0.75\sqrt{f'_c} = 50$ then $f'_c = 4444$ psi. This means that when $f'_c < 4500$ psi, the minimum $A_v = 50 b_w s / f_{yt}$ controls and when $f'_c \geq 4500$ psi, then the minimum $A_v = 0.75\sqrt{f'_c}(b_w s / f_{yt})$ controls. This increase in the minimum area of shear reinforcement for $f'_c \geq 4500$ psi is to prevent sudden shear failure when inclined cracking occurs.

It is a common practice to increase the depth of a slab, footing, or shallow beam to increase its shear capacity. Stirrups may not be effective in shallow members, because their compression zones have relatively small depths and may not satisfy the anchorage requirements of stirrups. For beams that are not shallow, reinforcement is not required when V_u is less than $\phi V_c/2$.

The minimum shear reinforcement area can be achieved by using no. 3 stirrups placed at maximum spacing, S_{max}. If $f_y = 60$ ksi and U-shaped (two legs) no. 3 stirrups are used, then Eq. 8.20 becomes

$$S_{max} = \frac{A_v f_{yt}}{(0.75\sqrt{f'_c}) b_w} \leq \frac{A_v f_{yt}}{50 b_w} \tag{8.21}$$

$$\begin{aligned}
&\text{For } f'_c < 4500 \text{ psi}, && S_{\max}(\text{in.}) = 0.22(60{,}000)/50b_w = 264/b_w.\\
&\text{For } f'_c = 4500 \text{ psi}, && S_{\max}(\text{in.}) = 262/b_w.\\
&\text{For } f'_c = 5000 \text{ psi}, && S_{\max}(\text{in.}) = 249/b_w.\\
&\text{For } f'_c = 6000 \text{ psi}, && S_{\max}(\text{in.}) = 227/b_w.
\end{aligned} \tag{8.22}$$

If U-shaped no. 4 stirrups are used, then for $f'_c < 4500$ psi,

$$\begin{aligned}
&S_{\max}(\text{in.}) = \frac{0.4(60{,}000)}{50b_w} = \frac{480}{b_w}\\
&\text{For } f'_c = 4500 \text{ psi}, \quad S_{\max}(\text{in.}) = 476/b_w.\\
&\text{For } f'_c = 5000 \text{ psi}, \quad S_{\max}(\text{in.}) = 453/b_w.\\
&\text{For } f'_c = 6000 \text{ psi}, \quad S_{\max}(\text{in.}) = 413/b_w.
\end{aligned} \tag{8.23}$$

Note that $S_{\max}$ shall not exceed 24 in., nor $d/2$.

Table 8.1 gives $S_{\max}$ based on Eqs. 8.22 and 8.23. Final spacings should be rounded to the lower inch. For example, $S = 20.3$ in. becomes 20 in.

8.6.3 Maximum Shear Carried by Web Reinforcement V_s

To prevent a shear-compression failure, where the concrete may crush due to high shear and compressive stresses in the critical region on top of a diagonal crack, the ACI Code, Section 11.4.7.9, requires that V_s shall not exceed $(8\sqrt{f'_c})b_w d$. If V_s exceeds this value, the section should be increased. Based on this limitation,

$$\text{If } f'_c = 3 \text{ ksi, then } V_s \le 0.438 b_w d \text{ (kips)} \quad \text{or } V_s/b_w d \le 438 \text{ psi.}$$

$$\text{If } f'_c = 4 \text{ ksi, then } V_s \le 0.506 b_w d \text{ (kips)} \quad \text{or } V_s/b_w d \le 506 \text{ psi.}$$

$$\text{If } f'_c = 5 \text{ ksi, then } V_s \le 0.565 b_w d \text{ (kips)} \quad \text{or } V_s/b_w d \le 565 \text{ psi.}$$

8.6.4 Maximum Spacing of Stirrups

To ensure that a diagonal crack will always be intersected by at least one stirrup, the ACI Code, Section 11.4.5, requires that the spacings between stirrups shall not exceed $d/2$, nor 24 in., provided that $V_s \le (4\sqrt{f'_c})b_w d$. This is based on the assumption that a diagonal crack develops at 45° and extends a horizontal distance of about d. In regions of high shear, where V_s exceeds $(4\sqrt{f'_c})b_w d$, the maximum spacing between stirrups must not exceed $d/4$. This limitation is necessary to ensure that the diagonal crack will be intersected by at least three stirrups. When V_s exceeds the maximum value of $8\sqrt{f'_c}b_w d$, this limitation of maximum stirrup spacing does not apply, and the dimensions of the concrete cross-section should be increased.

Table 8.1 Values of $S_{\max} = (A_v f_y/50b_w) = 24$ in. when $f_{yt} = 60$ ksi and $f'_c < 4500$ psi

b_w (in.)	10	11	12	13	14	15	16	18	20	22	24	b_w
$S_{\max}$ (in.) no. 3 stirrups	24	24	22	20.3	18.9	17.6	16.5	14.7	13.2	12	11	$\frac{264}{b_w}$
S_{max} (in.) no. 4 stirrups	24	24	24	24	24	24	24	24	24	21.8	20	$\frac{480}{b_w}$

A second limitation for the maximum spacing of stirrups may also be obtained from the condition of minimum area of shear reinforcement. A minimum A_v is obtained when the spacing s is maximum (Eq. 8.21).

A third limitation for maximum spacing is 24 in. $V_s \le (4\sqrt{f'_c})b_w d$ and 12 in. when V_s is greater than $(4\sqrt{f'_c})b_w d$ but less than or equal to $(8\sqrt{f'_c})b_w d$. The least value of all maximum spacings must be adopted. The ACI Code maximum spacing requirement ensures closely spaced stirrups that hold the longitudinal tension steel in place within the beam, thereby increasing their dowel capacity, V_d (Fig. 8.5).

8.6.5 Yield Strength of Shear Reinforcement

The ACI Code, Section 11.4.2, requires that the design yield strength of shear reinforcement shall not exceed 60 ksi (420 MPa). The reason behind this decision is to limit the crack width caused by the diagonal tension and to ensure that the sides of the crack remain in close contact to improve the interface shear transfer, V_a (Fig. 8.5). For welded deformed wire fabric, the design yield strength shall not exceed 80 ksi (560 MPa).

8.6.6 Anchorage of Stirrups

The ACI Code, Section 12.13.1, requires that shear reinforcement be carried as close as possible to the compression and tension extreme fibers, within the Code requirements for concrete cover, because near ultimate load the flexural tension cracks penetrate deep into the beam. Also, for stirrups to achieve their full yield strength, they must be well anchored. Near ultimate load, the stress in a stirrup reaches its yield stress at the point where a diagonal crack intercepts that stirrup. The ACI Code requirements for stirrup anchorage, Section 12.13, are as follows:

1. Each bend in the continuous portion of a simple U-stirrup or multiple U-stirrup shall enclose a longitudinal bar (ACI Code, Section 12.13.3). See Fig. 8.8*a*.
2. The code allows the use of a standard hook of 90°, 135°, or 180° around longitudinal bars for no. 5 bars or D31 wire stirrups. If no. 6, 7, or 8 stirrups with $f_{yt} > 40$ ksi are used, the Code (Section 12.13.2) requires a standard hook plus an embedment length of $0.014 d_b f_{yt}/(\lambda\sqrt{f'_c})$ between midheight of the member and the outside of the hook. If the bars are bent at 90°, extensions shall not be less than $12d_b$. For no. 5 bars or smaller stirrups, the extension is $6d_b$ (ACI Code, Section 7.1.3). See Fig. 8.8*b*.
3. If spliced double U-stirrups are used to form closed stirrups, the lap length shall not be less than $1.3l_d$ (ACI Code, Section 12.13.5). See Fig. 8.8*c*.
4. Welded wire fabric is used for shear reinforcement in the precast industry. Anchorage details are given in the ACI Code, Section 12.13.2.3, and in its commentary.
5. Closed stirrups are required for beams subjected to torsion or stress reversals (ACI Code, Section 7.11).
6. Beams at the perimeter of the structure should contain closed stirrups to maintain the structural integrity of the member (ACI Code, Section 7.13.2.2).

8.6.7 Stirrups Adjacent to the Support

The ACI Code, Section 11.1.3, specifies that shear reinforcement provided between the face of the support and the critical section at a distance d from it may be designed for the same shear V_u at the critical section. It is common practice to place the first stirrup at a distance $S/2$ from the face of the support, where s is the spacing calculated by Eq. 8.16 for V_u at the critical section.

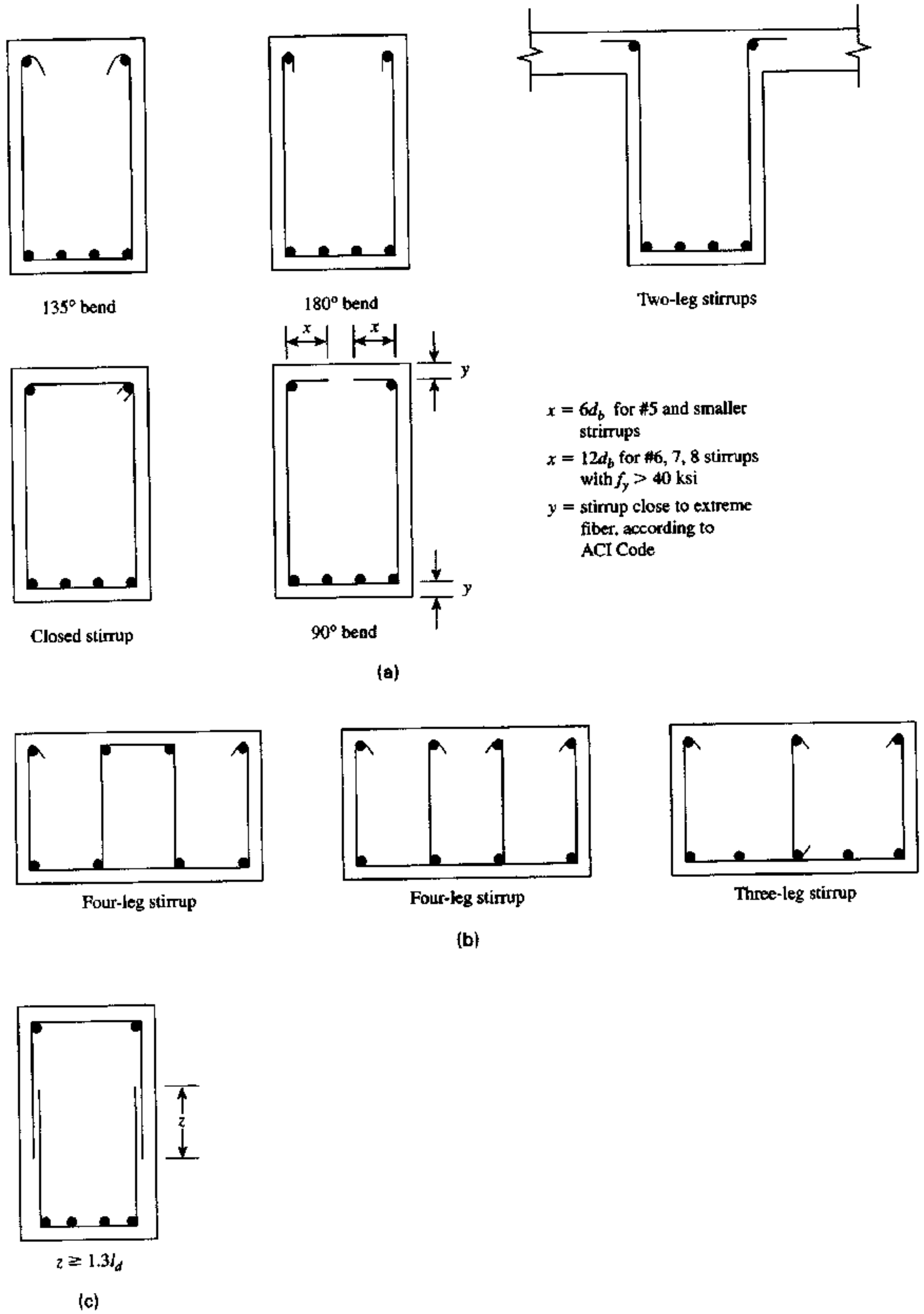


Figure 8.8 Stirrup types: (*a*) U-stirrups enclosing longitudinal bars, anchorage lengths, and closed stirrups, (*b*) multileg stirrups, and (*c*) spliced stirrups.

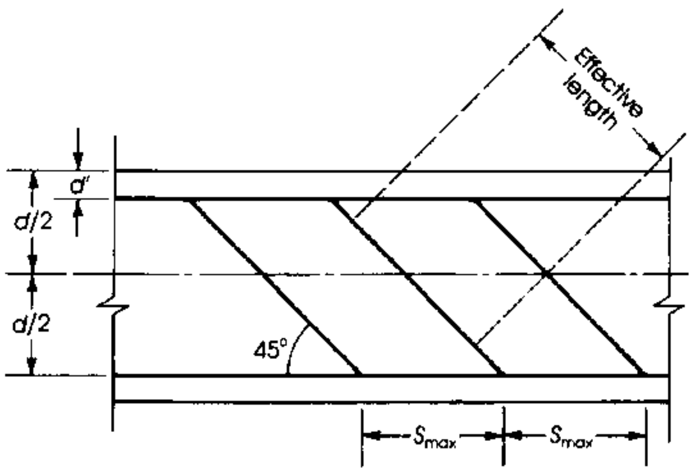


Figure 8.9 Effective length and spacing of bent bars.

8.6.8 Effective Length of Bent Bars

Only the center three-fourths of the inclined portion of any longitudinal bar shall be considered effective for shear reinforcement. This means that the maximum spacing of bent bars is $0.75(d - d')$. From Fig. 8.9, the effective length of the bent bar is $0.75(d - d')/(\sin 45°) = 0.75(1.414)(d - d') = 1.06(d - d')$. The maximum spacing S is equal to the horizontal projection of the effective length of the bent bar. Thus $S_{max} = 1.06(d - d')\cos 45°$, or $S_{max} = 0.707[1.06(d - d')] = 0.75(d - d')$.

8.7 DESIGN OF VERTICAL STIRRUPS

Stirrups are needed when $V_u > \frac{1}{2}\phi V_c$. Minimum stirrups are used when V_u is greater than $\frac{1}{2}\phi V_c$ but less than ϕV_c. This is achieved by using no. 3 stirrups placed at maximum spacing. When V_u is greater than ϕV_c stirrups must be provided. The spacing of stirrups may be less than the maximum spacing and can be calculated using Eq. 8.16: $S = A_v f_{yt} d / V_s$.

The stirrups that are commonly used in concrete sections are made of two-leg no. 3 or no. 4 U-stirrups with $f_{yt} = 60\,\text{ksi}$. If no. 3 stirrups are used, then Eq. 8.16 becomes

$$\frac{S}{d} = \frac{A_v f_y}{V_s} = \frac{0.22(60)}{V_s} = \frac{13.2}{V_s} \tag{8.24}$$

If no. 4 stirrups are used, then

$$\frac{S}{d} = \frac{A_v f_y}{V_s} = \frac{0.4(60)}{V_s} = \frac{24}{V_s} \tag{8.25}$$

The ratio of stirrup spacings relative to the effective depth of the beam, d, depends on V_s. The values of S/d for different values of V_s when $f_y = 60\,\text{ksi}$ are given in Tables 8.2 and 8.3 for no. 3 and no. 4 U-stirrups, respectively. The same values are plotted in Figs. 8.10 and 8.11. The following observations can be made:

1. If no. 3 stirrups are used, $S = d/2$ when $V_s \le 26.4\,\text{K}$. When V_s increases, S/d decreases in a nonlinear curve to reach 0.132 at $V_s = 100\,\text{K}$. If the minimum spacing is limited to

Table 8.2 S/d Ratio for Different Values of V_s (f_{yt} = 60 ksi, $S/d = 13.2/V_s$), No. 3 Stirrups

V_s (K)	26.4	30	40	50	52.8	60	70	80	90	100	125
S/d	0.5	0.44	0.33	0.264	0.25	0.22	0.19	0.165	0.15	0.132	0.106

Table 8.3 S/d Ratio for Different Values of V_s(f_{yt} = 60 ksi, $S/d = 24/V_s$), No. 4 Stirrups

V_s(K)	48	50	60	70	80	90	96	100	110	120	150	175
S/d	0.50	0.48	0.40	0.34	0.3	0.27	0.25	0.24	0.22	0.20	0.16	0.137

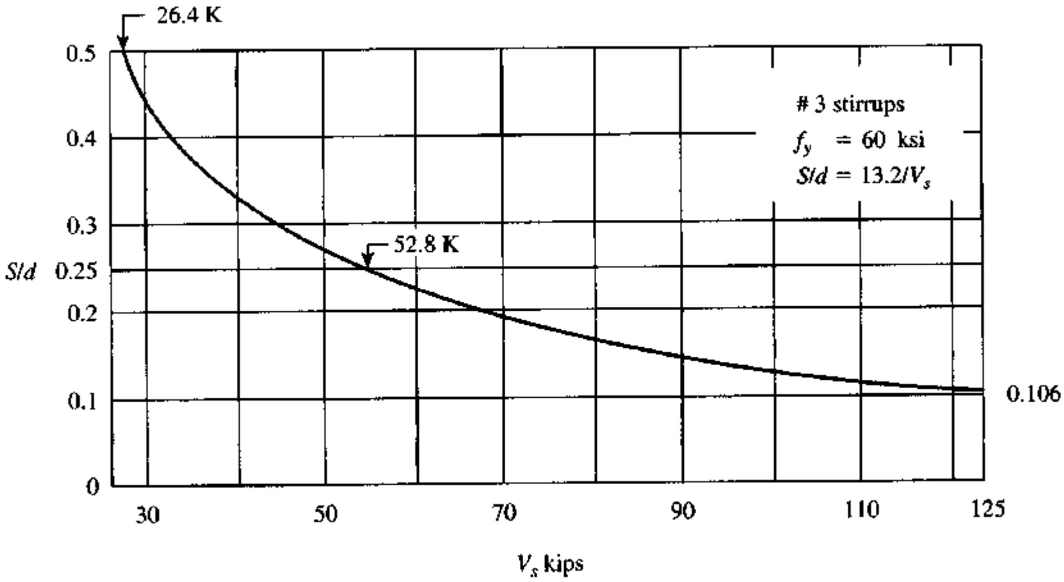


Figure 8.10 V_s versus S/d for no. 3 stirrups and f_{yt} = 60 ksi.

3 in., then d must be equal to or greater than 22.7 in. to maintain that 3-in. spacing. When V_s is equal to or greater than 52.8 K, then $S \leq d/4$.

2. If no. 4 U-stirrups are used, $S = d/2$ when $V_s \leq 48$ K. When V_s increases, S/d decreases to reach 0.16 at $V_s = 150$ K. If the minimum spacing is limited to 3 in., then $d \geq 18.75$ in. to maintain the 3-in. spacing. When V_s is equal to or greater than 96 K, then $S \leq d/4$.
3. If grade 40 U-stirrups are used (f_{yt} = 40 ksi), multiply the S/d values by $\frac{2}{3}$ or, in general, $f_{yt}/60$.

8.8 DESIGN SUMMARY

The design procedure for shear using vertical stirrups according to the ACI Code can be summarized as follows:

1. Calculate the factored shearing force, V_u, from the applied forces acting on the structural member. The critical design shear value is at a section located at a distance d from the face of the support.

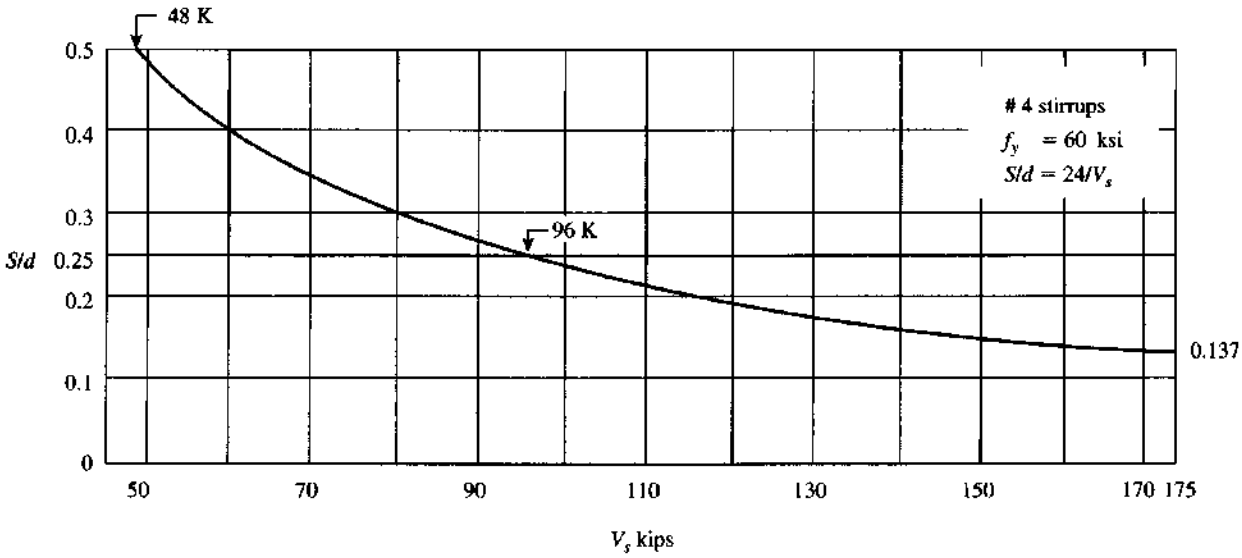


Figure 8.11 V_s versus S/d for no. 4 stirrups and f_{yt} = 60 ksi.

2. Calculate $\phi V_c = \phi 2\lambda\sqrt{f'_c}b_w d$, or

$$\phi V_c = \phi\left[1.9\lambda\sqrt{f'_c} + 2500\rho_w \frac{V_u d}{M_u}\right] b_w d \leq \phi 3.5\lambda\sqrt{f'_c}b_w d$$

Then calculate $\frac{1}{2}\phi V_c$

3. a. If $V_u < \frac{1}{2}\phi V_c$, no shear reinforcement is needed.
   b. If $\frac{1}{2}\phi V_c < V_u \leq \phi V_c$, minimum shear reinforcement is required. Use no. 3 U-stirrups spaced at maximum spacings, as explained in step 7.
   c. If $V_u > \phi V_c$, shear reinforcement must be provided according to steps 4 through 8.
4. If $V_u > \phi V_c$, calculate the shear to be carried by shear reinforcement:

$$V_u = \phi V_c + \phi V_s \text{ or } V_s = \frac{V_u - \phi V_c}{\phi}$$

5. Calculate $V_{c_1} = (4\sqrt{f'_c})b_w d$ and $V_{c_2} = (8\sqrt{f'_c})b_w d = 2V_{c_1}$. Compare the calculated V_s with the maximum permissible value of $V_{c_2} = (8\sqrt{f'_c})b_w d$. If V_s is less than V_{c_2}, proceed in the design; if not, increase the dimensions of the section.
6. Calculate the stirrup spacings based on the calculated $S_1 = A_v f_{yt} d/V_s$ or use Figs. 8.10 and 8.11 or Tables 8.2 and 8.3.
7. Determine the maximum spacing allowed by the ACI Code. The maximum spacing is the least of S_2 and S_3:
   a. $S_2 = d/2 \leq 24$ in. if $V_s \leq= V_{c_1} = (4\sqrt{f'_c})b_w d$.
      $S_2 = d/4 \leq 12$ in. if $V_{c_1} < V_s \leq V_{c_2}$.
   b. $S_3 = A_v f_{yt}/50b_w \geq A_v f_{yt}/(0.75\sqrt{f'_c}b_w)$
      S_{max} is the smaller of S_2 and S_3. Values of S_3 are shown in Table 8.1.
8. If S_1 calculated in step 6 is less than S_{max} (the smaller of S_2 and S_3), then use S_1 to the nearest smaller $\frac{1}{2}$ in. If $S_1 > S_{max}$, then use S_{max} as the adopted S.

9. The ACI Code did not specify a minimum spacing. Under normal conditions, a practical minimum S may be assumed to be equal to 3 in. for $d \leq 20$ in. and 4 in. for deeper beams. If S is considered small, either increase the stirrup bar number or use multiple-leg stirrups (Fig. 8.8).
10. For circular sections, the area used to compute V_c = diameter times the effective depth d, where $d = 0.8$ the diameter, ACI Code, Section 11.2.3.

Example 8.1

A simply supported beam has a rectangular section $b = 12$ in., $d = 21.5$ in., and $h = 24$ in. and is reinforced with four no. 8 bars. Check if the section is adequate for each of the following factored shear forces. If it is not adequate, design the necessary shear reinforcement in the form of U-stirrups. Use $f'_c = 4$ ksi and $f_{yt} = 60$ ksi. Assume normal-weight concrete.

a. $V_u = 12$ K **(b)** $V_u = 24$ K **(c)** $V_u = 54$ K **(d)** $V_u = 77$ K **(e)** $V_u = 128$ K

Solution

In general, $b_w = b = 12$ in., $d = 21.5$ in., and

$$\phi V_c = \phi(2\lambda\sqrt{f'_c})bd = 0.75(2)(1)(\sqrt{4000})(12)(21.5) = 24.5 \text{ K}$$

$$\tfrac{1}{2}\phi V_c = 12.25 \text{ K}$$

$$V_{c_1} = (4\sqrt{f'_c})bd = (4\sqrt{4000})(12)(21.5)/1000 = 65.3 \text{ K}$$

$$V_{c_2} = (8\sqrt{f'_c})bd = 130.6 \text{ K}$$

a. $V_u = 12$ K $< \frac{1}{2}\phi V_c = 12.25$ K, section is adequate, and shear reinforcement is not required.

b. $V_u = 24$ K $> \frac{1}{2}\phi V_c$, but it is less than $\phi V_c = 24.5$ K. Therefore, $V_s = 0$ and minimum shear reinforcement is required. Choose no. 3 U-stirrup (two legs) at maximum spacing. $A_v = 2(0.11) = 0.22$ in^2. Maximum spacing is the least of

$$S_2 = d/2 = 21.5/2 = 10.75 \text{ in., say, 10.5 in. (controls).}$$

$$S_3 = A_v f_{yt}/50b_w = 0.22(60{,}000)/50(12) = 22 \text{ in. (or use Table 8.1)}$$

$S_4 = 24$ in. Use no. 3 U-stirrups spaced at 10.5 in.

c. $V_u = 54$ K $> \phi V_c$. Shear reinforcement is needed. Calculation may be organized in steps:
Calculate $V_s = (V_u - \phi V_c)/\phi = (54 - 24.5)/0.75 = 39.3$ K.
Check if $V_s \leq V_{c_1} = (4\sqrt{f'_c})b_w d = 65.3$ K. Because $V_s < 65.3$ K, then $S_{max} = d/2$, and the $d/4$ condition does not apply.
Choose no. 3 U-stirrups and calculate the required spacings based on V_s.

$$S_1 = \frac{A_v f_{yt} d}{V_s} = \frac{0.22(60)(21.5)}{39.3} = 7.26 \text{ in.} \qquad \text{say, 7 in.}$$

Calculate maximum spacings: $S_2 = 10.5$ in., $S_3 = 22$ in., and $S_4 = 24$ in. and maximum $S = 10.5$ in. (calculated in (b)).
Because $S = 7$ in. $< S_{max} = 10.5$ in., then use no. 3 U-stirrups spaced at 7 in.

d. $V_u = 77$ K $> \phi V_c$, so stirrups must be provided.
Calculate $V_s = (V_u - \phi V_c)/\phi = (77 - 24.5)/0.75 = 70$ K.
Check if $V_s \leq V_{c_1} = 4\sqrt{f'_c}b_w d = 65.3$ K. Because $V_s > 65.3$ K, then $S_{max} = d/4 = 12$ in. must be used.
Check if $V_s \leq V_{c_2} = 8\sqrt{f'_c}b_w d = 130.6$ K. Because $V_{c_1} < V_s < V_{c_2}$, then stirrups can be used without increasing the section.

Choose no. 3 U-stirrups and calculate S_1 based on V_s:

$$S_1 = \frac{A_v f_{yt} d}{V_s} = \frac{0.22(60)(21.5)}{70} = 4.1 \text{ in.,} \qquad \text{say, 4 in.}$$

Calculate maximum spacings: $S_2 = d/4 = 21.5/4 = 5.3$ in., say, 5.0 in.; $S_3 = 22$ in.; and $S_4 = 12$ in. Hence $S_{max} = 5$-in. controls.

Because $S = 4$ in. $< S_{max} = 5$ in., then use no. 3 stirrups spaced at 4 in.

e. $V_u = 128$ K $> \phi V_c$, so shear reinforcement is required.
Calculate $V_s = (V_u - \phi V_c)/\phi = (128 - 24.5)/0.75 = 138$ K.
Because $V_s > V_{c_2} = 130.2$ K, the section is not adequate. Increase one or both dimensions of the beam section.

Notes : Table 8.2 and Fig. 8.10 can be used to calculate the spacing S for (c) and (d).

1. For (c), $V_s = 39.3$ K, from Fig. 8.10 (or Table 8.2 for no. 3 U-stirrups), $S/d = 0.34$ and $S_1 = 7.3$ in., which is less than $d/2 = 10.5$ in. Note that S_{max} based on V_s is $d/2$ and not $d/4$. Also, from Table 8.1, $S_3 = A_v f_{yt}/50b_w = 22$ in.
2. For (d), $V_s = 70$ K, $S/d = 0.19$ and $S_1 = 4.1$ in. $V_s = 70$ is greater than 52.8 K, and $S_{max} = d/4$ is required.

Example 8.2

A 17-ft-span simply supported beam has a clear span of 16 ft and carries uniformly distributed dead and live loads of 4.5 K/ft and 3.75 K/ft, respectively. The dimensions of the beam section and steel reinforcement are shown in Fig. 8.12. Check the section for shear and design the necessary shear reinforcement. Given $f'_c = 3$ ksi normal-weight concrete and $f_{yt} = 60$ ksi.

Solution

Given b_w (web) = 14 in., $d = 22.5$ in.

1. Calculate factored shear from external loading:

   factored uniform load = 1.2(4.5) + 1.6(3.75) = 11.4K/ft

   $$V_u(\text{at face of support}) = \frac{11.4(16)}{2} = 91.2\text{K}$$

   Design V_u (at d distance from the face of support) = 91.2 − 22.5(11.4)/12 = 69.83 K.
2. Calculate ϕV_c:

   $$\phi V_c = \phi(2\lambda\sqrt{f'_c})b_w d = \frac{0.75(2)(1)(\sqrt{3000})(14)(22.5)}{1000} = 25.88 \text{ K}$$

   $$\tfrac{1}{2}\phi V_c = 12.94 \text{ K}$$

   Calculate $V_{c_1} = (4\sqrt{f'_c})b_w d = (4\sqrt{3000})(14)(22.5)/1000 = 69$ K. Calculate $V_{c_2} = (8\sqrt{f'_c})b_w d = 138$ K.
3. Design $V_u = 69.83$ K $> \phi V_c = 25.88$ K; therefore, shear reinforcement must be provided. The distance x' at which no shear reinforcement is needed (at $\frac{1}{2}\phi V_c$) is

   $$x' = \left(\frac{91.2 - 12.94}{91.2}\right)(8) = 6.86 \text{ ft} = 82 \text{ in.}$$

   (from the triangles of shear diagram, Fig. 8.12).

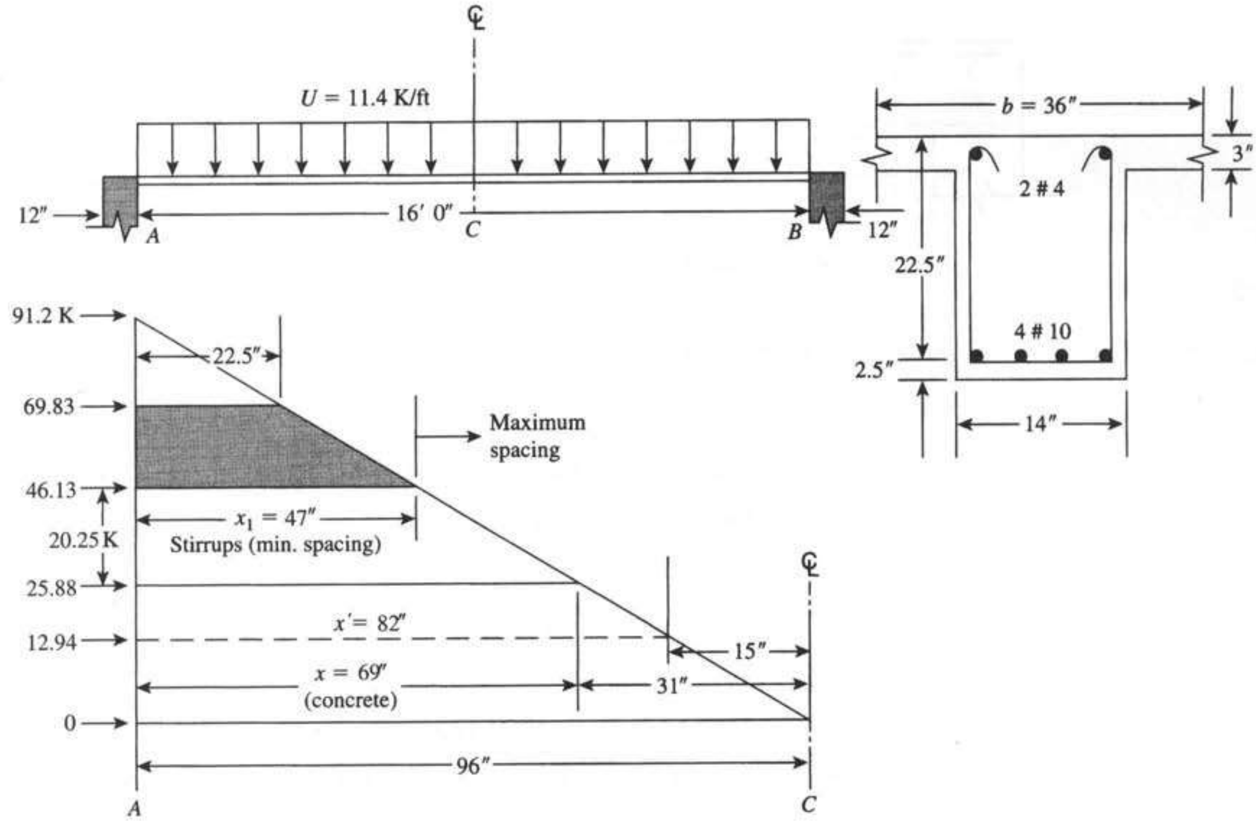


Figure 8.12 Example 8.2.

4. Calculate $V_s = (V_u - \phi V_c)/\phi = (69.83 - 25.88)/0.75 = 58.6$ K. Because V_s is less than $V_{c_1} = (4\sqrt{f'_c})b_w d$, then $S_{max} = d/2$ must be considered (or refer to Fig. 8.10 or Table 8.2: $V_s < 52.8$ K).
5. Design of stirrups: Choose no. 3 U-stirrups, $A_v = 2(0.11) = 0.22\,\text{in.}^2$ Calculate S_1 based on $V_s = 58.6$ K, $S_1 = A_v f_y d/V_s = 13.2\ d/V_s = 5.07$ in., say, 5 in. (or get $s/d = 0.225$ from Table 8.2 or Fig. 8.10).
6. Calculate maximum spacings: $S_2 = d/2 = 22.5/2 = 11.25$ in., say, 11.0 in.; $S_3 = A_v f_{yt}/50b_w = 0.22(60{,}000)/50(14) = 18.9$ in. (or use Table 8.1); $S_4 = 24$ in.; $S_{max} = 11$ in. controls.
7. Because $S_1 = 5$ in. $< S_{max} = 11$ in., use no. 3 U-stirrups spaced at 5 in.
8. Calculate V_s for maximum spacings of 11 in.:

$$V_s = \frac{A_v f_{yt} d}{S} = \frac{0.22(60)(22.5)}{11} = 27 \text{ K}$$

$$\phi V_s = 20.25 \text{ K}$$

$$\phi V_c + \phi V_s = 25.88 + 20.25 = 46.13 \text{ K}$$

The distance x_1 at which $S = 11$ can be used is

$$\left(\frac{91.2 - 46.13}{91.2}\right)(96) = 47 \text{ in.}$$

Because x_1 is relatively small, use $S = 5$ in. for a distance greater than or equal to 47 and then use $S = 11$ for the rest of the beam. *Note*: If x_1 is long, then an intermediate spacing between 5 in. and 11 in. may be added.

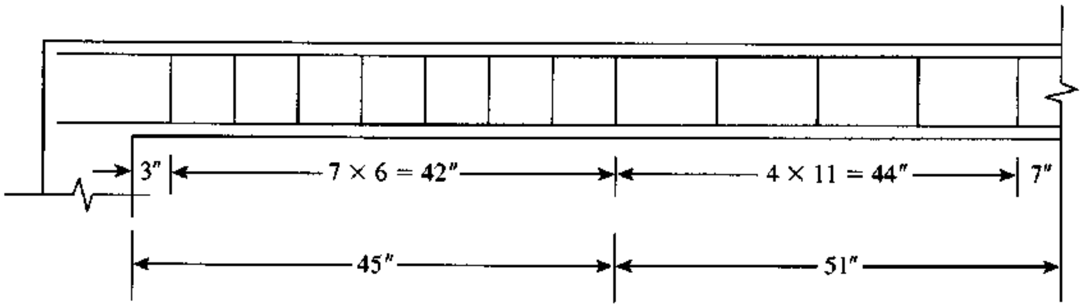


Figure 8.13 Example 8.2: distribution of stirrups.

9. Distribute stirrups as follows: Place the first stirrup at $S/2$ from the face of the support.

First stirrup at $S/2 = 5/2 = 2$ in.

Nine stirrups at $S = 5 = 45$ in.

Total $= 45 + 2$ in. $= 47$ in.

Four stirrups at $S = 11 = 44$ in.

Total $= 91$ in. > 82 in. (minimum length required)

The total number of stirrups for the beam is $2(1 + 9 + 4) = 28$. Distribution of stirrups is shown in Fig. 8.13, whereas calculated shear forces are shown in Fig. 8.12.

10. Place two no. 4 bars at the top of beam section to act as stirrup hangers.

8.9 SHEAR FORCE DUE TO LIVE LOADS

In Example 8.2, it was assumed that the dead and live loads are uniformly distributed along the full span, producing zero shear at midspan. Actually, the dead load does exist along the full span, but the live load may be applied to the full span or part of the span, as needed to develop the maximum shear at midspan or at any specific section. Figure 8.14*a* shows a simply supported beam with a uniform load acting on the full span. The shear force varies linearly along the beam, with maximum shear acting at support A.

In the case of live load, $W_2 = 1.6W_L$, the maximum shear force acts at support A when W_2 is applied on the full span, Fig. 8.14*a*. The maximum shear at midspan develops if the live load is placed on half the beam, BC (Fig. 8.14*b*), producing V_u at midspan equal to $W_2L/8$. Consequently, the design shear force is produced by adding the maximum shear force due to live load (placed at different lengths of the span) to the dead load shear force (Fig. 8.14*c*) to give the shear distribution shown in Fig. 8.14*d*. It is a common practice to consider the maximum shear at support A to be $W_uL/2 = (1.2W_D + 1.6W_L)L/2$, whereas V_u at midspan is $W_2L/8 = (1.6W_L)L/8$ with a straight-line variation along AC and CB, as shown in Fig. 8.14*d*. The design for shear in this case will follow the same procedure explained in Example 8.2. If the approach is applied to the beam in Example 8.2, then V_u (at A) $= 91.2$ K and V_u (at midspan) $= (1.6 \times 3)(16/8) = 10$ K.

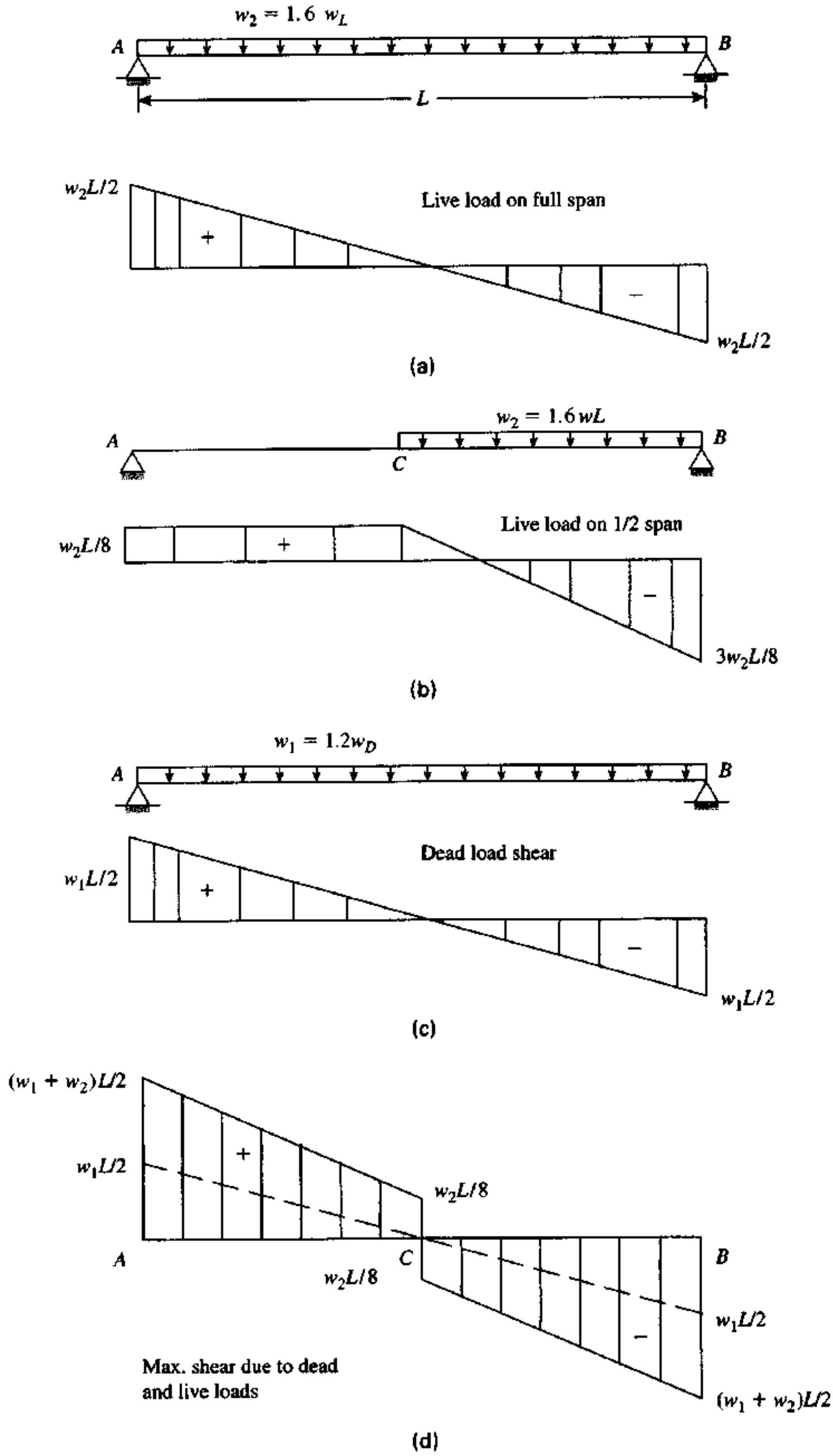


Figure 8.14 Effect of live load application on part of the span.

Example 8.3

A 10-ft-span cantilever beam has a rectangular section and carries uniform and concentrated factored loads (self-weight is included), as shown in Fig. 8.15. Using $f_c' = 4$ ksi normal-weight concrete and $f_y = 60$ ksi, design the shear reinforcement required for the entire length of the beam according to the ACI Code.

Solution

1. Calculate the shear force along the beam due to external loads.

$$V_u \text{ (at support)} = 5.5(10) + 20 + 8 = 83 \text{ K}$$

$$V_{ud} \text{ (at } d \text{ distance)} = 83 - 5.5\left(\frac{20.5}{12}\right) = 73.6 \text{ K}$$

$$V_u \text{ (at 4 ft left)} = 83 - 4(5.5) = 61 \text{ K}$$

$$V_u \text{ (at 4 ft right)} = 61 - 20 = 41 \text{ K}$$

$$V_u \text{ (at free end)} = 8 \text{ K}$$

The shear diagram is shown in Fig. 8.15.

2. Calculate ϕV_c:

$$\phi V_c = 2\lambda\sqrt{f_c'}bd = 2(0.75)(1)\sqrt{4000}(12)(20.5) = 23.34 \text{ K}$$

$$\frac{1}{2}\phi V_c = 11.67 \text{ K}$$

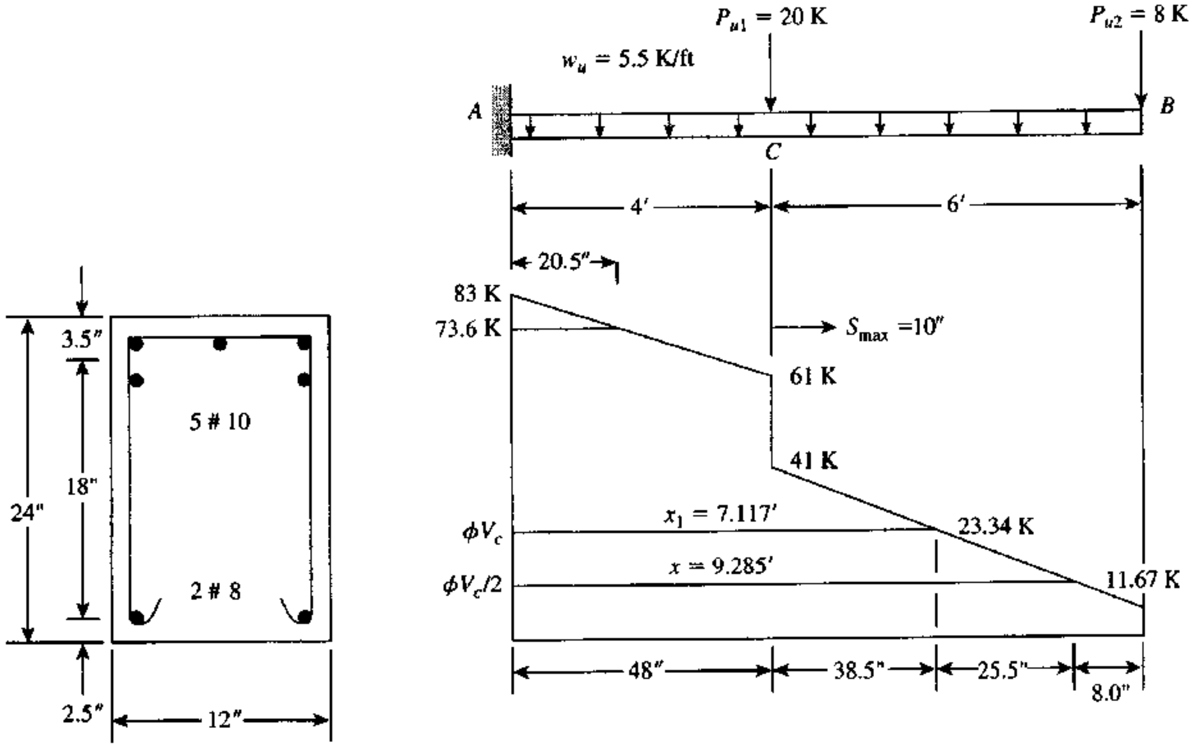


Figure 8.15 Example 8.3.

Because $V_{ud} > \phi V_c$ shear reinforcement is required. Calculate

$$V_{c_1} = 4\sqrt{f_c'}bd = 4\sqrt{4000}(12)(20.5) = 62.2 \text{ K}$$

$$V_{c_2} = 8\sqrt{f_c'}bd = 2V_1 = 124.4 \text{ K}$$

The distance x at which no shear reinforcement is needed (at $\frac{1}{2}\phi V_c = 11.67$ K), measured from support A:

$$x = 4 + \left(\frac{41 - 11.67}{41 - 8}\right)6 = 9.33 \text{ ft} = 112 \text{ in.}$$

(8.0 in. from free end). Similarly, x_1 for ϕV_c is 7.21 ft from A (33.5 in. from the free end).

3. Part AC: Design shear $V_u = V_{ud} = 73.6$ K. Calculate $V_s = (V_u - \phi V_c)/\phi = (73.6 - 23.34)/0.75 = 67$ K. Because $V_{c_1} < V_s < V_{c_2}$, $S_{max} \le d/4$ must be considered (or check Fig. 8.10).
4. Design stirrups: Choose no. 3 U-stirrups, $A_v = 0.22$ in.^{2} Calculate S_1 (based on V_s):

$$S_1 = \frac{A_v f_{yt} d}{V_s} = \frac{13.2d}{V_s} = \frac{13.2(20.5)}{67} = 4.0 \text{ in.}$$

Use 4.0 in. (or get $s/d = 0.22$ from Fig. 8.10).

5. Calculate maximum spacings: $S_2 = d/4 = 20.5/4 = 5.12$ in., so use 5.0 in.

$$S_3 = \frac{A_v f_{yt}}{50b_w} = 22 \text{ in.} \quad \text{(from Table 8.1 for } b = 12 \text{ in.)}$$

$$S_4 = 12 \text{ in.}$$

Then $S_{max} = 5.0$ in.

6. Because $S = 4$ in. $< S_{max} = 5.1$ in., use no. 3 stirrups spaced at 4 in.
7. At C, design shear $V_u = 61$ K $> \phi V_c$. Then $V_s = (61 - 23.34)/0.75 = 50.2$ K, $S_1 = A_v f_{yt} d / V_s = 5.4$ in.

$$V_s = 50.2 \text{ K} < V_{c_1} = 62.2 \text{ K} \quad S_2 = \frac{d}{2} = \frac{20.5}{2} = 10.25 \text{ in.} \quad \text{(or 10 in.)}$$

$S_1 = 5.4$ in. $< S_2$; then $S_1 = 5.4$ or 5.0 in. controls.

8. Because spacings of 5.5 in. and 4.0 in. are close, use no. 3 U-stirrups spaced at 4 in. for part AC.
9. Part BC:
   a.
$$V_u = 41 \text{ K} > \phi V_c$$
$$V_s = (V_u - \phi V_c)/\phi = (41 - 23.34)/0.75 = 23.55 \text{ K} < V_{c_1} = 62.2 \text{ K}$$
   b. $S_1 = A_v f_{yt} d / V_s = (13.2)(20.5)/23.55 = 11.5$ in.
   c. $S_2 = d/2 = 20.5/2 = 10.25$ in. (or less than $S_3 = 22$ in. or $S_4 = 24$ in.). Let $S_{max} = 10$ in. Choose no. 3 stirrups spaced at 10 in. for part BC.
10. Distribution of stirrups measured from support A: Place the first stirrup at

$$\frac{S}{2} = \frac{4}{2} = 2 \text{ in.}$$

$$12 \times 4 \text{ in.} = \underline{48 \text{ in.}}$$

$$50 \text{ in.}$$

$$6 \times 10 \text{ in.} + 1 \times 8 \text{ in.} = \underline{68 \text{ in.}}$$

$$\text{Total} \quad 118 \text{ in.}$$

Distance left to the free end is 7 in., which is less than 8.0 in., where no stirrups are needed. Distribution of stirrups is shown in Fig. 8.16. Total number of stirrups is 20.

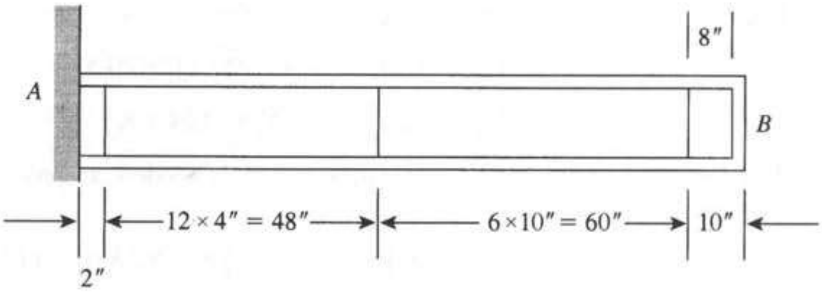


Figure 8.16 Example 8.3: distribution of stirrups.

8.10 SHEAR STRESSES IN MEMBERS OF VARIABLE DEPTH

The shear stress, v, is a function of the effective depth, d; therefore, shear stresses vary along a reinforced concrete beam with variable depth [10]. In such a beam (Fig. 8.17), consider a small element dx. The compression force C at any section is equal to the moment divided by its arm, or $C = M/y$. The first derivative of C is

$$dC = \frac{ydM - Mdy}{y^2}$$

If C_1 is greater than C_2, then $C_1 - C_2 = dC = vb\,dx$

$$vbdx = \frac{ydM - Mdy}{y^2} = \frac{dM}{y} - \frac{M}{y^2}dy$$

$$v = \frac{1}{yb}\left(\frac{dM}{dx}\right) - \frac{M}{by^2}\left(\frac{dy}{dx}\right)$$

Because $y = jd$, dM/dx is equal to the shearing force V and $d(jd)/dx$ is the slope,

$$v = \frac{V}{bjd} - \frac{M}{b(jd)^2}\left[\frac{d}{dx}(jd)\right] \quad \text{and} \quad v = \frac{V}{bjd} \pm \frac{M}{b(jd)^2}(\tan\alpha) \qquad (8.26)$$

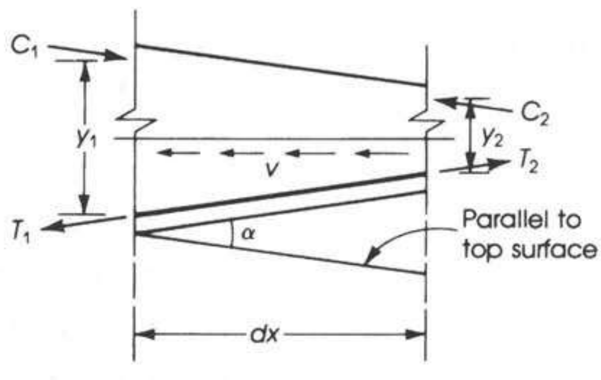


Figure 8.17 Shear stress in a beam with variable depth.

where V and M are the external shear and moment, respectively, and α is the slope angle of one face of the beam relative to the other face. The plus sign is used when the beam depth decreases as the moment increases, whereas the minus sign is used when the depth increases as the moment increases. This formula is used for small slopes, where the angle α is less than or equal to 30°.

A simple form of Eq. 8.26 can be formed by eliminating the j value:

$$v = \frac{V}{bd} \pm \frac{M}{bd^2}(\tan\alpha) \tag{8.27}$$

For the strength design method, the following equation may be used:

$$v_u = \frac{V_u}{\phi bd} \pm \frac{M_u}{\phi bd^2}(\tan\alpha) \tag{8.28}$$

For the shearing force,

$$\phi V_n = V_u \pm \frac{M_u}{d}(\tan\alpha) \tag{8.29}$$

Figure 8.18 shows a cantilever beam with a concentrated load P at the free end. The moment and the depth d increase toward the support. In this case a negative sign is used in Eqs. 8.27, 8.28, and 8.29. Similarly, a negative sign is used for section t in the simply supported beam shown, and a positive sign is used for section Z, where moment increases as the depth decreases.

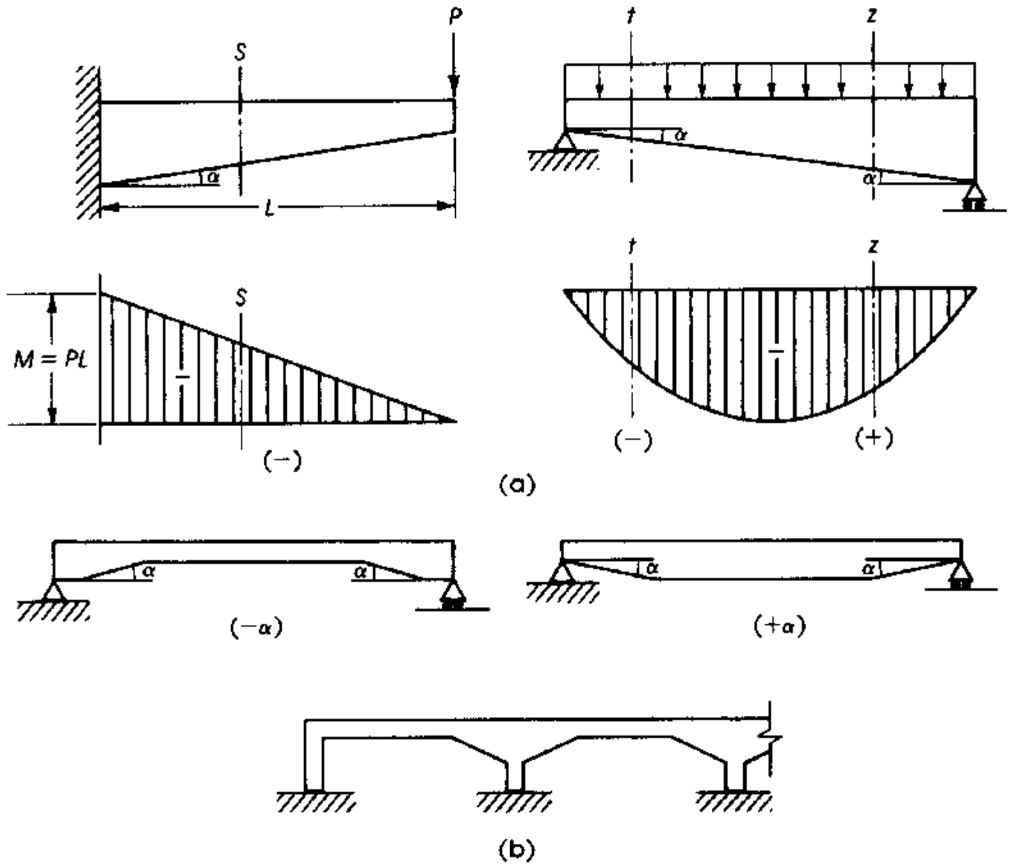


Figure 8.18 Beams with variable depth: (*a*) moment diagrams and (*b*) typical forms.

In many cases, the variation in the depth of beams occurs on parts of the beams near their supports (Fig. 8.18).

Tests [11] on beams with variable depth indicate that beams with greater depth at the support fail mainly by shear compression. Beams with smaller depth at the support fail generally by an instability type of failure, caused by the propagation of the major crack in the beam upward and then horizontally to the beam's top section. Tests also indicate that for beams with variable depth (Fig. 8.18) with an inclination α of about 10° and subjected to shear and flexure, the concrete shear strength, V_{cv}, may be computed by

$$V_{cv} = V_c(1 + \tan\alpha) \tag{8.30}$$

where

V_{cv} = shear strength of beam with variable depth

V_c = ACI Code Eq. 11.5

$= \left(1.9\lambda\sqrt{f_c'} + 2500\rho_w \frac{V_u d_s}{M_u}\right) b_w d_s \leq 3.5\lambda\sqrt{f_c'} b_w d_s$

α = angle defining the orientation of reinforcement, considered positive for beams of small depth at the support and negative for beams with greater depth at the support (Fig. 8.18)

d_s = effective depth of the beam at the support

The simplified ACI Code, Eq. 11.3,can also be used to compute V_c:

$$V_c = (2\lambda\sqrt{f_c'})b_w d_s \tag{8.31}$$

Example 8.4

Design the cantilever beam shown in Fig. 8.19 under the factored loads applied if the total depth at the free end is 12 in., and it increases toward the support. Use a steel percentage $\rho = 1.5\%$, $f_c' = 4$ ksi normal-weight concrete, $f_y = 60$ ksi, and $b = 10$ in.

Solution

1. M_u (support) $= (2.5/2)(8)^2(12) + (14)(8)(12) = 2304$ K·in.
2. For $\rho = 1.5\%$, $R_u = 703$ psi (from Table 4.1).

$$d = \sqrt{\frac{M}{R_u b}} = \sqrt{\frac{2304}{0.703 \times 10}} = 18.1 \text{ in.}$$

$A_s = 0.015 \times 10 \times 18.1 = 2.72\,\text{in.}^2$ (use three no. 9 bars); let actual $d = 19.5$ in., $h = 22$ in.

3. Design for shear: Maximum shear at the support is $14 + 20 = 34$ K. Because the beam section is variable, moment effect shall be considered; because the beam depth increases as the moment increases, a minus sign is used in Eq.8.28.

$$v_u = \frac{V_u}{\phi bd} - \frac{M_u}{\phi bd^2}(\tan\alpha)$$

To find $\tan\alpha$, let d at the free end be 9.5 in., and d at the support be 19.5 in.:

$$\tan\alpha = \frac{19.5 - 9.5}{8 \times 12} = 0.1042$$

$$v_u \text{ (at the support)} = \frac{34{,}000}{(0.75 \times 10 \times 19.5)} - \frac{2304 \times 1000 \times 0.1042}{[0.75 \times 10 \times (19.5)^2]}$$

$$= 148 \text{ psi}$$

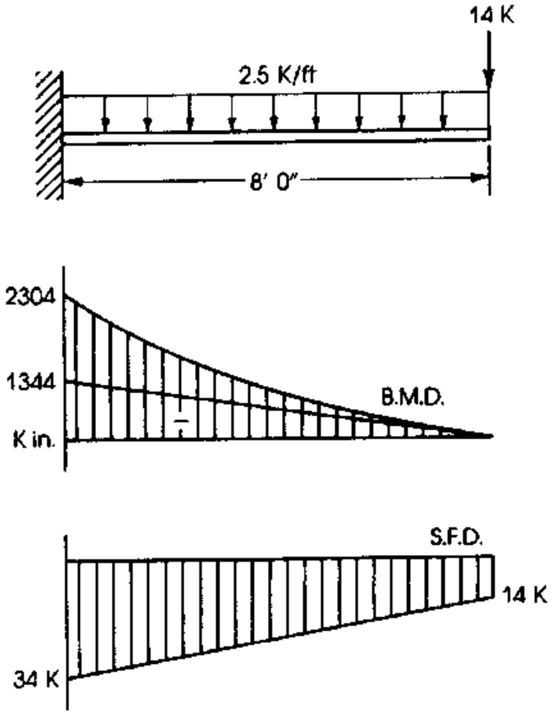


Figure 8.19 Example 8.4 with bending moment diagram (*middle*) and shear force diagram (*bottom*).

4. Shear stress at the free end is $V_u/\phi bd(M_u = 0)$.

$$v_u = \frac{14{,}000}{0.75 \times 10 \times 9.5} = 196 \text{ psi}$$

5. At a distance $d = 18$ in. from the face of the support, the effective depth is 17.6 in. (from geometry),

$$V_u = 34 - 2.5 \times \frac{18}{12} = 30.25 \text{ K}$$

$$M_u \text{ (at 18 in. from support)} = 14 \times 78 + \frac{2.5}{12} \times \frac{(78)^2}{2}$$

$$= 1726 \text{ K·in.}$$

$$v_u = \frac{30.25}{0.75 \times 10 \times 18} - \frac{1726 \times 1000 \times 0.1042}{0.75 \times 10 \times (18)^2}$$

$$= 150 \text{ psi}$$

6. At midspan (48 in. from the support),

$$d = 14.5 \text{ in.}$$

$$V_u = 14 + 10 = 24 \text{ K}$$

$$M_u = 14 \times 48 + \frac{2.5}{12} \times \frac{(48)^2}{2} = 912 \text{ K·in.}$$

$$v_u = \frac{24{,}000}{0.75 \times 10 \times 14.5} - \frac{912 \times 1000 \times 0.1042}{0.75 \times 10 \times (14.5)^2} = 160 \text{ psi}$$

Similarly, at 6 ft from the support (2 ft from the free end),

$$d = 12 \text{ in.} \quad V_u = 19 \text{ K} \quad M_u = 396 \text{ K·in.}$$

$$v_u = 173 \text{ psi}$$

At 1 ft from the free end,

$$d = 10.75 \text{ in.} \quad V_u = 16.5 \text{ K} \quad M_u = 183 \text{ K·in.}$$

$$v_u = 182 \text{ psi}$$

These values are shown in Fig. 8.20.

7. Shear stress resisted by concrete is

$$2\lambda\sqrt{f'_c} = (2)(1)\sqrt{4000} = 126.5 \text{ psi}$$

Minimum shear stress to be resisted by shear reinforcement

$$v_{us} = 196 - 126.6 = 69.5 \text{ psi}$$

(V_u and consequently v_{us} have already been increased by the ratio $1/\phi$ in Eq. 8.28).

8. Choose no. 3 stirrups with two legs.

$$A_v = 2 \times 0.11 = 0.22 \text{ in.}^2$$

$$S \text{ (required)} = \frac{A_v f_{yt}}{v_s b_w} = \frac{0.22 \times 60{,}000}{69.5 \times 10} = 19 \text{ in.}$$

$$S_{max} \left(\text{for } \frac{d}{2}\right) = 9.5 \text{ in. to } 4.5 \text{ in. at the free end}$$

$$S_{max} \text{ (for minimum} A_v) = \frac{A_v f_{yt}}{50 b_w} = \frac{0.22 \times 60{,}000}{50 \times 10} = 26.4 \text{ in.}$$

9. Check for maximum spacing ($d/4$): $v_{us} \le 4\sqrt{f'_c}$.

$$4\sqrt{f'_c} = (4)\sqrt{4000} = 253 > 69.5 \text{ in.}$$

10. Distribution of stirrups (distances from the free end):

1 stirrup at 2 in. = 42 in.

10 stirrups at 4.5 in. = 45 in.

3 stirrups at 7 in. = 21 in.

3 stirrups at 8 in. = <u>24 in.</u>

Total = 92 in.

There is 4 in. left to the face of the support.

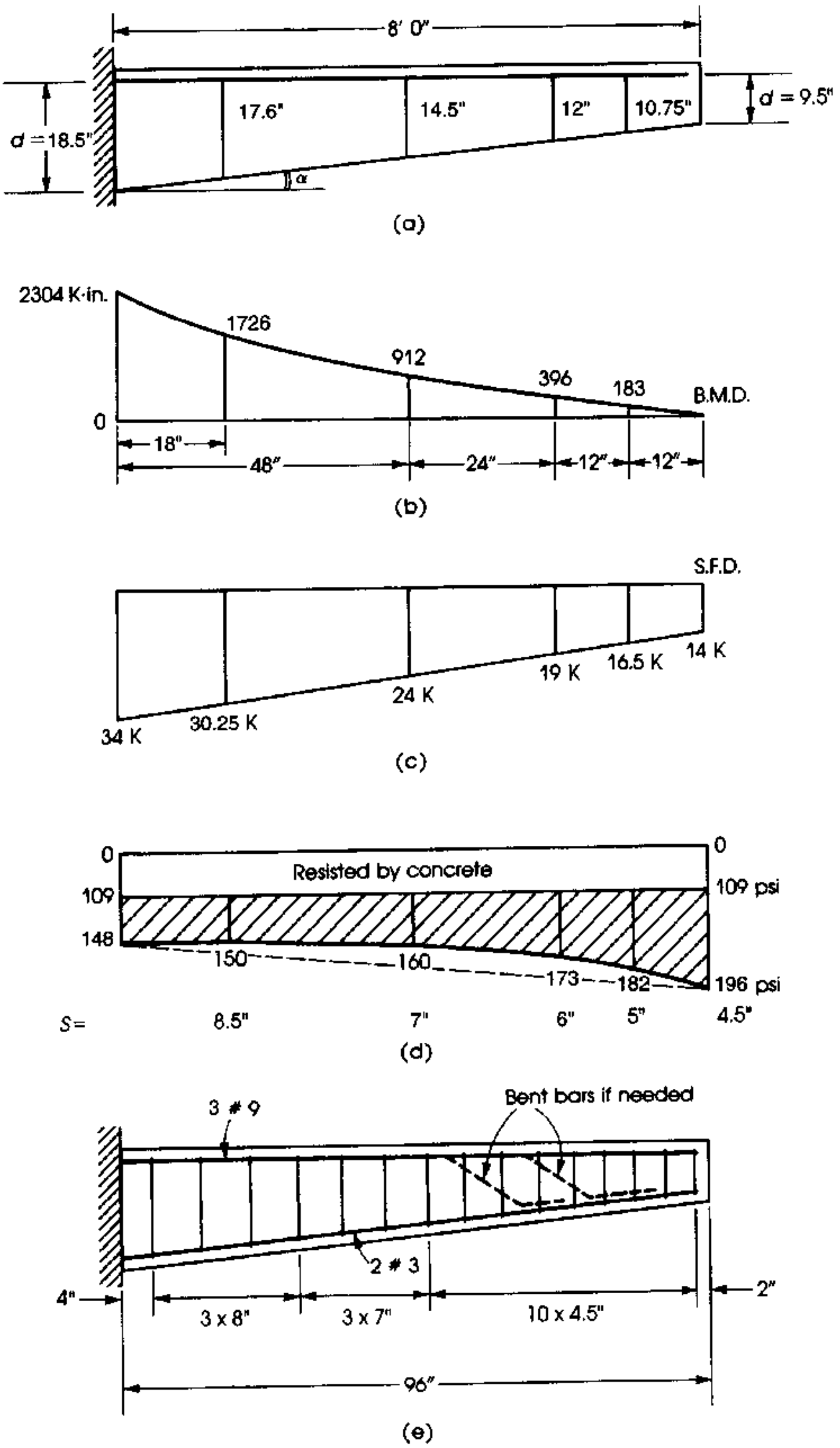


Figure 8.20 Example 8.4: web reinforcement for a beam of variable depth.

8.11 DEEP FLEXURAL MEMBERS

Flexural members should be designed as deep beams if the ratio of the clear span, l_n (measured from face to face of the supports; Fig. 8.21), to the overall depth, h, is less than 4 (ACI Code, Section 11.8). The members should be loaded on one face and supported on the opposite face so that compression struts can develop between the loads and supports (Fig. 8.22). If the loads are applied through the bottom or sides of the deep beam, shear design equations for ordinary beams given earlier should be used. Examples of deep beams are short-span beams supporting heavy loads, vertical walls under gravity loads, shear walls, and floor slabs subjected to horizontal loads.

The definition of deep flexural members is also presented in the ACI Code, Section 10.7.1. It indicates that flexural members where the ratio of the clear span, l_n, to the overall depth, h (Fig. 8.21), is less than 4 and regions loaded with concentrated loads within twice the member depth from the face of the support are considered deep flexural members. Such beams should be designed taking into account nonlinear distribuion of stress and lateral buckling (Fig. 8.22*a*).

Figure 8.22*a* shows the elastic stress distribution at the midspan section of a deep beam, and Fig. 8.22*b* shows the principal trajectories in top-loaded deep beams. Solid lines indicate tensile stresses, whereas dashed lines indicate compressive stress distribution. Under heavy loads, inclined vertical cracks develop in the concrete in a direction perpendicular to the principal tensile stresses and almost parallel to the dashed trajectories (Fig. 8.22*c*). Hence, both horizontal and vertical reinforcement is needed to resist principal stresses. Moreover, tensile flexural reinforcement is needed within about the bottom one-fifth of the beam along the tensile stress trajectories (Fig. 8.22*b*). In general, the analysis of deep beams is complex and can be performed using truss models or more accurately using a finite-element approach or similar methods. A simplified provision for the shear design of deep beams can be presented in steps as follows:

1. Critical section: If the critical section for shear design in deep beams supporting top vertical loads is located at a distance X from the face of the support, then the distance X can be determined as follows (Fig. 8.23):
   a. For deep beams supporting uniformly distributed loads, $X = 0.15l_n$, where l_n = clear span.
   b. For concentrated loads, $X_1 = 0.5a_1$ (left support) or $X_2 = 0.5a_2$ (right support) (Fig. 8.23), where a_1 and a_2 equal the shear span near each support. The shear span is the distance between the concentrated load and the face of the support.
   In all cases, the distances X, X_1, and X_2 must not exceed the effective depth, d.

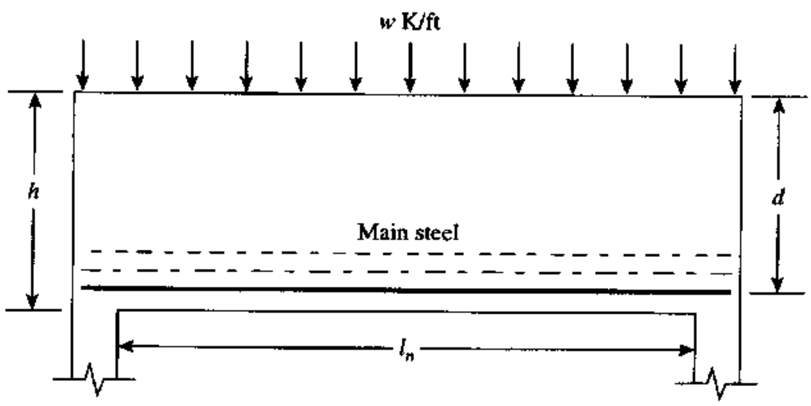


Figure 8.21 Single-span deep beam ($l_n/d < 5$).

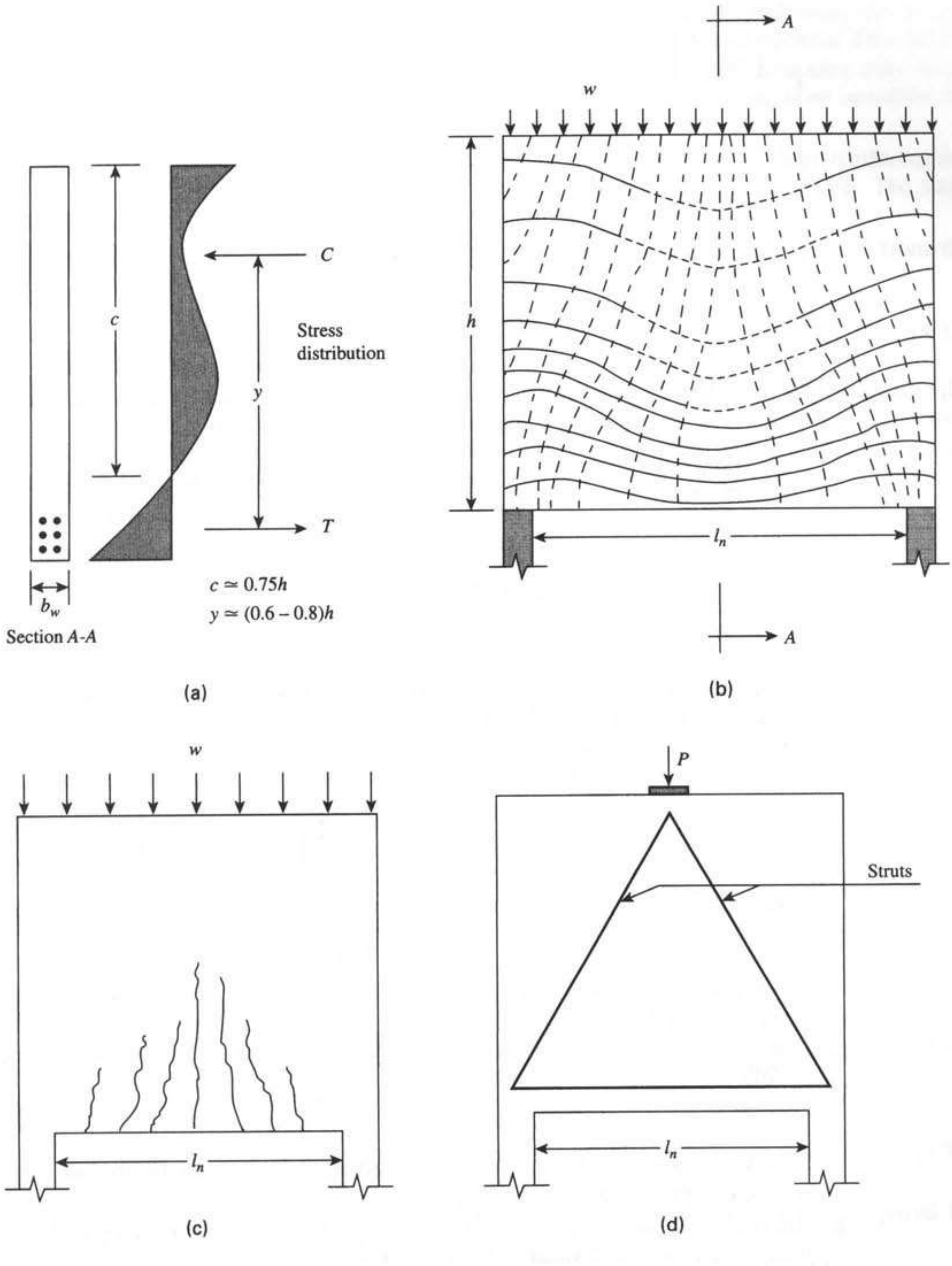


Figure 8.22 Stress distribution and cracking: (*a*) elastic stress distribution, (*b*) stress trajectories (tension, solid lines, and compression, dashed lines), (*c*) cracks pattern, and (*d*) truss model for a concentrated load applied at the wall upper surface.

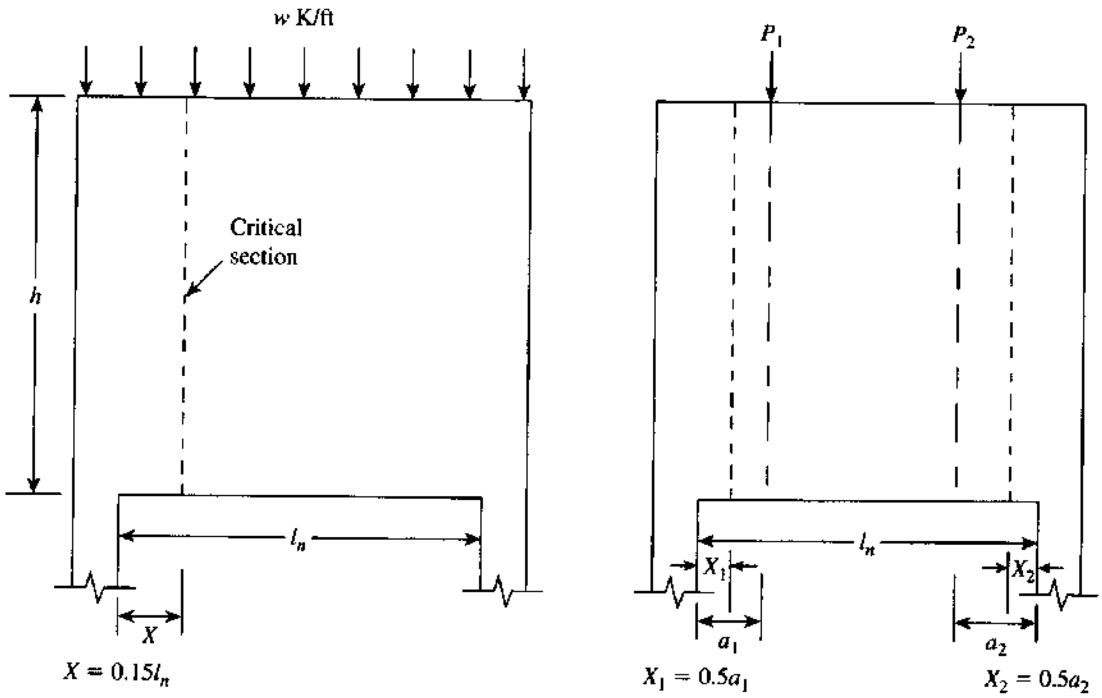


Figure 8.23 Critical sections for shear design.

2. Maximum shear strength ϕV_n: The maximum shear strength, ϕV_n, for deep flexural members shall not exceed the following values ($\phi = 0.75$):

$$\text{For } \frac{l_n}{d} < 2, \quad \phi V_n = \phi 8\sqrt{f_c'} b_w d \tag{8.32a}$$

$$\text{For } 2 \leq \frac{l_n}{d} \leq 5, \quad \phi V_n = \phi \frac{2}{3}\left(10 + \frac{l_n}{d}\right)\sqrt{f_c'} b_w d \tag{8.32b}$$

or let

$$\phi V_n = \phi 10\sqrt{f_c'} b_w d \tag{8.33}$$

for both cases, ACI Code, Section 11.7.3. If V_u exceeds ϕV_n, then the section dimensions must be increased.

3. a. Concrete shear strength, V_c: The nominal shear strength, V_c, of concrete can be estimated as follows:

$$V_c = 2\lambda\sqrt{f_c'} b_w d \tag{8.34}$$

This V_c is similar to the concrete shear strength for regular beams, as in the previous sections of this chapter.

b. Alternatively, another expression may be used that takes into account the effect of the factored moment and shear at the critical section:

$$V_c = \left(3.5 - \frac{2.5M_u}{V_u d}\right)\left(1.9\lambda\sqrt{f_c'} + \frac{2500\rho_w V_u d}{M_u}\right) b_w d \tag{8.35}$$

but V_c should not exceed $6\sqrt{f_c'} b_w d$.

The value of $(3.5 - 2.5\, M_u/V_u d)$ may not be greater than 2.5 and must not be less than 1.0. The values of M_u and V_u are taken at the critical design section. This higher shear strength of Eq. 8.35 is used with the idea that minor unsightly cracking may occur in the deep beam and can be tolerated. Cracks may start to develop at about one-third the factored load.

4. Shear reinforcement: When the factored shear force, V_u, exceeds ϕV_c, shear reinforcement must be provided, considering that $V_u = \phi(V_c + V_s)$, or $V_s = (V_u - \phi V_c)/\phi$. The steps are as follows:

   a. Determine V_s: The force resisted by shear reinforcement V_s is determined from the following expression:

$$V_s = \left[\frac{A_v}{S_v}\left(\frac{1 + l_n/d}{12}\right) + \frac{A_{vh}}{S_h}\left(\frac{11 - l_n/d}{12}\right)\right] f_y d \tag{8.36}$$

   where A_v = total area of vertical shear reinforcement spaced at S_v and perpendicular to the main flexural tensile reinforcement on both faces of the beam and A_{vh} = total area of horizontal shear reinforcement spaced at S_h parallel to the main flexural tensile reinforcement on both faces of the beam.

   b. Spacing of shear reinforcement is

$$\text{Maximum vertical spacings} \quad S_v \le \frac{d}{5} \le 12 \text{ in.}$$

$$\text{Maximum horizontal spacings} \quad S_h \le \frac{d}{5} \le 12 \text{ in.}$$

   c. Minimum shear reinforcement: The area of vertical shear reinforcement is $A_v = 0.0025 b_w S_v$. The area of horizontal shear reinforcement is $A_{vh} = 0.0015\, b_w S_h$.

   d. The shear reinforcement required at the critical section should be extended throughout the length and depth of the deep beam.

   e. For continuous deep beams, the same shear reinforcement may be used in all spans if the spans are almost equal with similar loading.

5. Flexural reinforcement of deep beams: The flexural behavior of deep beams is complex and requires nonlinear analysis of stresses and strains along the depth of the beam. For a preliminary design, the following simplified approach may be used:

$$\phi M_n = \phi A_s f_y y$$

   where y = moment arm = $(d - a/2)$. Because the value of $(d - a/2)$ is not easy to calculate, the moment arm y may be taken approximately equal to $0.6h$ for $l_n/h = 1.0$ and equal to $0.8h$ for $l_n/h = 2.0$. Linear interpolation may be used to estimate y when l_n/h varies between 1.0 and 2.0. Therefore,

$$A_s = \frac{M_u}{\phi y f_y} \tag{8.37}$$

   The value of A_s may not be less than the minimum flexural reinforcement required for regular beams given next, assuming $d = 0.9h$:

$$\text{Minimum } A_s = \left(\frac{3\sqrt{f_c'}}{f_y}\right) b_w d \ge \left(\frac{200}{f_y}\right) b_w d \tag{8.38}$$

   The second term controls when $f_c' < 4500$ psi. Note that, and f_y are in psi.

The flexural tension reinforcement shoul d be placed within $h/4$ to $h/5$ of the beam and should be adequately spaced along the bottom tension zone. Tension bars should be well anchored to the supports.

For more accurate analysis and design and for continuous deep beams, a rigorous non-linear approach should be used to determine the proper amount and distribution of the tension reinforcement.

Example 8.5

A simply supported deep beam has a span = 14 ft, a clear span of $l_n = 12$ ft, a total height of $h = 8$ ft, and width of $b = 16$ in. The deep beam supports a uniform service dead load of 41 K/ft (including self-weight) and a live load of 22 K/ft on top of the beam. Design the beam for moment and shear using $f'_c = 4$ ksi normal-weight concrete, and $f_y = 60$ ksi. Refer to Fig. 8.24.

Solution

1. Design for moment:

$$W_u = 1.2W_D + 1.6W_L = 1.2(41) + 1.6(22) = 84.4 \text{ K/ft}$$

$$M_u = \frac{W_u L^2}{8} = \frac{84.4(14)^2}{8} = 2067.8 \text{ K·ft}$$

$$\frac{l_n}{h} = \frac{12}{8} = 1.5$$

Determine the moment arm, y. For $l_n/h = 1.0$, $y = 0.6d$, and for $l_n/h = 2.0$, $y = 0.8d$; hence for $l_n/h = 1.5$, $y = 0.7d$ (by interpolation) $= 0.7(8 \times 12) = 67.2$ in.

$$A_s = \frac{M_u}{\phi y f_y} = \frac{2067.8 \times 12}{0.9(67.2)(60)} = 6.84 \text{ in.}^2$$

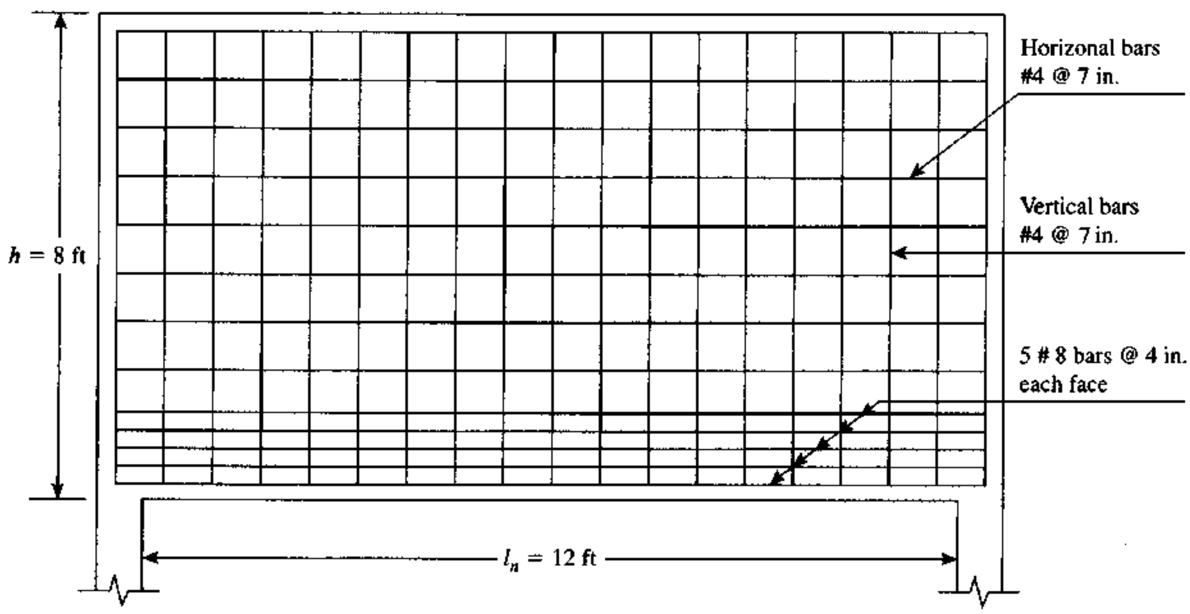


Figure 8.24 Example 8.5.

Assume $d = 0.9h = 0.9(8 \times 12) = 86.4$ in. Because $f_c' < 4500$ psi,

$$A_s \text{ (minimum)} = \left(\frac{200}{f_y}\right) b_w d = \frac{200(16)(86.4)}{60{,}000} = 4.6 \text{ in.}^2$$

Therefore, $A_s = 6.84$ in.^{2} controls. Choose 10 no. 8 bars (7.85 in.^{2}), five on each face, distributed within $h/5 = 8(12)/5 = 19.2$ in. of the tension zone of the beam. Spacing of bars = 19.2/5 = 3.84 in., or 4 in. Bars should be well anchored into the supports.

2. Design for shear:

a. Calculate V_u and M_u at the distance $x = 0.15l_n = d$ from the face of the support.

$$0.15l_n = 0.15(12 \times 12) = 21.6 \text{ in.} = 1.8 \text{ ft} < d = 86.4 \text{ in.}$$

$$\text{Design } V_u = 84.4(12/2) - 84.4(1.8) = 354.5 \text{ K}$$

$$M_u = 84.4(6)(1.8) - \frac{84.4(1.8)^2}{2} = 774.8 \text{ K·ft}$$

$$\frac{M_u}{V_u d} = \frac{774.8(12)}{354.5(86.4)} = 0.304$$

b. Calculate V_c:

$$3.5 - 2.5\frac{M_u}{V_u d} = 3.5 - 2.5(0.304) = 2.74 > 2.5$$

So, use 2.5 In this case, determine $M_u/V_u d$ to be used to calculate V_c: $2.5 = 3.5 - 2.5\, M_u/(V_u d)$, and $M_u/(V_u d) = 0.4$.

$$\frac{V_u d}{M_u} = \frac{1}{0.4} = 2.5$$

$$\rho_w = \frac{A_s}{b_w d} = \frac{7.85}{16 \times 86.4} = 0.00496$$

$$V_c = 2.5\left(1.9\lambda\sqrt{f_c'} + \frac{2500\rho_w V_u d}{M_u}\right) b_w d$$

$$= 2.5[(1.9)(1)\sqrt{4000} + (2500)(0.00496)(2.5)](16)(86.4)$$

$$= 522.4 \text{ K}$$

$$V_c \leq 6\sqrt{f_c'} b_w d = 6\sqrt{4000}(16)(86.4) = 524.6 \text{ K}$$

Hence, $V_c = 522.4$ controls and $\phi V_c = 392$ K.

c. Calculate $V_s = (V_u - \phi V_c)/\phi$. Because $\phi V_c = 392 \text{ K} > V_u = 354.5$ K, then $V_s = 0$, and only minimum shear reinforcement is required.

d. Calculate shear reinforcement: Assume no. 4 bars placed on both faces in the horizontal and vertical directions; then $A_v = A_{vh} = 2(0.2) = 0.4$ in.^{2} Maximum allowable spacing of vertical bars is $S_v = d/5 = 18$ in. $S_v = 86.4/5 = 17.3$ in. > 12 in. use $S_u = 12$ in. Maximum allowable spacing of horizontal bars is $S_h = d/5 = 18$ in. $S_h = 86.4/5 = 17.3$ in > 12 in.; use $S_h = 12$ in. Minimum A_v (vertical) $= 0.0025 b_w S_v = 0.0025(16)(72) = 0.48$ in.^{2} > 0.4 in.^{2} Minimum A_{vh} (horizontal) $= 0.0015 b_w S_h = 0.0015(16)(12) = 0.288$ in.^{2} < 0.4 in.^{2} Reduce spacing to $S_v = 0.4/(0.0025 \times 16) = 10$ in. Therefore, use no. 4 vertical bars spaced at 10 in., and use no. 4 horizontal bars spaced at 12 in.

3. If $V_c = 2\lambda\sqrt{f_c'} b_w d$ is used, then $V_c = (2)(1)\sqrt{4{,}000}(16)(86.4) = 174.9$ K and $\phi V_c = 0.75 V_c$ = 131 K, which is less than $V_u = 354.5$ K. Hence, shear reinforcement is required.

$$V_s = \frac{V_u - \phi V_c}{\phi} = \frac{354.1 - 131}{0.75} = 297.5 \text{ K}$$

Assuming no. 4 bars placed on both faces in the vertical and horizontal directions, then $A_v = A_{vh} = 2(0.2) = 0.4\,\text{in.}^2$ Assuming that the spacings of bars in both directions are equal, $S_v = S_h = S$, and $l_n/d = 12 \times 12/86.4 = 1.67$, then

$$V_s = \left[\frac{A_v}{S_v}\left(\frac{1+l_n/d}{12}\right) + \frac{A_{\text{vh}}}{S_h}\left(\frac{11-l_n/d}{12}\right)\right] f_y d \tag{8.36}$$

$$297.5 = \left[\frac{0.4}{S}\left(\frac{1+1.67}{12}\right) + \frac{0.4}{S}\left(\frac{11-1.67}{12}\right)\right](60)(86.4)$$

$S = 7\,\text{in.}$, which is less than the maximums $S_v = 17.3\,\text{in.}$ and $S_h = 18\,\text{in.}$ Use $S = 7\,\text{in.}$ for both vertical and horizontal spacing.

$$\text{Minimum } A_v \text{ (vertical)} = 0.0025(16)(7) = 0.28\ \text{in.}^2 < 0.4\ \text{in.}^2$$

$$\text{Minimum } A_{\text{vh}} \text{ (horizontal)} = 0.0015(16)(7) = 0.168\ \text{in.}^2 < 0.4\ \text{in.}^2$$

Then use no. 4 bars spaced at 7 in. on both faces in the horizontal and vertical directions. A welded wire fabric mesh may be adopted to replace the preceding bar arrangements. It can be seen that this solution is more conservative than that given in step 2. Reinforcement details are shown in Fig. 8.24 on page 281.

Example 8.6: Strut and Tie Deep Beam

A simply supported deep beam has a clear span = 12 ft, a total height = 6 ft, and a width = 18 in. The beam supports an 18-in. square column at midspan carrying a dead load = 300 K, and a live load = 240 K. Design the beam using the strut and tie model, using $f'_c = 4$ ksi and $f_y = 60$ ksi. (Refer to Fig. 8.25).

Solution

1. Calculate the factored loads (Fig. 8.25):

$$\text{Weight of the beam} = 15 \times 6 \times 1.5 \times 0.150 = 20\text{K}$$

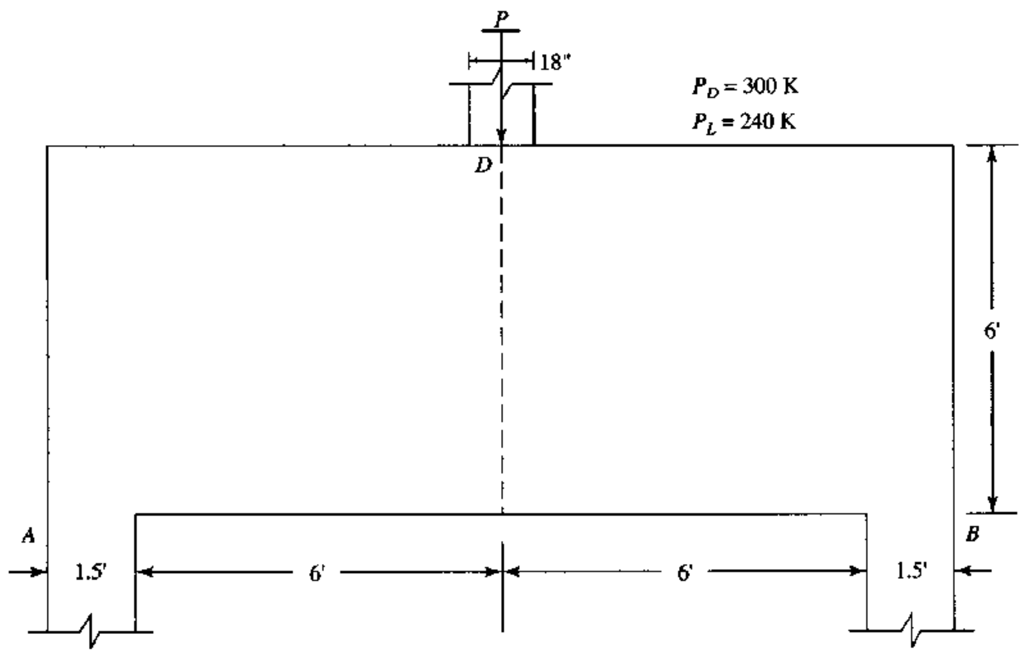


Figure 8.25 Example 8.6

Since the weight of the beam is small relative to the concentrated loads at midspan, add it to the concentrated load at midspan.

$$P_u = 1.2D + 1.6L = 1.2(300 + 20) + 1.6(240) = 768 \text{ K}$$

$$R_A = R_B = 768/2 = 384 \text{ K}$$

2. Check if the beam is deep according to to the ACI Code, Section 11.8: Clear span, $l_n = 12$ ft, $h = 6$ ft, and $l_n/h = 2 < 4$, a deep beam.
3. Calculate the maximum shear strength of the beam cross-section: Let V_u at $A = R_A = 384$ K, and assume $d = 0.9h = 0.9 \times 72 = 64$ in.

$$V_n = 10\sqrt{f_c'}b_w d = 10 \times \sqrt{4000}(18 \times 64) = 728.6 \text{ K}$$

$$\phi V_n = 0.75(728.6) = 546 \text{ K} > V_u \qquad \text{(o.k.)}$$

4. Select a truss model.
A triangular truss model is chosen. Assume that the nodes act at the centerline of the supports and at 6.0 in from the lower or upper edge of the beam (Fig. 8.26). The strut and tie model consists of a tie AB and two struts AD and BD. Also, the reactions at A and B and the load P_u at D represent vertical struts.

$$\text{Length of the diagonal strut } AD = \sqrt{(60)^2 + (80)^2} = 100.8 \text{ in.}$$

Let the angle between the strut and the tie $= \theta$, $\tan\theta = 60/81 = 0.7407$, and $\theta = 36.5$ degrees >26 degrees, which is o.k.

5. Calculate the forces in the truss members: The compression force in strut $AD = F_{AD} = F_{BD}$ $= 384\ (100.8/60) = 645$ K. The tension force in the tie $AB = F_{AB} = 384\ (100.8/81) = 478$ K.
6. Calculate the effective strength, f_{cu}. Assume that confining reinforcement is provided to resist the splitting forces. Struts AD and BD represent the bottle-shape compression members, and therefore, $\beta_s = 0.75$.

$$f_{ce} = 0.85\beta_s f_c' = 0.85 \times 0.75 \times 4 = 2.55 \text{ ksi}$$

The vertical struts at A, B, and D have uniform sections, and therefore $\beta_s = 1.0$.

$$f_{ce} = 0.85 \times 1.0 \times 4 = 3.4 \text{ ksi}$$

The nodal zone D has a C-C-C force and therefore, $\beta_s = 1.0$. The effective strength at nodal zone D is:

$$f_{ce} = 0.85 \times 1.0 \times 4 = 3.4 \text{ ksi}$$

Since the struts AD and BD connect to the other nodes, $f_{ce} = 2.55$ ksi controls to all nodal zones.

7. Design of nodal zones:
   a. Design of nodal zone at A : Assume that the faces of the nodal zone have the same stress of 2.55 ksi and the faces are perpendicular to their respective forces.

   $$\phi F_n \geq F_u \quad \text{or} \quad \phi f_{ce} A_{cs} \geq F_u$$

   where $\phi = 0.75$ for struts, ties, and nodes. The length of the horizontal face ab (Fig. 8.27a) is equal to $F_u/(\phi f_{ce} b) = 384/(0.75 \times 2.55 \times 18) = 11.2$ in. From geometry, the length $ac =$ 11.2 (478/384) = 13.94 in., say 14 in. Similarly, the length $bc = 11.2\ (645/384) = 18.8$ in. The center of the nodal zone is located at $14/2 = 7$ in. from the bottom of the beam, which is close to 6.0 in., assumed earlier.
   b. Design of nodal zone at D (Fig. 8.27b): The length of the horizontal face $de = 768/(0.75 \times 2.55 \times 18) = 22.3$ in. The length of $df = ef = 22.3(645/768) = 18.7$ in. The length of $fg = 15.0$ in., and the center of the nodal zone is located at $15/3 = 5.0$ in from the top, which is close to the assumed 6.0 in.

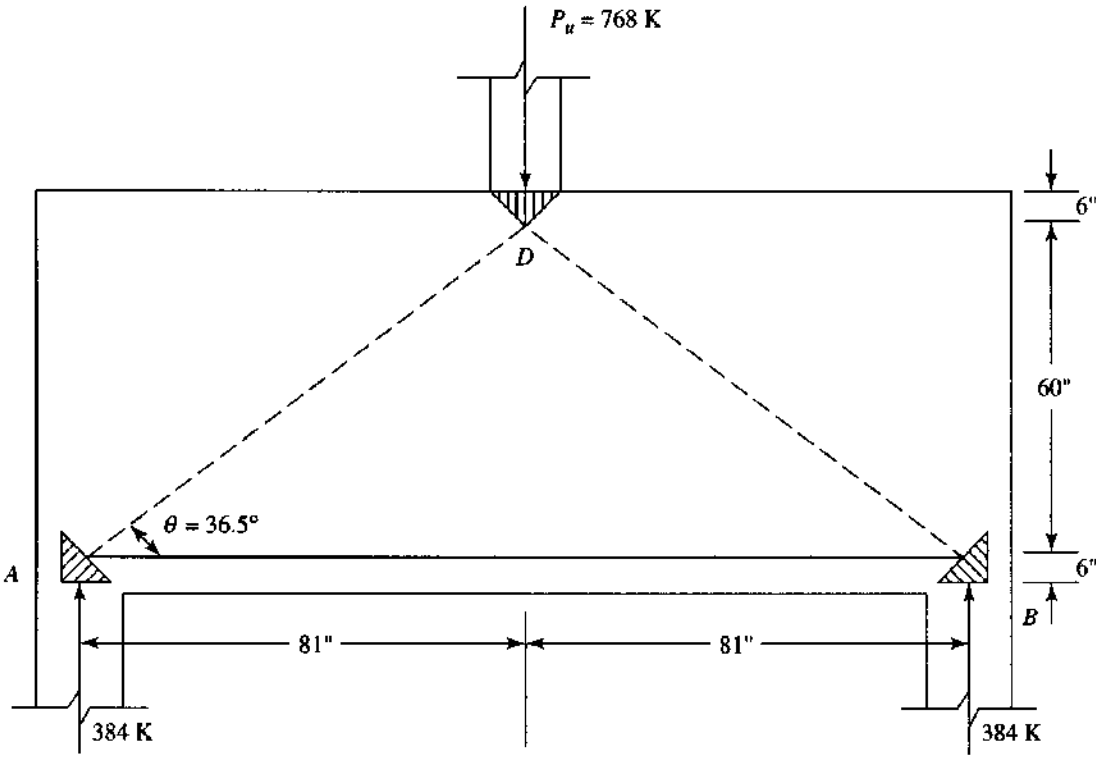


Figure 8.26 Example 8.6: idealized beam.

8. Design of vertical and horizontal reinforcement:

a. Vertical bars: The angle between the vertical bars and strut = 53.5°, from Fig. 8.27*a*. Use No. 5 bars spaced at 12 in., two branches, $A_s = 2\ (0.31) = 0.62$ in₂. Sin 53.5° = 0.804.

$$(A_{si}/b_s s)\sin\gamma_i = (0.62/18 \times 12)(0.804) = 0.0023$$

b. Horizontal bars: The angle between the horizontal bars and strut = 36.5°, (Fig. 8.27*a*). Use No. 5 bars spaced at 12 in., two branches, $A_s = 0.62\,\text{in.}^2$. Sin 36.5° = 0.595.

$$(A_{si}/b_s s)\sin\gamma_i = (0.62/18 \times 12)(0.595) = 0.0017$$

c. Total $(A_{si}/b_s s)\sin\gamma_i = 0.0023 + 0.0017 = 0.004 > 0.003$, which is o.k.

9. Design of the horizontal tie AB :

a. Calculate A_s:

$$F_u = \phi A_s f_y \quad A_s = 478/(0.75 \times 60) = 10.6\ \text{in.}^2$$

Use 12 no. 9 bars, $A_s = 12\,\text{in.}^2$ in three rows as shown in Fig. 8.27*c*.

b. Calculate anchorage length: Anchorage length is measured from the point beyond the extended nodal zone, Fig. 8.28. Tan 36.5 = 7/*x*. Then $x = 9.5$ in. Available anchorage length = 9.5 + 5.6 + 9 − 1.5 in. (cover) = 22.6 in. Development length of no. 9 bars required = 47.5 in. (Table 7.1), which is greater than 22.6 in. Use a standard 90° hook enclosed within the column reinforcement.

$$l_{dh} = 0.02\psi_e f_y d_b/\lambda\sqrt{f_c'} \tag{7.15}$$

$$\psi_e = \lambda = 1.0 \qquad d_b = 1.128\ \text{in.}$$

$$l_{dh} = 0.02(1)(60{,}000)(1.128)/\big((1)(\sqrt{4000})\big) = 21.4\ \text{in} < 22.6\ \text{in.}$$

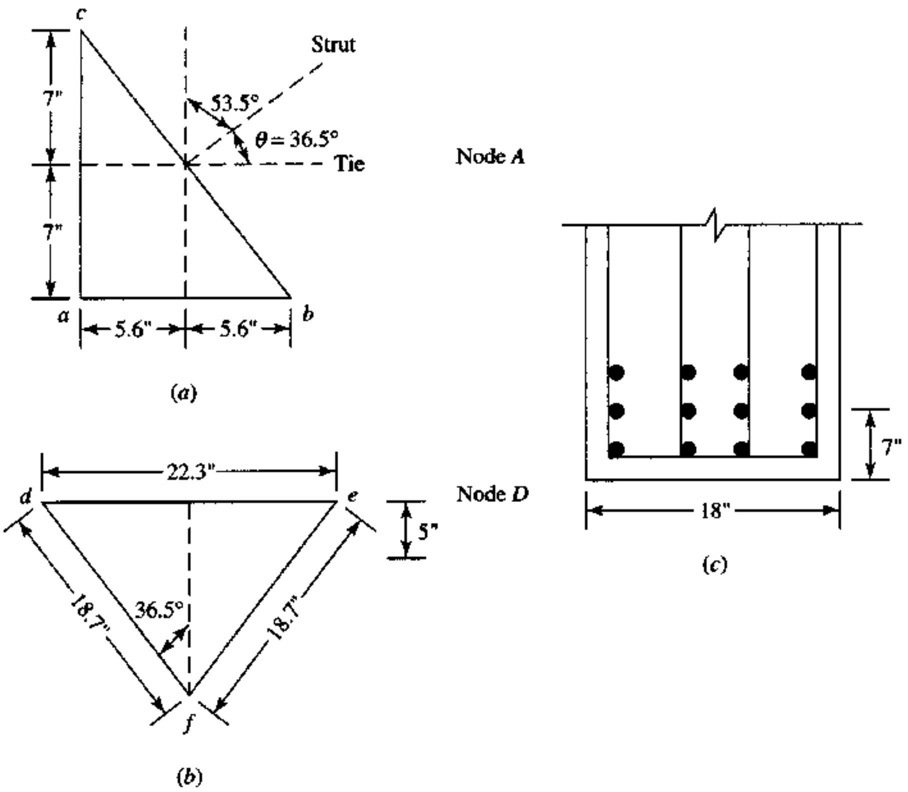


Figure 8.27 Example 8.6: nodal zones, (*a*) at node *A*, (*b*) at node *D*, and (*c*) reinforcement details.

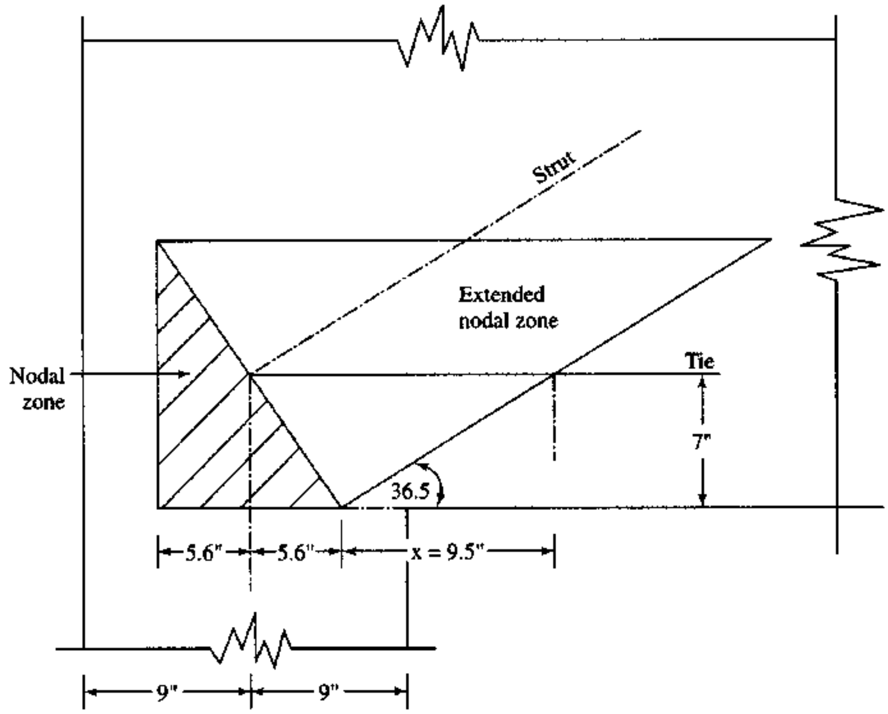


Figure 8.28 Example 8.6: development of tie reinforcement.

Table 8.4 Shear Reinforcement Formulas

U.S. Customary Units	SI Units
V_u = design shear	V_u = design shear
(Maximum design V_u is at a distance d from the face of the support.)	
$V_c = (2.0\lambda\sqrt{f'_c})\, b_w d$	$V_c = (0.17\lambda\sqrt{f'_c})\, b_w d$
$V_c = \left[1.9\lambda\sqrt{f'_c} + \left(2500\rho_w \frac{V_u d}{M_u}\right)\right] b_w d$	$V_c = \left[0.16\lambda\sqrt{f'_c} + \left(17.2\rho_w \frac{V_u d}{M_u}\right)\right] b_w d$
$\rho_w = \frac{A_s}{b_w d} \quad \frac{V_u d}{M_u} \le 1.0$	$\rho_w = \frac{A_s}{b_w d} \quad \frac{V_u d}{M_u} \le 1.0$
$V_c \le (3.5\lambda\sqrt{f'_c})\, b_w d$	$V_c \le (0.29\lambda\sqrt{f'_c})\, b_w d$
$V_u = \phi V_c + \phi V_s$	$V_u = \phi V_c + \phi V_s$
Vertical stirrups	
$\phi V_s = V_u - \phi V_c$	$\phi V_s = V_u - \phi V_c$
$S = \frac{A_v f_{yt} d}{V_s}$	$S = \frac{A_v f_{yt} d}{V_s}$
Minimum $A_v = \frac{50 b_w S}{f_{yt}} \le 0.75\sqrt{f'_c}\left(\frac{b_w S}{f_{yt}}\right)$	Minimum $A_v = \frac{0.35 b_w S}{f_{yt}} \le 0.0062\sqrt{f'_c}\left(\frac{b_w S}{f_y}\right)$
Maximum $S = \frac{A_v f_{yt}}{50 b_w} \ge \frac{A_v f_{yt}}{0.75\sqrt{f'_c} b_w}$	Maximum $S = \frac{A_v f_{yt}}{0.35 b_w} \ge \frac{A_v f_y}{0.062\sqrt{f'_c} b_w}$
For vertical web reinforcement	
Maximum $S = \frac{d}{2} \le$ 24in.	Maximum $S = \frac{d}{2} \le$ 600 mm
if $V_s \le 4.0\sqrt{f'_c}(b_w d)$	if $V_s \le 0.33\sqrt{f'_c}(b_w d)$
Maximum $S = d/4 = 12$ in.	Maximum $S = d/4 = 300$ mm
if $V_s > 4.0\sqrt{f'_c}(b_w d)$	if $V_s > 0.33\sqrt{f'_c}(b_w d)$
$V_s \le 8\sqrt{f'_c}(b_w d)$	$V_s \le 0.67\sqrt{f'_c}(b_w d)$
Otherwise increase the dimensions of the section.	
Series of bent bars or inclined stirrups	
$A_v = \frac{V_s S}{f_{yt}\, d(\sin\alpha + \cos\alpha)}$	$A_v = \frac{V_s S}{f_{yt}\, d(\sin\alpha + \cos\alpha)}$
For $\alpha = 45°$, $S = \frac{1.4 A_v f_y d}{V_s}$	For $\alpha = 45°$, $S = \frac{1.4 A_v f_y d}{V_s}$
For a single bent bar or group of bars, parallel and bent in one position	
$A_v = \frac{V_s}{f_{yt}\sin\alpha}$	$A_v = \frac{V_s}{f_{yt}\sin\alpha}$
For $\alpha = 45°$, $A_y = \frac{1.4 V_s}{f_{yt}}$	For $\alpha = 45°$, $A_v = \frac{1.4 V_s}{f_{yt}}$
$V_s \le (3\sqrt{f'_c})\, b_w d$	$V_s \le (0.25\sqrt{f'_c})\, b_w d$

8.12 EXAMPLES USING SI UNITS

The general design requirements for shear reinforcement according to the ACI Code are summarized in Table 8.4, which gives the necessary design equations in both U.S. customary and SI units. The following example shows the design of shear reinforcement using SI units.

Example 8.7

A 6-m clear span simply supported beam carries a uniform dead load of 47.5 kN/m and a live load of 25 kN/m (Fig. 8.29). The dimensions of the beam section are $b = 350$ mm, $d = 550$ mm. The beam is reinforced with four bars of 25-mm diameter in one row. It is required to design the necessary shear reinforcement. Given: $f_c' = 28$ MPa and $f_y = 280$ MPa.

Solution

1. Factored load is

$$1.2D + 1.6L = 1.2 \times 47.5 + 1.6 \times 25 = 97 \text{ kN/m}$$

2. Factored shear force at the face of the support is

$$V_u = 97 \times \frac{6}{2} = 291 \text{ kN}$$

3. Maximum design shear at a distance d from the face of the support is

$$V_u \text{ (at distance} d) = 291 - 0.55 \times 97 = 237.65 \text{ kN}$$

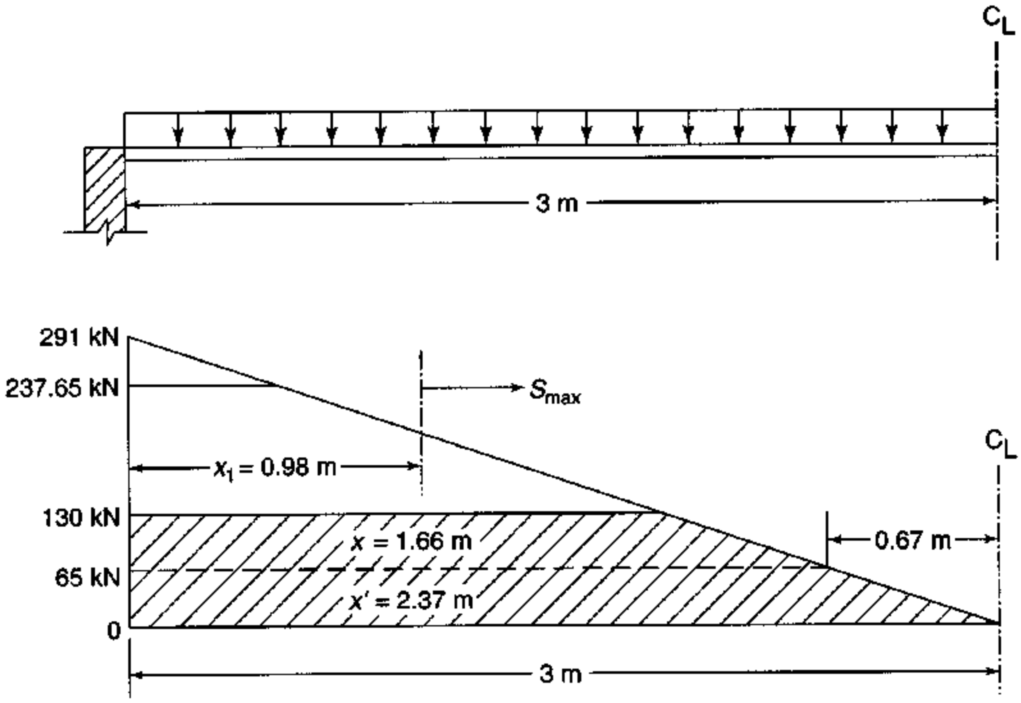


Figure 8.29 Example 8.7.

4. The nominal shear strength provided by concrete is

$$V_c = (0.17\lambda\sqrt{f'_c})bd = (0.17\sqrt{28}) \times 350 \times 550 = 173.2 \text{ kN}$$

$$V_u = \phi V_c + \phi V_s$$

$$\phi V_c = 0.75 \times 173.2 = 130 \text{ kN}$$

$$\frac{1}{2}\phi V_c = 65 \text{ kN}$$

$$\phi V_s = 237.65 - 130 = 107.65 \text{ kN}$$

$$V_s = \frac{107.65}{0.75} = 143.5 \text{ kN}$$

5. Distance from the face of the support at which $\frac{1}{2}\phi V_c = 65$ kN is

$$x' = \frac{(291 - 65)}{291}(3) = 2.33 \text{ m} \quad \text{(from triangles)}$$

6. Design of stirrups:
   a. Choose stirrups 10 mm in diameter with two branches ($A_s = 78.5\,\text{mm}^2$).

$$A_v = 2 \times 78.5 = 157\text{mm}^2$$

$$\text{Spacing } S_1 = \frac{A_y f_{yt} d}{V_s} = \frac{157 \times 280 \times 550}{143.5 \times 10^3} = 168.5 \text{ mm} < 600 \text{ mm}$$

Thus, use 160 mm. Check maximum spacing of stirrups:

$$\text{Maximum } S_2 = \frac{d}{2} = \frac{550}{2} = 275 \text{ mm}$$

$$S_3 = \frac{A_v f_{yt}}{0.35b} = \frac{157 \times 280}{0.35 \times 350} = 359 \text{ mm}$$

$S = S_1 = 160$ mm controls.

   b. Check for maximum spacing of $d/4$:

$$\text{If } V_s \le (0.33\sqrt{f'_c})bd, \quad S_{\max} = \frac{d}{2}.$$

$$\text{If } V_s > (0.33\sqrt{f'_c})bd, \quad S_{\max} = \frac{d}{4}.$$

$$bd(0.33\sqrt{f'_c}) = 0.33\sqrt{28} \times 350 \times 550 = 336.1\text{kN}$$

Actual $V_s = 143.5$ kN < 336.1 kN. Therefore, $S_{\max}$ is limited to $d/2 = 275$ mm.

7. The shear reinforcement, stirrups 10 mm in diameter and spaced at 160 mm, will be needed only for a distance $d = 0.55$ m from the face of the support. Beyond that, the shear stress V_s decreases to 0 at a distance $x = 1.66$ m when $\phi V_c = 130$ kN. It is not practical to provide stirrups at many different spacings. One simplification is to find out the distance from the face of support where maximum spacing can be used, and then only two different spacings may be adopted.

$$\text{Maximum spacing} = \frac{d}{2} = 275 \text{ mm}$$

$$V_s \text{ (for } s_{\max} = 275 \text{ mm}) = \frac{A_v f_{yt} d}{S} = \frac{157 \times 0.280 \times 550}{275} = 87.9 \text{ kN}$$

$$\phi V_s = 87.9 \times 0.75 = 65.94 \text{ kN}$$

The distance from the face of the support where $S_{\max} = 275$ mm can be used (from the triangles):

$$x_1 = \frac{291 - (130 + 65.94)}{291}(3) = 0.98 \text{ m}$$

Then, for 0.98 m from the face of support, use stirrups of 10-mm diameter at 160 mm, and for the rest of the beam, minimum stirrups (with maximum spacings) can be used.

8. Distribution of stirrups:

$$\text{one stirrup at } \frac{S}{2} = \frac{160}{2} = 80 \text{ mm}$$

$$\text{six stirrups at } 160 \text{ mm} = \underline{960 \text{ mm}}$$

$$\text{Total} = 1040 \text{ mm} = 1.04 \text{ m} > 0.98 \text{ m}$$

$$\text{six stirrups at } 270 \text{ mm} = \underline{1620 \text{ mm}}$$

$$\text{Total} = 2660 \text{ mm} = 2.66 \text{ m} < 3 \text{ m}$$

The last stirrup is (3 − 2.66) = 0.34 m = 340 mm from the centerline of the beam, which is adequate. A similar stirrup distribution applies to the other half of the beam, giving a total number of stirrups of 28.

The other examples in this chapter can be worked out in a similar way using SI equations.

SUMMARY

Sections 8.1–8.2

The shear stress in a homogeneous beam is $v = VQ/Ib$. The distribution of the shear stress above the neutral axis in a singly reinforced concrete beam is parabolic. Below the neutral axis, the maximum shear stress is maintained down to the level of the steel bars.

Section 8.3

The development of shear resistance in reinforced concrete members occurs by

- Shear resistance of the uncracked concrete
- Interface shear transfer

- Arch action
- Dowel action

Section 8.4

The shear stress at which a diagonal crack is expected is

$$v_c = \frac{V}{bd} = \left(1.9\lambda\sqrt{f_c'} + 2500\rho_w \frac{V_u d}{M_u}\right) \leq 3.5\sqrt{f_c'}$$

The nominal shear strength is

$$V_c = v_c b_w d = 2\lambda\sqrt{f_c'} b_w d$$

Sections 8.5–8.6

1. The common types of shear reinforcement are stirrups (perpendicular or inclined to the main bars), bent bars, or combinations of stirrups and bent bars.

$$V_u = \phi V_n = \phi V_c + \phi V_s \quad \text{and} \quad V_s = \frac{1}{\phi}(V_u - \phi V_c)$$

2. The ACI Code design requirements are summarized in Table 8.4.

Sections 8.7–8.8

Design of vertical stirrups and shear summary is given in these sections.

Sections 8.9–8.10

1. Variation of shear force along the span due to live load may be considered.
2. For members with variable depth,

$$\phi V_n = V_u \pm \frac{M_u(\tan\alpha)}{d} \tag{8.29}$$

Section 8.11

For deep beams, the shear capacity, V_c, may be determined from the following expressions:

$$V_c = 2\lambda\sqrt{f_c'} b_w d \tag{8.35}$$

or

$$V_c = \left(3.5 - \frac{2.5M_u}{V_u d}\right)\left(1.9\lambda\sqrt{f_c'} + \frac{2500\rho_w V_u d}{M_u}\right) b_w d \tag{8.36}$$

The critical section for shear design is at $X = 0.15l_n$ for uniform loads and $X = 0.5a$ for concentrated loads.

Also, refer to Section 5.7 in text.

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PROBLEMS

8.1 Design the necessary shear reinforcement (if needed) in the form of U-stirrups (two legs) for the T-section shown in Fig. 8.30. Use $f'_c = 4$ ksi (28 MPa) and $f_y = 60$ ksi (420 MPa).

a. $V_u = 22$ K (98 kN)

b. $V_u = 56$ K (246 kN)

c. $V_u = 69$ K (306 kN)

8.2 Repeat Problem 8.1 for the section shown in Fig. 8.31.

8.3 Design the necessary shear reinforcement (if needed) in the form of U-stirrups (two legs) for the rectangular section shown in Fig. 8.32 using $f'_c = 3$ ksi (21 MPa) and $f_{yt} = 60$ ksi (420 MPa).

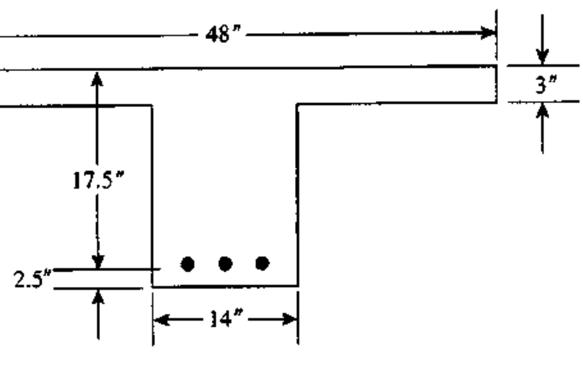


Figure 8.30 Problem 8.1.

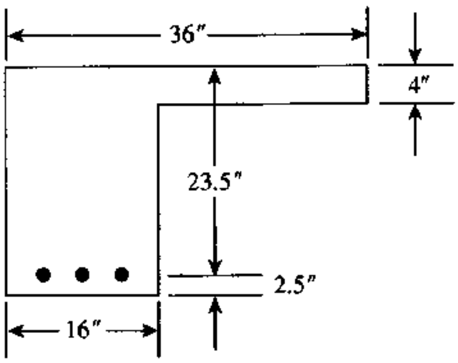


Figure 8.31 Problem 8.2.

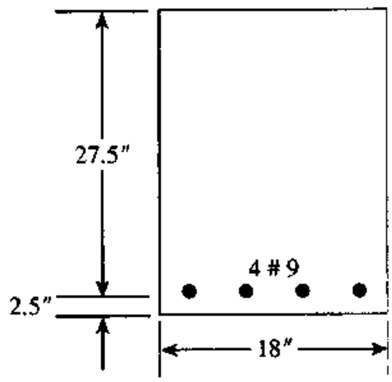


Figure 8.32 Problem 8.3.

a. V_u = 55 K (245 kN)
b. V_u = 110 K (490 kN)
c. V_u = 144 K (640 kN)

8.4 A 16-ft- (4.8-m-)span simply supported beam, Fig. 8.33; has a clear span of 15 ft (4.5 m) and is supported by 12 × 12-in. (300 × 300-mm) columns. The beam carries a factored uniform load of 11.1 K/ft (166 kN/m). The dimensions of the beam section and the flexural steel reinforcement are shown in Fig. 8.33. Design the necessary shear reinforcements using f'_c = 3ksi (21 MPa) and f_{yt} = 60 ksi (420 MPa). Show the distribution of stirrups along the beam.

8.5 An 18-ft- (5.4-m-)span simply supported beam carries a uniform dead load of 4 K/ft (60 kN/m) and a live load of 1.5 K/ft (22 kN/m). The beam has a width of b = 12 in. (300 mm) and a depth of d = 24 in. (600 mm) and is reinforced with six no. 9 bars (6 × 28 mm) in two rows. Check the beam for shear and design the necessary shear reinforcement. Given: f'_c = 3 ksi (21 MPa) and f_{yt} = 50 ksi (280 MPa).

8.6 Design the necessary shear reinforcement for a 14-ft (4.2-m) simply supported beam that carries a factored uniform load of 10 K/ft (150 kN/m) (including self-weight) and a factored concentrated load at midspan of P_u = 24 K (108 kN). The beam has a width of b = 14 in. (350 mm) and a depth of d = 16.5 (400 mm) and is reinforced with four no. 8 bars (4 × 25 mm). Given: f'_c = 4 ksi (28 MPa) and f_{yt} = 60 ksi (420 MPa).

8.7 A cantilever beam with 7.4-ft (2.20-m) span carries a uniform dead load of 2.5 K/ft (36 kN/m) (including self-weight) and a concentrated live load of 18 K (80 kN) at a distance of 3 ft (0.9 m) from the face of the support. Design the beam for moment and shear. Given: f'_c = 3 ksi (21 MPa), f_{yt} = 60 ksi (420 MPa), and b = 12 in. (200 mm), and use $\rho = 3/4\rho_{max}$.

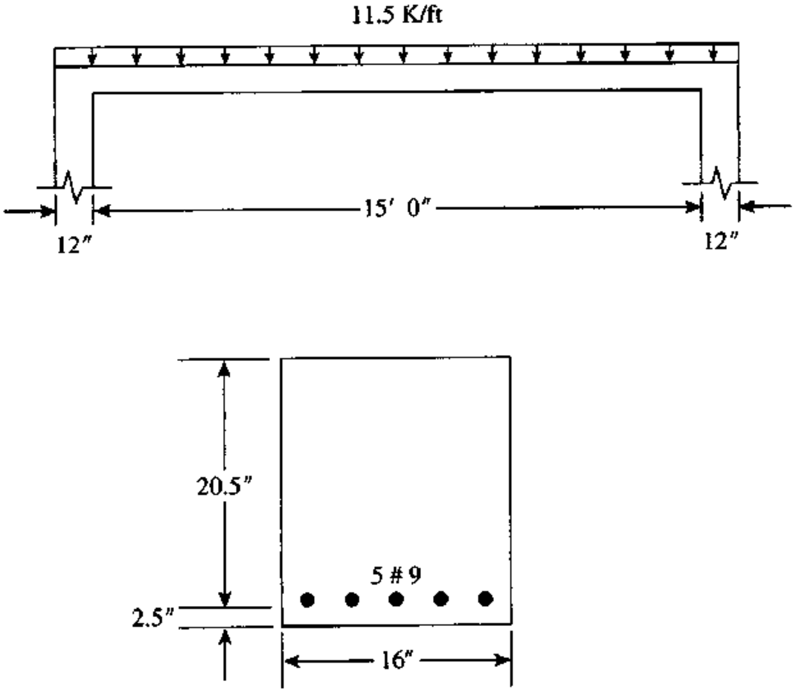


Figure 8.33 Problem 8.4.

8.8 Design the critical sections of an 11-ft-(3.3-m-)span simply supported beam for bending moment and shearing forces using $\rho = 0.016$. Given: $f_c' = 3$ ksi (21 MPa), $f_{yt} = 60$ ksi (420 MPa), and $b = 10$ in. (250 mm). Dead load is 2.75 K/ft (40 kN/m) and live load is 1.375 K/ft (20 kN/m).

8.9 A rectangular beam is to be designed to carry a factored shearing force of 75 kips (335 kN). Determine the minimum beam section if controlled by shear ($V_c = 2\lambda\sqrt{f_c'}bd$) using the minimum shear reinforcement as specified by the ACI Code and no. 3 stirrups. Given: $f_c' = 4$ ksi (28 MPa), $f_{yt} =$ 40 ksi (280 MPa), and $b = 16$ in. (400 mm).

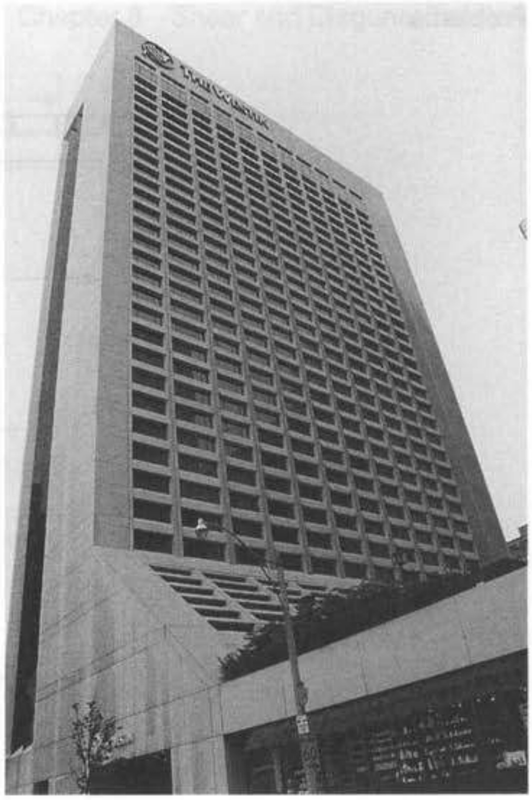
8.10 Redesign Problem 8.5 using $f_{yt} = 60$ ksi.

8.11 Redesign the shear reinforcement of the beam in Problem 8.6 if the uniform factored load of 6 K/ft (90 kN/m) is due to dead load and the concentrated load $P_u = 24$ k (108 kN) is due to a moving live load. Change the position of the live load to cause maximum shear at the support and at midspan.

8.12 Design a cantilever beam that has a span of 9 ft (2.7 m) to carry a factored triangular load that varies from 0 load at the free end to maximum load of 8 K/ft (120 kN/m) at the face of the support. The beam shall have a variable depth, with minimum depth at the free end of 10 in. (250 mm) and increasing linearly toward the support. Use steel percentage $\rho = 0.016$ for flexural design. Given: $f_c' = 4$ ksi (28 MPa), $f_{yt} = 60$ ksi (420 MPa) or flexural reinforcement, $f_{yt} = 40$ ksi (280 MPa) for stirrups, and $b = 11$ in. (275 mm).

CHAPTER 9

ONE-WAY SLABS



The Westin Hotel, Toronto, Canada.

9.1 TYPES OF SLABS

Structural concrete slabs are constructed to provide flat surfaces, usually horizontal, in building floors, roofs, bridges, and other types of structures. The slab may be supported by walls, by reinforced concrete beams usually cast monolithically with the slab, by structural steel beams, by columns, or by the ground. The depth of a slab is usually very small compared to its span. See Fig. 9.1.

Structural concrete slabs in buildings may be classified as follows:

1. *One-way slabs:* If a slab is supported on two opposite sides only, it will bend or deflect in a direction perpendicular to the supported edges. The structural action is one way, and the loads are carried by the slab in the deflected short direction. This type of slab is called a *one-way slab* (Fig. 9.1*a*). If the slab is supported on four sides and the ratio of the long side to the short side is equal to or greater than 2, most of the load (about 95% or more) is carried in the short direction, and one-way action is considered for all practical purposes (Fig. 9.1*b*). If the slab is made of reinforced concrete with no voids, then it is called a *one-way solid slab*. Fig. 9.1*c*, *d*, and *e* shows cross-sections and bar distribution.
2. *One-way joist floor system:* This type of slab is also called a *ribbed slab*. It consists of a floor slab, usually 2 to 4 in. (50 to 100 mm) thick, supported by reinforced concrete ribs (or joists). The ribs are usually tapered and are uniformly spaced at distances that do not exceed 30 in. (750 mm). The ribs are supported on girders that rest on columns. The spaces between the ribs may be formed using removable steel or fiberglass form fillers (pans), which may be used many times (Fig. 9.2). In some ribbed slabs, the spaces between ribs may be filled with permanent fillers to provide a horizontal slab.

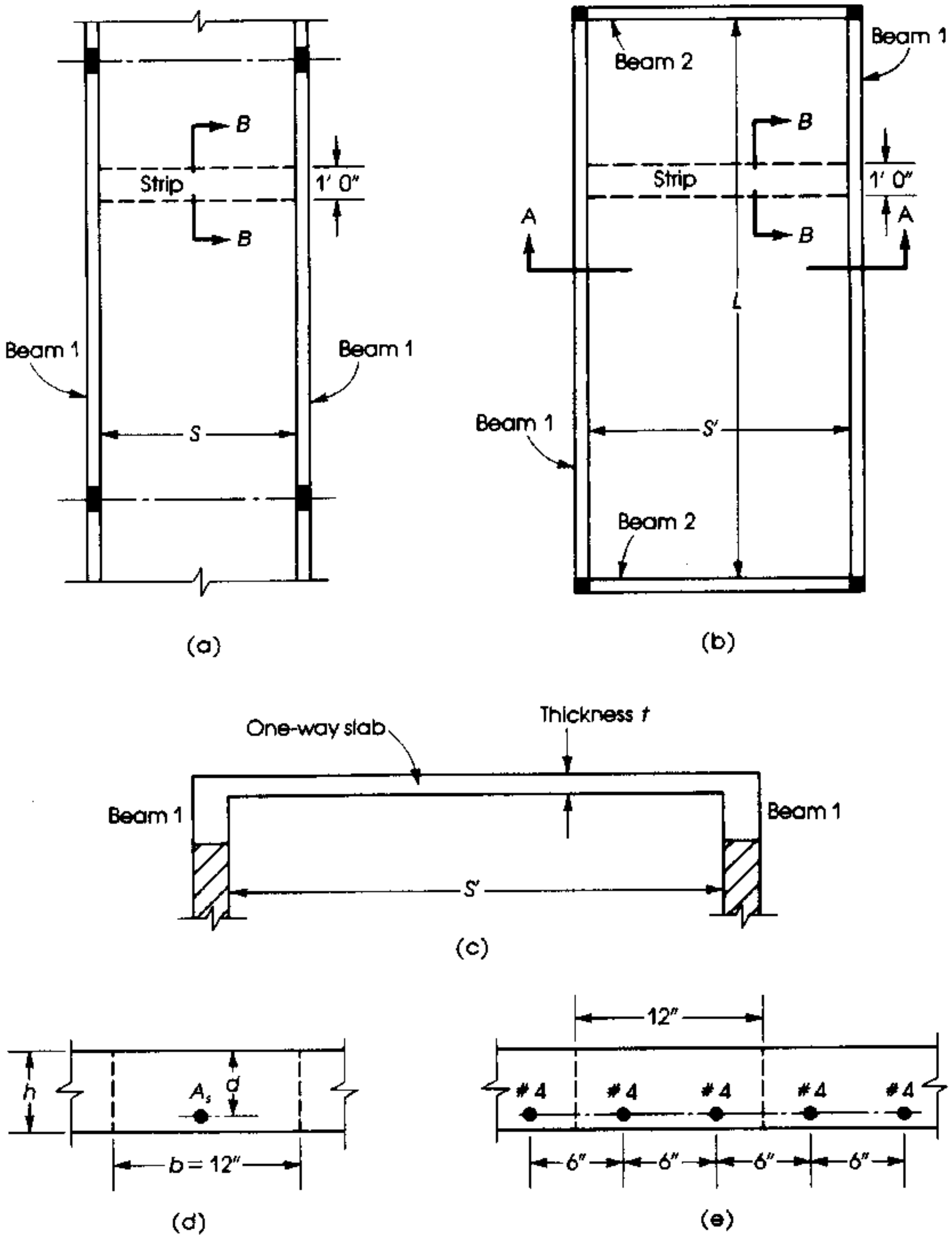
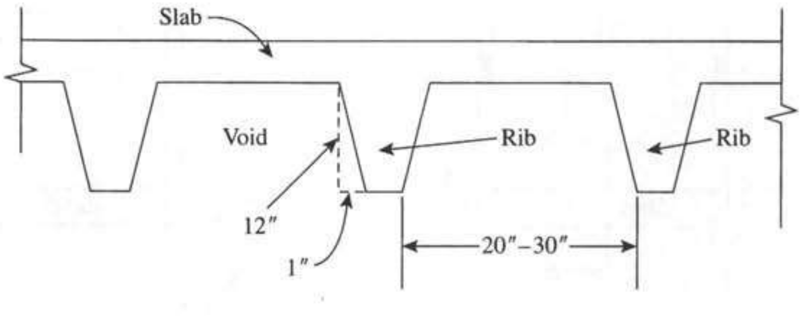
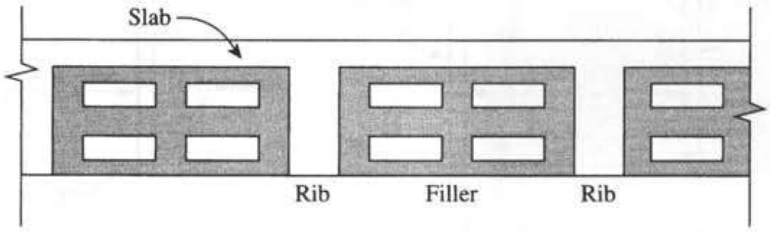


Figure 9.1 One-way slabs.

3. *Two-way floor systems:* When the slab is supported on four sides and the ratio of the long side to the short side is less than 2, the slab will deflect in double curvature in both directions. The floor load is carried in two directions to the four beams surrounding the slab (refer to Chapter 17). Other types of *two-way floor systems* are flat plate floors, flat slabs, and waffle slabs, all explained in Chapter 17. This chapter deals only with one-way floor systems.



(a)



(b)

Figure 9.2 Cross-sections of one-way ribbed slab: (*a*) without fillers and (*b*) with fillers.

9.2 DESIGN OF ONE-WAY SOLID SLABS

If the concrete slab is cast in one uniform thickness without any type of voids, it can be referred to as a *solid slab*. In a one-way slab, the ratio of the length of the slab to its width is greater than 2. Nearly all the loading is transferred in the short direction, and the slab may be treated as a beam. A unit strip of slab, usually 1 ft (or 1 m) at right angles to the supporting girders, is considered a rectangular beam. The beam has a unit width with a depth equal to the thickness of the slab and a span length equal to the distance between the supports. A one-way slab thus consists of a series of rectangular beams placed side by side (Fig. 9.1).

If the slab is one span only and rests freely on its supports, the maximum positive moment M for a uniformly distributed load of w psf is $M = (wL^2)/8$, where L is the span length between the supports. If the same slab is built monolithically with the supporting beams or is continuous over several supports, the positive and negative moments are calculated either by structural analysis or by moment coefficients as for continuous beams. The ACI Code, Section 8.3, permits the use of moment and shear coefficients in the case of two or more approximately equal spans (Fig. 9.3). This condition is met when the larger of two adjacent spans does not exceed the shorter span by more than 20%. For uniformly distributed loads, the unit live load shall not exceed three times the unit dead load. When these conditions are not satisfied, structural analysis is required. In structural analysis, the negative bending moments at the centers of the supports are calculated. The value that may be considered in the design is the negative moment at the face of the support. To obtain this value, subtract from the maximum moment value at the center of the support a quantity equal to $Vb/3$, where V is the shearing force calculated from the analysis and b is the width of the support:

$$M_f \text{ (at face of the support)} = M_c \text{ (at centerline of support)} - \frac{Vb}{3} \tag{9.1}$$

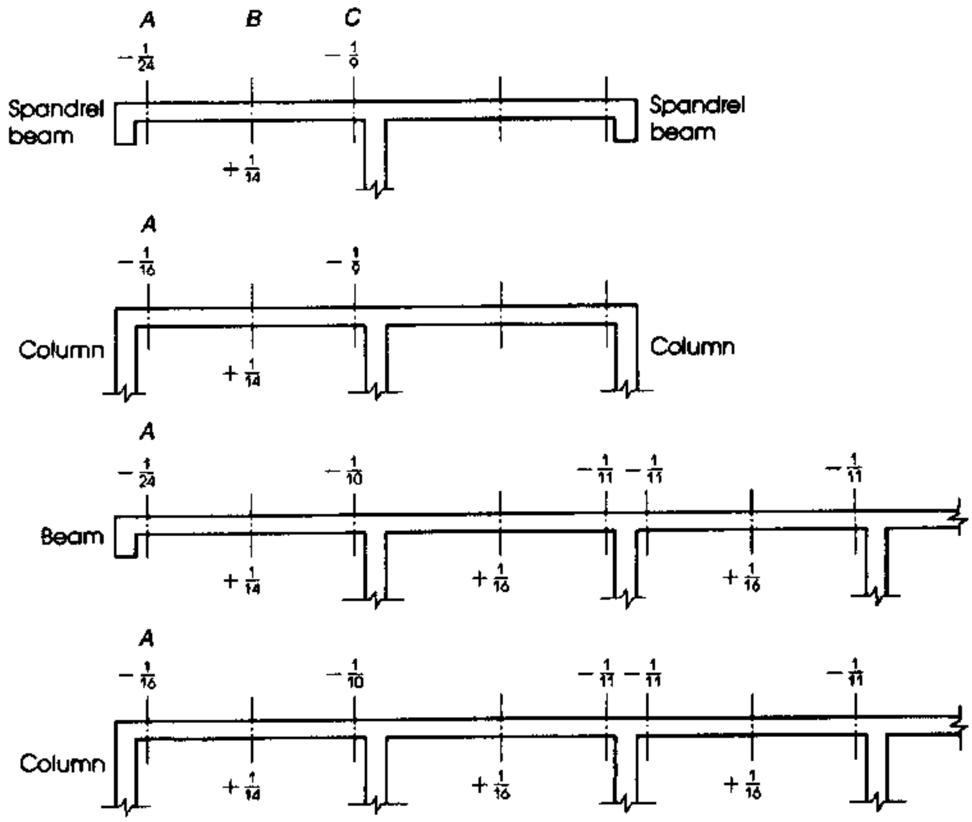


Figure 9.3 Moment coefficients for continuous beams and slabs (ACI Code, Section 8.3).

In addition to moment, diagonal tension and development length of bars should also be checked for proper design.

The conditions under which the moment coefficients for continuous beams and slabs given in Fig. 9.3 should be used can be summarized as follows:

1. Spans are approximately equal: Longer span $\leq$ 1.2 (shorter span).
2. Loads are uniformly distributed.
3. The ratio (live load/dead load) is less than or equal to 3.
4. For slabs with spans less than or equal to 10 ft, negative bending moment at face of all supports is $\left(\frac{1}{12}\right) w_u l_n^2$.
5. For an unrestrained discontinuous end at A, the coefficient is 0 at A and $+\frac{1}{11}$ at B.
6. Shearing force at C is $1.15 w_u l_n/2$ and at the face of all other support is $\frac{1}{2} w_u l_n$.
7. $M_u =$ (coefficient) $(w_u l_n^2)$ and $l_n =$ clear span.

9.3 DESIGN LIMITATIONS ACCORDING TO THE ACI CODE

The following limitations are specified by the ACI Code.

1. A typical imaginary strip 1 ft (or 1 m) wide is assumed.

2. The minimum thickness of one-way slabs using grade 60 steel according to the ACI Code, Table 9.5*a*, for solid slabs and for beams or ribbed one-way slabs should be equal to the following:
   - For simply supported spans: solid slabs, $h = L/20$ (ribbed slabs, $h = L/16$).
   - For one-end continuous spans: solid slabs, $h = L/24$ (ribbed slabs, $h = L/18.5$).
   - For both-end continuous spans: solid slabs, $h = L/28$ (ribbed slabs, $h = L/21$).
   - For cantilever spans: solid slabs, $h = L/10$ (ribbed slabs, $h = L/8$).
   - For f_y other than 60 ksi, these values shall be multiplied by $0.4 + 0.01 f_y$, where f_y is in ksi. This minimum thickness should be used unless computation of deflection indicates a lesser thickness can be used without adverse effects.
3. Deflection is to be checked when the slab supports are attached to construction likely to be damaged by large deflections. Deflection limits are set by the ACI Code, Table 9.5*b*.
4. It is preferable to choose slab depth to the nearest $\frac{1}{2}$ in. (or 10 mm).
5. Shear should be checked, although it does not usually control.
6. Concrete cover in slabs shall not be less than $\frac{3}{4}$ in. (20 mm) at surfaces not exposed to weather or ground. In this case, $d = h - \left(\frac{3}{4} \text{ in.}\right) -$ (half-bar diameter). Refer to Fig. 9.1*d*.
7. In structural slabs of uniform thickness, the minimum amount of reinforcement in the direction of the span shall not be less than that required for shrinkage and temperature reinforcement (ACI Code, Section 7.12).
8. The principal reinforcement shall be spaced not farther apart than three times the slab thickness nor more than 18 in. (ACI Code, Section 7.6.5).
9. Straight-bar systems may be used in both tops and bottoms of continuous slabs. An alternative bar system of straight and bent (trussed) bars placed alternately may also be used.
10. In addition to main reinforcement, steel bars at right angles to the main must be provided. This additional steel is called *secondary, distribution, shrinkage*, or *temperature reinforcement*.

9.4 TEMPERATURE AND SHRINKAGE REINFORCEMENT

Concrete shrinks as the cement paste hardens, and a certain amount of shrinkage is usually anticipated. If a slab is left to move freely on its supports, it can contract to accommodate the shrinkage. However, slabs and other members are joined rigidly to other parts of the structure, causing a certain degree of restraint at the ends. This results in tension stresses known as *shrinkage stresses*. A decrease in temperature and shrinkage stresses is likely to cause hairline cracks. Reinforcement is placed in the slab to counteract contraction and distribute the cracks uniformly. As the concrete shrinks, the steel bars are subjected to compression.

Reinforcement for shrinkage and temperature stresses normal to the principal reinforcement should be provided in a structural slab in which the principal reinforcement extends in one direction only. The ACI Code, Section 7.12.2, specifies the following minimum steel ratios: For slabs in which grade 40 or 50 deformed bars are used, $\rho = 0.2\%$, and for slabs in which grade 60 deformed bars or welded bars or welded wire fabric are used, $\rho = 0.18\%$. In no case shall such reinforcement be placed farther apart than five times the slab thickness or more than 18 in.

For temperature and shrinkage reinforcement, the whole concrete depth h exposed to shrinkage shall be used to calculate the steel area. For example, if a slab has a total depth of $h = 6$ in.

and $f_y = 60$ ksi, then the area of steel required per 1-ft width of slab is $A_s = 6(12)(0.0018) = 0.129\,\text{in.}^2$. The spacings of the bars, S, can be determined as follows:

$$S = \frac{12A_b}{A_s} \tag{9.2}$$

where A_b = area of the bar chosen and A_s = calculated area of steel.

For example, if no. 3 bars are used ($A_b = 0.11\,\text{in.}^2$), then $S = 12(0.11)/0.129 = 10.33$ in., say, 10 in. If no. 4 bars are chosen ($A_b = 0.2\,\text{in.}^2$), then $S = 12(0.2)/0.129 = 18.6$ in., say, 18 in. Maximum spacing is the smaller of five times slab thickness (30 in.) or 18 in. Then no. 4 bars spaced at 18 in. are adequate (or no. 3 bars at 10 in.). These bars act as secondary reinforcement and are placed normal to the main reinforcement calculated by flexural analysis. Note that areas of bars in slabs are given in Table A.14.

9.5 REINFORCEMENT DETAILS

In continuous one-way slabs, the steel area of the main reinforcement is calculated for all critical sections, at midspans, and at supports. The choice of bar diameter and detailing depends mainly on the steel areas, spacing requirements, and development length. Two bar systems may be adopted.

In the straight-bar system (Fig. 9.4), straight bars are used for top and bottom reinforcement in all spans. The time and cost to produce straight bars is less than that required to produce bent bars; thus, the straight-bar system is widely used in construction.

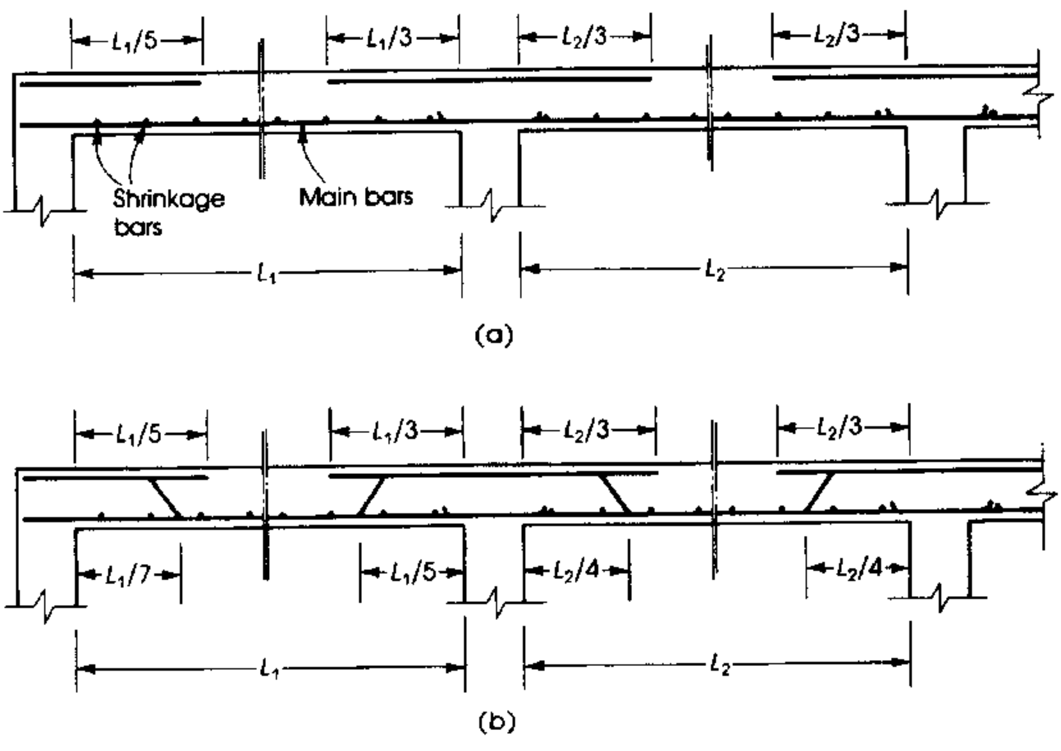


Figure 9.4 Reinforcement details in continuous one-way slabs: (*a*) straight bars and (*b*) bent bars.

In the bent-bar, or trussed, system, straight and bent bars are placed alternately in the floor slab. The location of bent points should be checked for flexural, shear, and development length requirements. For normal loading in buildings, the bar details at the end and interior spans of one-way solid slabs may be adopted as shown in Fig. 9.4.

9.6 DISTRIBUTION OF LOADS FROM ONE-WAY SLABS TO SUPPORTING BEAMS

In one-way floor slab systems, the loads from slabs are transferred to the supporting beams along the long ends of the slabs. The beams transfer their loads in turn to the supporting columns.

From Fig. 9.5 it can be seen that beam B_2 carries loads from two adjacent slabs. Considering a 1-ft length of beam, the load transferred to the beam is equal to the area of a strip 1 ft wide and S feet in length multiplied by the intensity of load on the slab.

This load produces a uniformly distributed load on the beam:

$$U_B = U_S \cdot S$$

The uniform load on the end beam, B_1, is half the load on B_2, because it supports a slab from one side only.

The load on column C_4 is equal to the reactions from two adjacent B_2 beams,

$$\text{Load on column } C_4 = U_B L = U_S L S$$

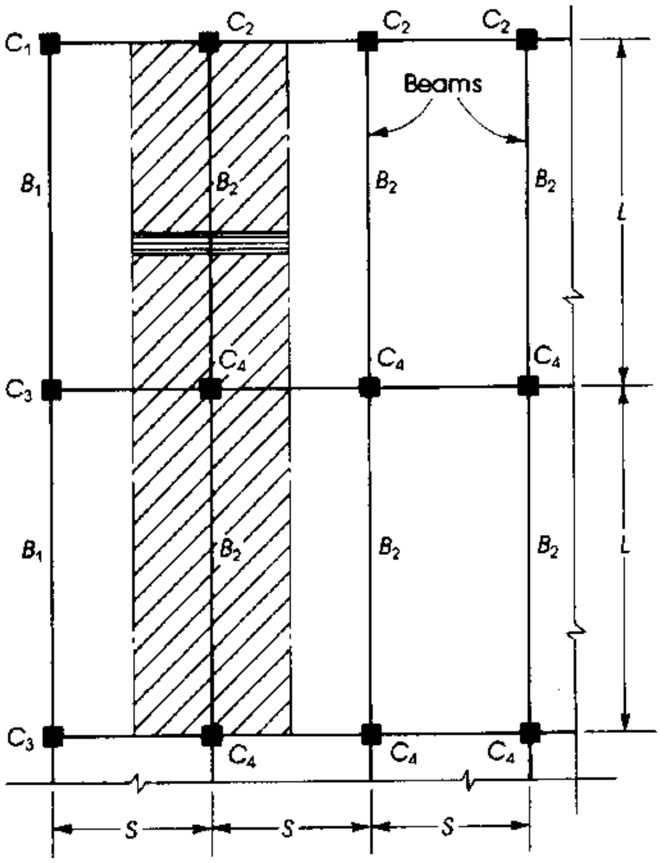


Figure 9.5 Distribution of loads on beams.

The load on column C_3 is one-half the load on column C_4, because it supports loads from slabs on one side only. Similarly, the loads on columns C_2 and C_1 are

$$\text{Load on } C_2 = U_S \frac{L}{2} S = \text{load on } C_3$$

$$\text{Load on } C_1 = U_S \left(\frac{L}{2}\right)\left(\frac{S}{2}\right)$$

From this analysis, it can be seen that each column carries loads from slabs surrounding the column and up to the centerline of adjacent slabs: up to $L/2$ in the long direction and $S/2$ in the short direction.

Distribution of loads from two-way slabs to their supporting beams and columns is discussed in Chapter 17.

Example 9.1

Calculate the design moment strength of a one-way solid slab that has a total depth of $h = 7$ in. and is reinforced with no. 6 bars spaced at $S = 7$ in. Use $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Determine the effective depth, d:

$$d = h - \frac{3}{4} \text{ in. (cover)} - \text{half-bar diameter} \quad \text{(See Fig. 9.1}d\text{)}.$$

$$d = 7 - \frac{3}{4} - \frac{6}{16} = 5.875 \text{ in.}$$

2. Determine the average A_s provided per 1-ft width (12 in.) of slab. The area of no. 6 bar is $A_b = 0.44 \text{ in.}^2$.

$$A_s = \frac{12A_b}{S} = \frac{12(0.44)}{7} = 0.754 \text{ in.}^2/\text{ft}$$

Areas of bars in slabs are given in Table A.14 in Appendix A.

3. Compare the steel ratio used with ρ_{max} and ρ_{min}. For $f'_c = 3$ ksi and $f_y = 60$ ksi, $\rho_{max} = 0.01356$ and $\rho_{min} = 0.00333$. ρ (used) $= 0.754/(12 \times 5.875) = 0.0107$, which is adequate ($\phi = 0.9$).
4. Calculate $\phi M_n = \phi A_s f_y (d - a/2)$.

$$a = A_s f_y/(0.85 f'_c b) = 0.754(60)/(0.85 \times 3 \times 12) = 1.48 \text{ in.}$$

$$\phi M_n = 0.9(0.754)(60)(5.875 - 1.48/2) = 209 \text{ K·in.} = 17.42 \text{ K·ft}$$

Example 9.2

Determine the allowable uniform live load that can be applied on the slab of the previous example if the slab span is 16 ft between simple supports and carries a uniform dead load (excluding self-weight) of 100 psf.

Solution

1. The design moment strength of the slab is 17.42 K·ft per 1-ft width of slab.

$$M_u = \phi M_n = 17.42 = \frac{W_u L^2}{8} = \frac{W_u (16)^2}{8}$$

The factored uniform load is $W_u = 0.544 \text{ K/ft}^2 = 544$ psf.

2.
$$W_u = 1.2D + 1.6L$$
$$D = 100 \text{ psf} + \text{self-weight} = 100 + \frac{7}{12}(150) = 187.5 \text{ psf}$$
$$544 = 1.2(187.5) + 1.6L \quad L = 200 \text{ psf}$$

Example 9.3

Design a 12-ft simply supported slab to carry a uniform dead load (excluding self-weight) of 120 psf and a uniform live load of 100 psf. Use $f_c' = 3$ ksi, $f_y = 60$ ksi, $\lambda = 1$, and the ACI Code limitations.

Solution

1. Assume a slab thickness. For $f_y = 60$ ksi, the minimum depth to control deflection is $L/20 = 12(12)/20 = 7$ in. Assume a total depth of $h = 7$ in. and assume $d = 6$ in. (to be checked later).
2. Calculate factored load: weight of slab $= \frac{7}{12}(150) = 87.5$ psf.

$$W_u = 1.2D + 1.6L = 1.2(87.5 + 120) + 1.6(100) = 409 \text{ psf}$$

For a 1-ft width of slab, $M_u = W_u L^2/8$.

$$M_u = \frac{0.409(12)^2}{8} = 7.362 \text{ K·ft}$$

3. Calculate A_s: For $M_u = 7.362$ K·ft, $b = 12$ in., and $d = 6$ in., $R_u = M_u/bd^2 = 7.362(12{,}000)/(12)(6)^2 = 205$ psi. From tables in Appendix A, $\rho = 0.0040 < \rho_{\max} = 0.01356$, $\phi = 0.9$.

$$A_s = \rho bd = 0.0040(12)(6) = 0.28 \text{ in.}^2$$

Choosing no. 4 bars ($A_b = 0.2$ in.^{2}), and $S = 12A_b/A_s = 12(0.2)/0.28 = 8.6$ in. Check actual $d = h - \frac{3}{4} - \frac{4}{16} = 6$ in. It is acceptable. Let $S = 8$ in. and $A_s = 0.3$ in^2.

4. Check the moment capacity of the final section.

$$a = \frac{A_s f_y}{(0.85 f_c' b)} = \frac{0.3(60)}{0.85 \times 3 \times 12} = 0.59 \text{ in.}$$
$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) = 0.9(0.3)(60)(6 - 0.59/2) = 92.42 \text{ K·in.} = 7.7 \text{ K·ft} > M_u$$
$$= 7.362 \text{ K·ft}$$

5. Calculate the secondary (shrinkage) reinforcement normal to the main steel. For $f_y = 60$ ksi,

$$\rho_{\min} = 0.0018$$
$$A_{sh} = \rho bh = 0.0018(12)(7) = 0.1512 \text{ in.}^2$$

Choose no. 4 bars, $A_b = 0.2$ in.^{2}, $S = 12A_b/A_s = 12(0.2)/0.1512 = 15.9$ in. Use no. 4 bars spaced at 15 in.

6. Check shear requirements: V_u at a distance d from the support is $0.409\left(\frac{12}{2} - \frac{6}{12}\right) = 2.25$ K.

$$\phi V_c = \phi 2\lambda\sqrt{f_c'}bd = \frac{0.75(2)(1)(\sqrt{3000})(12 \times 6)}{1000} = 5.9 \text{ K}$$
$$\tfrac{1}{2}\phi V_c = 2.95 \text{ K} > V_u, \text{ so the shear is adequate.}$$

7. Final section: $h = 7$ in., main bars = no. 4 spaced at 8 in., and secondary bars = no. 4 spaced at 15 in.

Example 9.4

The cross-section of a continuous one-way solid slab in a building is shown in Fig. 9.6. The slabs are supported by beams that span 12 ft between simple supports. The dead load on the slabs is that due to self-weight plus 77 psf; the live load is 130 psf. Design the continuous slab and draw a detailed section. Given: $f'_c = 3$ ksi and $f_y = 40$ ksi.

Solution

1. The minimum thickness of the first slab is $L/30$, because one end is continuous and the second end is discontinuous. The distance between centers of beams may be considered the span L, here equal to 12 ft. For $f_y = 40$ ksi,

$$\text{Minimum total depth} = \frac{L}{30} = \frac{12 \times 12}{30} = 4.8 \text{ in.}$$

$$\text{Minimum total depth for interior span} = \frac{L}{35} = 4.1 \text{ in.}$$

Assume a uniform thickness of 5 in., which is greater than 4.8 in.; therefore, it is not necessary to check deflection.

2. Calculate loads and moments in a unit strip:

$$\text{Dead load} = \text{weight of slab} + 60 \text{ psf}$$

$$= \left(\frac{5}{12} \times 150\right) + 77 = 139.5 \text{ psf}$$

$$\text{Factored load } (U) = 1.2D + 1.6L = 1.2 \times 139.5 + 1.6 \times 130 = 375.5 \text{ psf}$$

The clear span is 11.0 ft. The required moment in the first span is over the support and equals $UL^2/10$.

$$M_u = \frac{U(11)^2}{10} = (0.3755)\frac{121}{10} = 4.54 \text{ K·ft} = 54.5 \text{ K·in.}$$

3. Assume $\rho = 1.4\%$; then $R_u = 450$ psi $= 0.45$ ksi. This value is less than $\rho_{\max}$ of 0.0203 (Table 4.1), and greater than $\rho_{\min}$ of 0.005 ($\phi = 0.9$).

$$d = \sqrt{\frac{M_u}{R_u b}} = \sqrt{\frac{54.5}{0.45 \times 12}} = 3.18 \text{ in.}$$

$$A_s = \rho b d = 0.014(12)(3.18) = 0.53 \text{ in.}^2$$

Choosing no. 5 bars,

$$\text{Total depth} = d + \tfrac{1}{2} \text{ bar diameter} + \text{cover} = 3.18 + \tfrac{5}{16} + \tfrac{3}{4} = 4.25 \text{ in.}$$

Use slab thickness of 5 in., as assumed earlier.

$$\text{Actual } d \text{ used} = 5 - \tfrac{3}{4} - \tfrac{5}{16} = 3.9 \text{ in.}$$

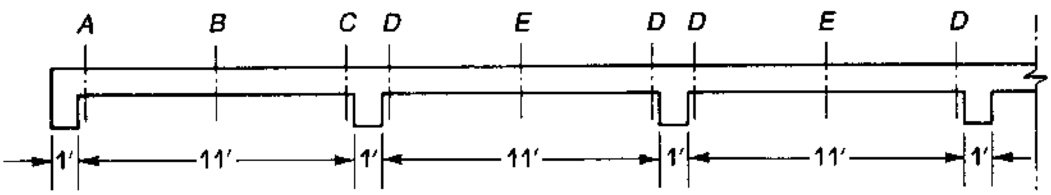


Figure 9.6 Example 9.4.

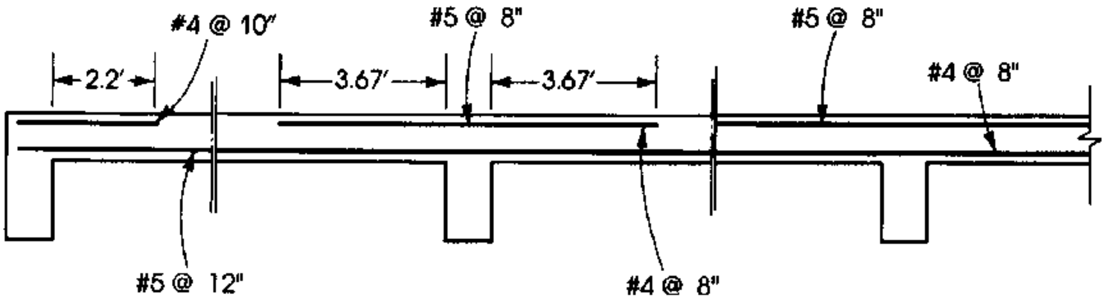


Figure 9.7 Example 9.4: Reinforcement details.

4. Moments and steel reinforcement required at other sections using $d = 3.9$ in. are as follows:

Location	Moment Coefficient	M_u (K-in.)	$R_u = M_u/bd^2$ (psi)	ρ (%)	A_s (in.^{2})	Bars and Spacings
A	$-\frac{1}{24}$	22.7	Small	0.50	0.23	No. 4 at 10 in.
B	$+\frac{1}{14}$	38.9	213	0.65	0.30	No. 5 at 12 in.
C	$-\frac{1}{10}$	54.5	300	0.90	0.44	No. 5 at 8 in.
D	$-\frac{1}{11}$	49.6	271	0.80	0.38	No. 5 at 8 in.
E	$+\frac{1}{16}$	34.1	187	0.55	0.26	No. 4 at 8 in.

The arrangement of bars is shown in Fig. 9.7.

5. Maximum shear occurs at the exterior face of the second support, section C.

$$V_u \text{ (at } C) = 1.15UL_n/2 = \frac{1.15(0.3755)(11)}{2} = 2.375 \text{ K/ft of width}$$

$$\phi V_c = \phi 2\lambda\sqrt{f'_c}bd = \frac{0.75(2)(1)(\sqrt{3000})(12)(3.9)}{1000} = 3.84 \text{ K}$$

This result is acceptable. Note that the provision of minimum area of shear reinforcement when V_u exceeds $\frac{1}{2}\phi V_c$ does not apply to slabs (ACI Code, Section 11.5.5).

Example 9.5

Determine the uniform factored load on an intermediate beam supporting the slabs of Example 9.4. Also calculate the axial load on an interior column; refer to the general plan of Fig. 9.5.

Solution

1. The uniform factored load per foot length on an intermediate beam is equal to the factored uniform load on slab multiplied by S, the short dimension of the slab. Therefore,

$$U \text{ (beam)} = U \text{ (slab)} \times S = 0.3755 \times 12 = 4.5 \text{ K/ft}$$

The weight of the web of the beam shall be added to this value. Span of the beam is 24 ft.

$$\text{Estimated total depth} = \frac{L}{20} \times 0.8 = \left(\frac{24}{20} \times 0.8\right) \times 12 = 11.5 \text{ in.} \quad \text{say, 12 in.}$$

Slab thickness is 5 in. and height of the web is $12 - 5 = 7$ in.

$$\text{Factored weight of beam web} = \left(\frac{7}{12} \times 150\right) \times 1.2 = 105 \text{ lb/ft}$$

$$\text{Total uniform load on beam} = 4.5 + 0.105 = 4.605 \text{ K/ft}$$

2. Axial load on an interior column:

$$P_u = 4.605 \times 24 \text{ ft} = 110.5 \text{ K}$$

9.7 ONE-WAY JOIST FLOOR SYSTEM

A one-way joist floor system consists of hollow slabs with a total depth greater than that of solid slabs. The system is most economical for buildings where superimposed loads are small and spans are relatively large, such as schools, hospitals, and hotels. The concrete in the tension zone is ineffective; therefore, this area is left open between ribs or filled with lightweight material to reduce the self-weight of the slab.

The design procedure and requirements of ribbed slabs follow the same steps as those for rectangular and T-sections explained in Chapter 3. The following points apply to design of one-way ribbed slabs:

1. Ribs are usually tapered and uniformly spaced at about 16 to 30 in. (400 to 750 mm). Voids are usually formed by using pans (molds) 20 in. (500 mm) wide and 6 to 20 in. (150 to 500 mm) deep, depending on the design requirement. The standard increment in depth is 2 in. (50 mm).
2. The ribs shall not be less than 4 in. (100 mm) wide and must have a depth of not more than 3.5 times the width. Clear spacing between ribs shall not exceed 30 in. (750 mm) (ACI Code, Section 8.13).
3. Shear strength, V_c, provided by concrete for the ribs may be taken 10% greater than that for beams. This is mainly due to the interaction between the slab and the closely spaced ribs (ACI Code, Section 8.13.8).
4. The thickness of the slab on top of the ribs is usually 2 to 4 in. (50 to 100 mm) and contains minimum reinforcement (shrinkage reinforcement). This thickness shall not be less than $\frac{1}{12}$ of the clear span between ribs or 1.5 in. (38 mm) (ACI Code, Section 8.13.5.2).
5. The ACI coefficients for calculating moments in continuous slabs can be used for continuous ribbed slab design.
6. There are additional practice limitations, which can be summarized as follows:
   - The minimum width of the rib is one-third of the total depth or 4 in. (100 mm), whichever is greater.
   - Secondary reinforcement in the slab in the transverse directions of ribs should not be less than the shrinkage reinforcement or one-fifth of the area of the main reinforcement in the ribs.
   - Secondary reinforcement parallel to the ribs shall be placed in the slab and spaced at distances not more than half of the spacings between ribs.
   - If the live load on the ribbed slab is less than 3 kN/m^2 (60 psf) and the span of ribs exceeds 5 m (17 ft), a secondary transverse rib should be provided at midspan (its direction

is perpendicular to the direction of main ribs) and reinforced with the same amount of steel as the main ribs. Its top reinforcement shall not be less than half of the main reinforcement in the tension zone. These transverse ribs act as floor stiffeners.

- If the live load exceeds 3 kN/m^2 (60 psf) and the span of ribs varies between 4 and 7 m (13 and 23 ft), one traverse rib must be provided, as indicated before. If the span exceeds 7 m (23 ft), at least two transverse ribs at one-third span must be provided with reinforcement, as explained before.

Example 9.6

Design an interior rib of a concrete joist floor system with the following description: Span of rib = 20 ft (simply supported), dead load (excluding own weight) = 16 psf, live load = 85 psf, $f'_c = 4$ ksi, and $f_y = 60$ ksi.

Solution

1. Design of the slab: Assume a top slab thickness of 2 in. that is fixed to ribs that have a clear spacing of 20 in. No fillers are used. The self-weight of the slab is $\frac{2}{12} \times 150 = 25$ psf.

$$\text{Total D.L.} = 16 + 16 = 41 \text{ psf}$$

$$U = 1.2D + 1.6L = 1.2 \times 41 + 1.6 \times 85 = 185 \text{ psf}$$

$$M_u = \frac{UL^2}{12} \quad \text{(Slab is assumed fixed to ribs.)}$$

$$= \frac{0.185}{12}\left(\frac{20}{12}\right)^2 = 0.043 \text{ K·ft} = 0.514 \text{ K·in.}$$

Considering that the moment in slab will be carried by plain concrete only, the allowable flexural tensile strength is $f_t = 5\sqrt{f'_c}$, with a capacity-reduction factor $\phi = 0.55$, $f_t = 5\sqrt{4000} = 316$ psi.

$$\text{Flexural tensile strength} = \frac{Mc}{I} = \phi f_t \quad I = \frac{bh^3}{12} = \frac{12(2)^3}{12} = 8 \text{ in.}^4 \quad c = \frac{h}{2} = \frac{2}{2} = 1 \text{ in.}$$

$$M = \phi f_t \frac{I}{c} = 0.55 \times 0.316 \times \frac{8}{1} = 1.39 \text{ K·in.}$$

This value is greater than $M_u = 0.514$ K·in., and the slab is adequate. For shrinkage reinforcement, $A_s = 0.0018 \times 12 \times 2 = 0.043$ in.^{2} Use no. 3 bars spaced at 12 in. laid transverse to the direction of the ribs. Welded wire fabric may be economically used for this low amount of steel reinforcement. Use similar shrinkage reinforcement no. 3 bars spaced at 12 in. laid parallel to the direction of ribs, one bar on top of each rib and one bar in the slab between ribs.

2. Calculate moment in a typical rib:

$$\text{Minimum depth} = \frac{L}{20} = \frac{20 \times 12}{20} = 12 \text{ in.}$$

The total depth of rib and slab is 10 + 2 = 12 in. Assume a rib width of 4 in. at the lower end that tapers to 6 in. at the level of the slab (Fig. 9.8). The average width is 5 in. Note that the increase in the rib width using removable forms has a ratio of about 1 horizontal to 12 vertical.

$$\text{Weight of rib} = \frac{5}{12} \times \frac{10}{12} \times 150 = 52 \text{ lb/ft}$$

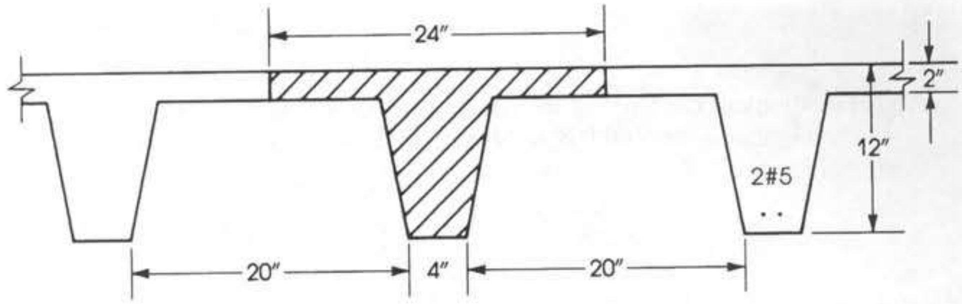
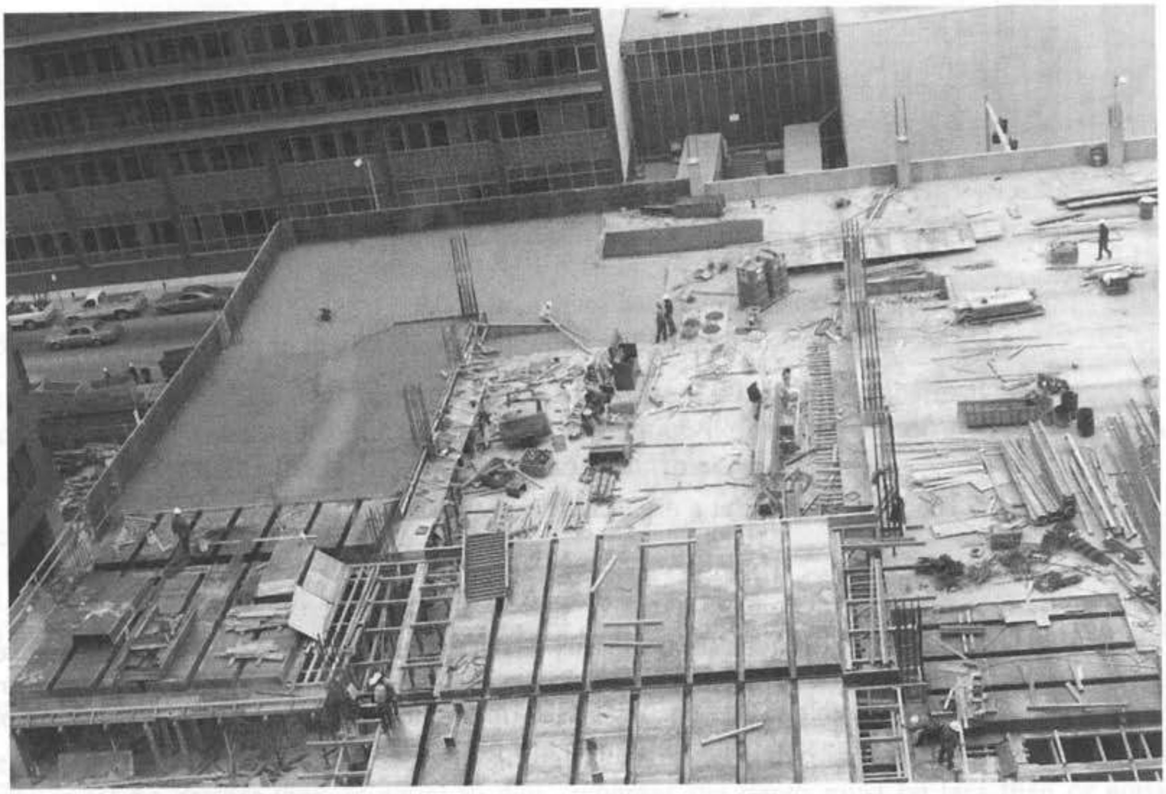


Figure 9.8 Example 9.6.



Rectangular steel pans used in one-way ribbed slab construction.

The rib carries a load from (20 + 4)-in.-wide slab plus its own weight:

$$U = \frac{24}{12} \times 185 + (1.2 \times 52) = 432.4 \text{ lb/ft}$$

$$M_u = \frac{UL^2}{8} = \frac{0.4324}{8}(20)^2 \times 12 = 259.4 \text{ K·in.}$$

3. Design of rib: The total depth is 12 in. Assuming no. 5 bars and concrete cover of $\frac{3}{4}$ in., the effective depth d is $12 - \frac{3}{4} - \frac{5}{16} = 10.9$ in. Check the moment capacity of the flange (assume tension-controlled section, $\phi = 0.9$):

$$\phi M_n \text{ (flange)} = \phi C \left(d - \frac{t}{2}\right), \text{ where } C = 0.85 f_c' bt$$

$$M_u = 0.9(0.85 \times 4 \times 24 \times 2)\left(10.9 - \frac{2}{2}\right) = 1454 \text{ K·in.}$$

The moment capacity of the flange is greater than the applied moment; thus, the rib acts as a rectangular section with $b = 24$ in., and the depth of the equivalent compressive block a is less than 2 in.

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f_c' b}\right)$$

$$259.4 = 0.9 A_s \times 60 \left(10.9 - \frac{A_s \times 60}{1.7 \times 4 \times 24}\right) \quad A_s = 0.45 \text{ in.}^2$$

$$a = \frac{A_s f_y}{0.85 \times f_c' b} = 0.33 \text{ in.} < 2 \text{ in.}$$

Use two no. 5 bars per rib ($A_s = 0.65$ in.^{2}).

$$A_{s\,\min} = 0.0033 b_w d = 0.0033(5)(10.9) = 0.18 \text{ in.}^2 < 0.45 \text{ in.}^2$$

Check

$$\rho = \frac{0.45}{24 \times 10.9} = 0.00172 < \rho_{\max} = 0.01806$$

which is a tension-controlled section, $\phi = 0.9$.

4. Calculate shear in the rib: The allowable shear strength of the rib web is

$$\phi V_c = \phi(1.1) \times 2\lambda\sqrt{f_c'} b_w d$$
$$= 0.75 \times 1.1 \times 2(1)\sqrt{4000} \times 5 \times 10.9 = 5687 \text{ lb}$$

The factored shear at a distance d from the support is

$$V_u = 432.4\left(10 - \frac{10.9}{12}\right) = 3931 \text{ lb}$$

This is less than the shear capacity of the rib. Minimum stirrups may be used, and in this case an additional no. 4 bar will be placed within the slab above the rib to hold the stirrups in place. It is advisable to add one transverse rib at midspan perpendicular to the direction of the ribs having the same reinforcement as that of the main ribs to act as a stiffener.

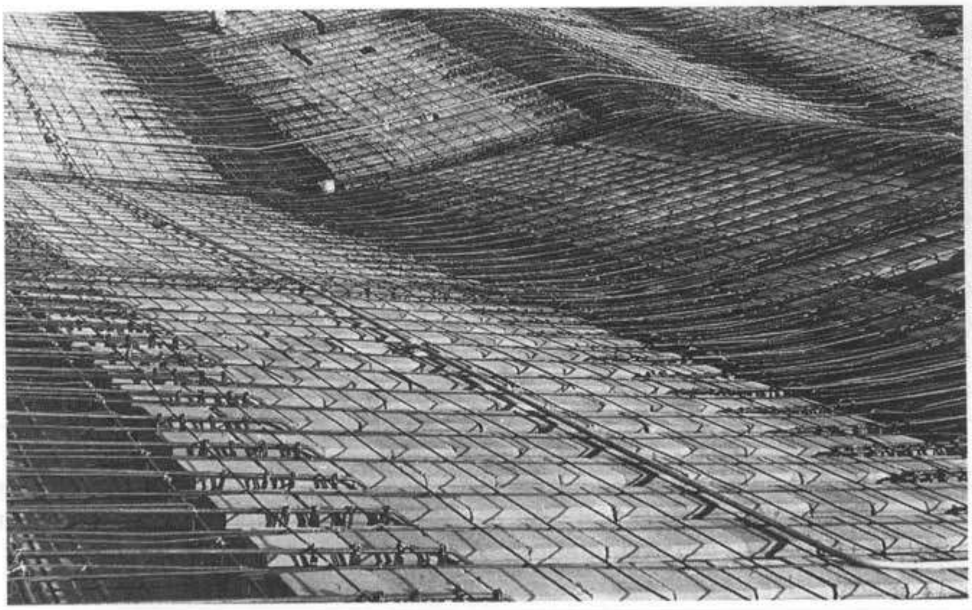
SUMMARY

Section 9.1

Slabs are of different types, one way (solid or joist floor systems) and two way (solid, ribbed, waffle, flat slabs, and flat plates).

Sections 9.2–9.3

1. The ACI Code moment and shear coefficients for continuous one-way slabs are given in Fig. 9.3.
2. The minimum thickness of one-way slabs using grade 60 steel is $L/20$, $L/24$, $L/28$, and $L/10$ for simply supported, one-end continuous, both-end continuous, and cantilever slabs, respectively.



One-way ribbed slab roof. The wide beams have the same total depth as the ribbed slab.

Section 9.4

The minimum shrinkage steel ratios, ρ_{min}, in slabs are 0.002 in. for slabs in which grade 40 or grade 50 bars are used and 0.0018 in. for slabs in which deformed bars of grade 60 are used.

Maximum spacings between bars $\leq$ 5 times rib thickness $\leq$ 18 in.

Sections 9.5–9.6

1. Reinforcement details are shown in Fig. 9.4.
2. Distribution of loads from one-way slabs to the supporting beams is shown in Fig. 9.5.

Section 9.7

The design procedure of ribbed slabs is similar to that of rectangular and T-sections. The width of ribs must be greater than or equal to 4 in., whereas the depth must be less than or equal to 3.5 times the width. The minimum thickness of the top slab is 2 in. or not less than one-twelfth of the clear span between ribs.

REFERENCES

1. Concrete Reinforcing Steel Institute. *CRSI Design Handbook*. Chicago, 2002.
2. Portland Cement Association. *Continuity in Concrete Building Frames*. Chicago, 1959.
3. American Concrete Institute. ACI Code 318-08, Building Code Requirements for Structural Concrete. Detroit, Michigan, 2008.

PROBLEMS

9.1 For each problem, calculate the factored moment capacity of each concrete slab section using $f_y = 60$ ksi.

Number	f'_c(ksi)	h (in.)	Bars and Spacings (in.)	Answer ϕM_n (K·ft)
(a)	3	5	No. 4 at 6	6.35
(b)	3	6	No. 5 at 8	9.29
(c)	3	7	No. 6 at 9	14.06
(d)	3	8	No. 8 at 12	21.01
(e)	4	$5\frac{1}{2}$	No. 5 at 10	6.93
(f)	4	6	No. 7 at 12	11.80
(g)	4	$7\frac{1}{2}$	No. 6 at 6	22.68
(h)	4	8	No. 8 at 12	21.23
(i)	5	5	No. 5 at 10	6.19
(j)	5	6	No. 5 at 8	9.66

9.2 For each slab problem, determine the required steel reinforcement, A_s, and the total depth, if required; then choose adequate bars and their spacings. Use $f_y = 60$ ksi for all problems, $b = 12$ in., and a steel ratio close to the steel ratio $\rho = A_s/bd$ given in some problems.

					One Answer	
Number	f'_c(ksi)	M_u (K·ft)	h (in.)	ρ (%)	h (in.)	Bars
(a)	3	5.4	6	—	6	No. 4 at 9 in.
(b)	3	13.8	$7\frac{1}{2}$	—	$7\frac{1}{2}$	No. 6 at 10 in.
(c)	3	24.4	—	0.85	9	No. 8 at 12 in.
(d)	3	8.1	5	—	5	No. 5 at 7 in.
(e)	4	22.6	—	1.18	$7\frac{1}{2}$	No. 7 at 8 in.
(f)	4	13.9	$8\frac{1}{2}$	—	$8\frac{1}{2}$	No. 6 at 12 in.
(g)	4	13.0	—	1.10	6	No. 6 at 8 in.
(h)	4	11.2	—	0.51	$7\frac{1}{2}$	No. 5 at 9 in.
(i)	5	20.0	9	—	9	No. 7 at 12 in.
(j)	5	10.6	—	0.90	6	No. 6 at 10 in.

9.3 A 16-ft- (4.8-m-)span simply supported slab carries a uniform dead load of 200 psf (10 kN/m^2) (excluding its own weight). The slab has a uniform thickness of 7 in. (175 mm) and is reinforced with no. 6 (20-mm) bars spaced at 5 in. (125 mm). Determine the allowable uniformly distributed load that can be applied on the slab if $f'_c = 4$ ksi (28 MPa) and $f_y = 60$ ksi (420 MPa)

9.4 Design a 10-ft (3-m) cantilever slab to carry a uniform total dead load of 170 psf (8.2 kN/m^2) and a concentrated live load at the free end of 2 K/ft (30 kN/m), when $f'_c = 4$ ksi (28 MPa) and $f_y =$ 60 ksi (420 MPa).

9.5 A 6-in. (150-mm) solid one-way slab carries a uniform dead load of 190 psf (9.2 kN/m^2) (including its own weight) and a live load of 80 psf (3.9 kN/m^2). The slab spans 12 ft (3.6 m) between 10-in.-(250-mm-)wide simple supports. Determine the necessary slab reinforcement using $f'_c = 4$ ksi (28 MPa) and $f_y = 50$ ksi (350 MPa).

9.6 Repeat Problem 9.4 using a variable section with a minimum total depth at the free end of 4 in. (100 mm).

9.7 Design a continuous one-way solid slab supported on beams spaced at 14 ft (4.2 m) on centers. The width of the beams is 12 in. (300 mm), leaving clear slab spans of 13 ft (3.9 m). The slab carries a uniform dead load of 126 psf (6.0 kN/m$_2$) (including self-weight of slab) and a live load of 120 psf (5.8 kN/m$_2$). Use $f'_c = 3$ ksi (21 MPa), $f_y = 40$ ksi (280 MPa), and the ACI coefficients. Show bar arrangements using straight bars for all top and bottom reinforcement.

9.8 Repeat Problem 9.7 using equal clear spans of 10 ft (3 m), $f'_c = 3$ ksi (21 MPa), and $f_y = 60$ ksi (420 MPa).

9.9 Repeat Problem 9.7 using $f'_c = 4$ ksi (28 MPa) and $f_y = 60$ ksi (420 MPa).

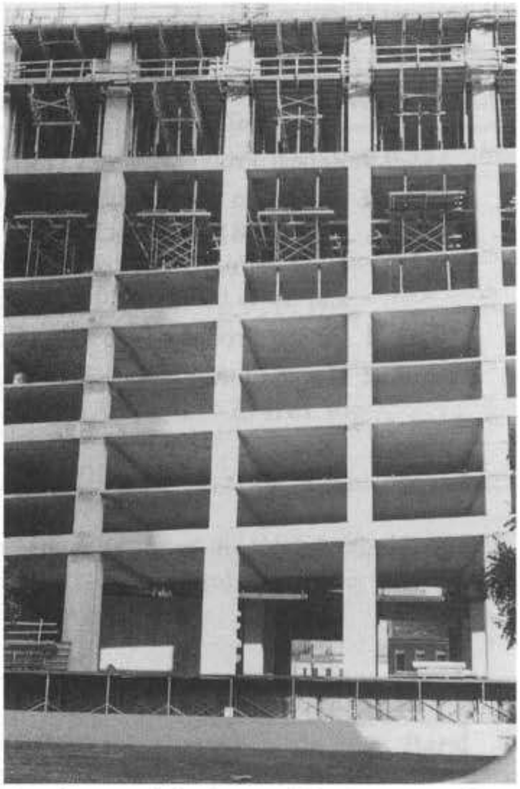
9.10 Design an interior rib of a concrete joist floor system with the following description: Span of ribbed slab is 18 ft (5.4 m) between simple supports; uniform dead load (excluding self-weight) is 30 psf (1.44 kN/m^2); live load is 100 psf (4.8 kN/m^2); support width is 14 in. (350 mm); $f'_c = 3$ ksi (21 MPa) and $f_y = 60$ ksi (420 MPa). Use 30-in.- (750-mm-)wide removable pans.

9.11 Repeat Problem 9.10 using 20-in.- (500-mm-)wide removable pans.

9.12 Use the information given in Problem 9.10 to design a continuous ribbed slab with three equal spans of 18 ft (5.4 m) each.

CHAPTER 10

AXIALLY LOADED COLUMNS



Continuous slabs in a parking structure, New Orleans, Louisiana.

10.1 INTRODUCTION

Columns are members used primarily to support axial compressive loads and have a ratio of height to the least lateral dimension of 3 or greater. In reinforced concrete buildings, concrete beams, floors, and columns are cast monolithically, causing some moments in the columns due to end restraint. Moreover, perfect vertical alignment of columns in a multistory building is not possible, causing loads to be eccentric relative to the center of columns. The eccentric loads will cause moments in columns. Therefore, a column subjected to pure axial loads does not exist in concrete buildings. However, it can be assumed that axially loaded columns are those with relatively small eccentricity, e, of about $0.1h$ or less, where h is the total depth of the column and e is the eccentric distance from the center of the column. Because concrete has a high compressive strength and is an inexpensive material, it can be used in the design of compression members economically. This chapter deals only with short columns; slender columns are covered in detail in Chapter 12.

10.2 TYPES OF COLUMNS

Columns may be classified based on the following different categories (Fig. 10.1):

1. Based on loading, columns may be classified as follows:

a. Axially loaded columns, where loads are assumed acting at the center of the column section.

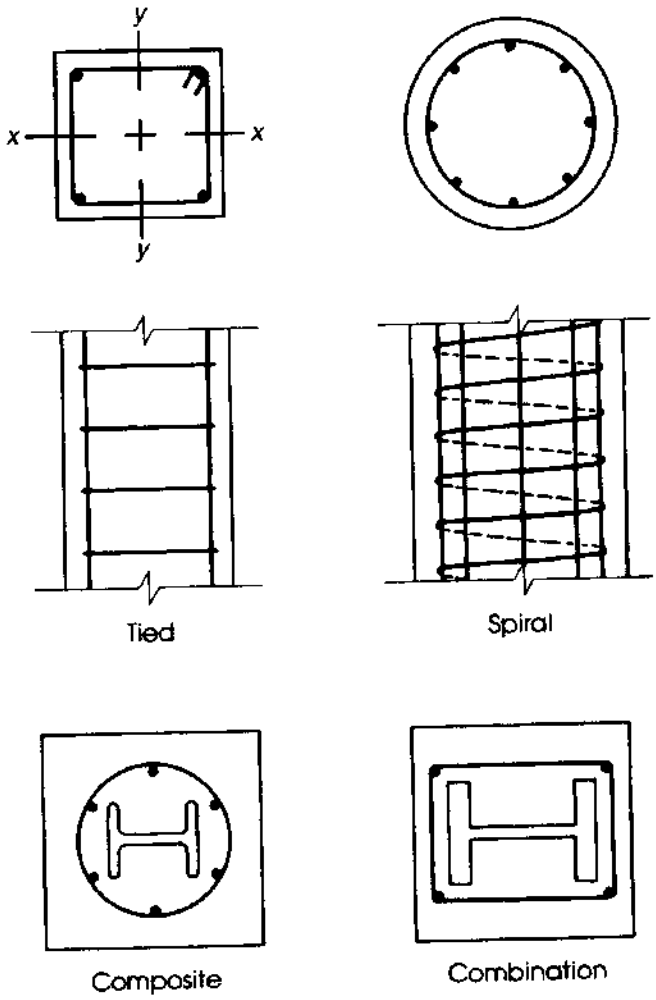


Figure 10.1 Types of columns.

b. Eccentrically loaded columns, where loads are acting at a distance e from the center of the column section. The distance e could be along the x- or y-axis, causing moments either about the x- or y-axis.

c. Biaxially loaded columns, where the load is applied at any point on the column section, causing moments about both the x- and y-axes simultaneously.

2. Based on length, columns may be classified as follows:

   a. Short columns, where the column's failure is due to the crushing of concrete or the yielding of the steel bars under the full load capacity of the column.

   b. Long columns, where buckling effect and slenderness ratio must be taken into consideration in the design, thus reducing the load capacity of the column relative to that of a short column.

3. Based on the shape of the cross-section, column sections may be square, rectangular, round, L-shaped, octagonal, or any desired shape with an adequate side width or dimensions.

4. Based on column ties, columns may be classified as follows:

   a. Tied columns containing steel ties to confine the main longitudinal bars in the columns. Ties are normally spaced uniformly along the height of the column.

   b. Spiral columns containing spirals (spring-type reinforcement) to hold the main longitudinal reinforcement and to help increase the column ductility before failure. In general, ties and spirals prevent the slender, highly stressed longitudinal bars from buckling and bursting the concrete cover.

5. Based on frame bracing, columns may be part of a frame that is braced against sidesway or unbraced against sidesway. Bracing may be achieved by using shear walls or bracings in the building frame. In braced frames, columns resist mainly gravity loads, and shear walls resist lateral loads and wind loads. In unbraced frames, columns resist both gravity and lateral loads, which reduce the load capacity of the columns.
6. Based on materials, columns may be reinforced, prestressed, composite (containing rolled steel sections such as I-sections), or a combination of rolled steel sections and reinforcing bars. Concrete columns reinforced with longitudinal reinforcing bars are the most common type used in concrete buildings.

10.3 BEHAVIOR OF AXIALLY LOADED COLUMNS

When an axial load is applied to a reinforced concrete short column, the concrete can be considered to behave elastically up to a low stress of about $(\frac{1}{3})f'_c$. If the load on the column is increased to reach its ultimate strength, the concrete will reach the maximum strength and the steel will reach its yield strength, f_y. The nominal load capacity of the column can be written as follows:

$$P_o = 0.85 f'_c A_n + A_{st} f_y \tag{10.1}$$

where A_n and A_{st} = the net concrete and total steel compressive areas, respectively.

$$A_n = A_g - A_{st}$$

$$A_g = \text{gross concrete area}$$

Two different types of failure occur in columns, depending on whether ties or spirals are used. For a tied column, the concrete fails by crushing and shearing outward, the longitudinal steel bars fail by buckling outward between ties, and the column failure occurs suddenly, much like the failure of a concrete cylinder.

A spiral column undergoes a marked yielding, followed by considerable deformation before complete failure. The concrete in the outer shell fails and spalls off. The concrete inside the spiral is confined and provides little strength before the initiation of column failure. A hoop tension develops in the spiral, and for a closely spaced spiral, the steel may yield. A sudden failure is not expected. Figure 10.2 shows typical load deformation curves for tied and spiral columns. Up to point a, both columns behave similarly. At point a, the longitudinal steel bars of the column yield, and the spiral column shell spalls off. After the factored load is reached, a tied column fails suddenly (curve b), whereas a spiral column deforms appreciably before failure (curve c).

10.4 ACI CODE LIMITATIONS

The ACI Code presents the following limitations for the design of compression members:

1. For axially as well as eccentrically loaded columns, the ACI Code sets the strength-reduction factors at $\phi = 0.65$ for tied columns and $\phi = 0.75$ for spirally reinforced columns. The

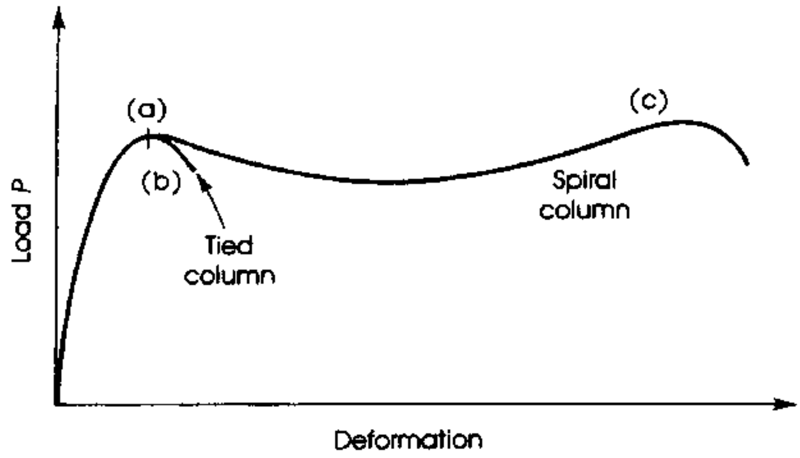


Figure 10.2 Behavior of tied and spiral columns.

difference of 0.05 between the two values shows the additional ductility of spirally reinforced columns.

The strength-reduction factor for columns is much lower than those for flexure ($\phi = 0.9$) and shear ($\phi = 0.75$). This is because in axially loaded columns, the strength depends mainly on the concrete compression strength, whereas the strength of members in bending is less affected by the variation of concrete strength, especially in the case of an under-reinforced section. Furthermore, the concrete in columns is subjected to more segregation than in the case of beams. Columns are cast vertically in long, narrow forms, but the concrete in beams is cast in shallow, horizontal forms. Also, the failure of a column in a structure is more critical than that of a floor beam.

2. The minimum longitudinal steel percentage is 1%, and the maximum percentage is 8% of the gross area of the section (ACI Code, Section 10.9.1). Minimum reinforcement is necessary to provide resistance to bending, which may exist, and to reduce the effects of creep and shrinkage of the concrete under sustained compressive stresses. Practically, it is very difficult to fit more than 8% of steel reinforcement into a column and maintain sufficient space for concrete to flow between bars.
3. At least four bars are required for tied circular and rectangular members and six bars are needed for circular members enclosed by spirals (ACI Code, Section 10.9.2). For other shapes, one bar should be provided at each corner, and proper lateral reinforcement must be provided. For tied triangular columns, at least three bars are required. Bars shall not be located at a distance greater than 6 in. clear on either side from a laterally supported bar. Figure 10.3 shows the arrangement of longitudinal bars in tied columns and the distribution of ties. Ties shown in dotted lines are required when the clear distance on either side from laterally supported bars exceeds 6 in. The minimum concrete cover in columns is 1.5 in.
4. The minimum ratio of spiral reinforcement, ρ_s, according to the ACI Code, Eq. 10.5, and as explained in Section 10.9.3, is limited to

$$\rho_s = 0.45\left(\frac{A_g}{A_{ch}} - 1\right)\frac{f'_c}{f_{yt}} \tag{10.2}$$

where

A_g = gross area of section

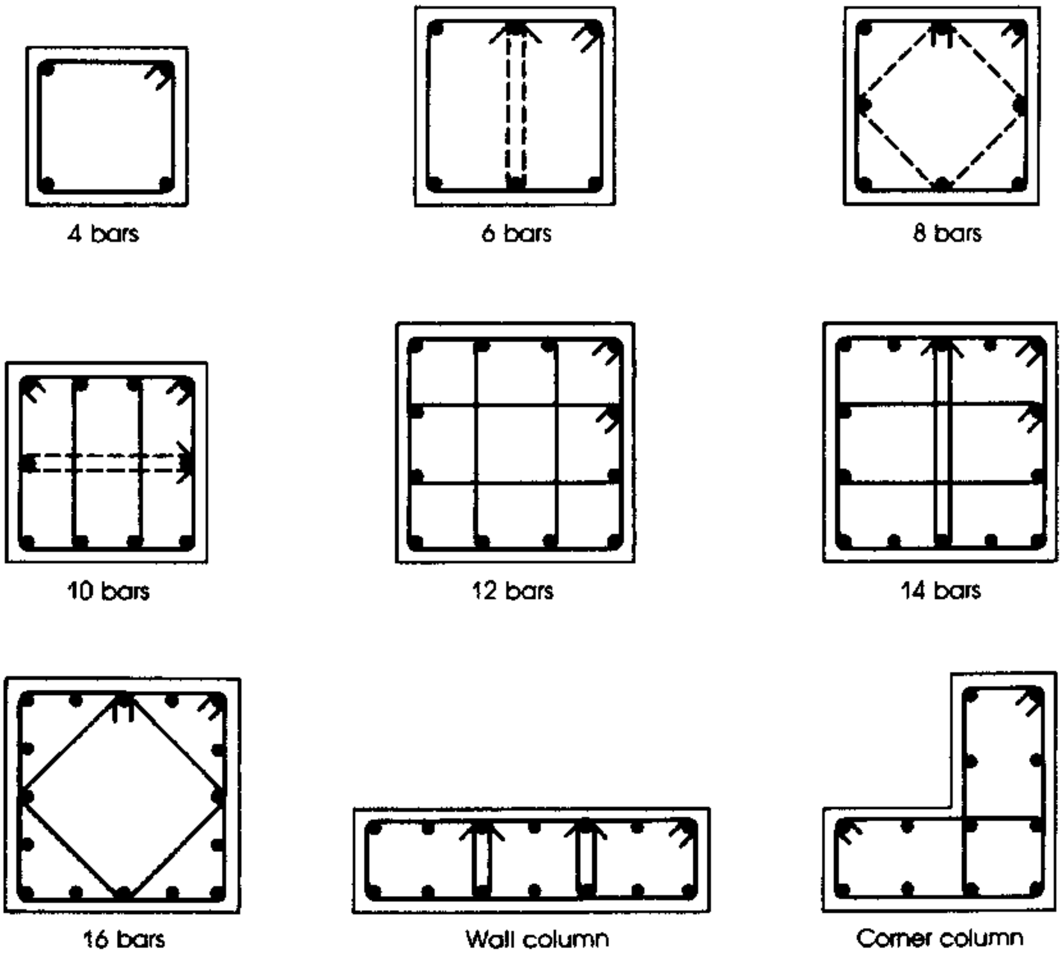


Figure 10.3 Arrangement of bars and ties in columns.

A_{ch} =area of core of spirally reinforced column measured to the outside diameter of spiral

f_{yt} = yield strength of spiral reinforcement (60 ksi; ACI Code, Section 10.9.3)

5. The minimum diameter of spirals is $\frac{3}{8}$ in., and their clear spacing should not be more than 3 in. nor less than 1 in., according to the ACI Code, Section 7.10.4. Splices may be provided by welding or lapping the deformed uncoated spiral bars by 48 diameters or a minimum of 12 in. Lap splices for plain uncoated bar or wire = $72d_p \leq 12$ in. The same applies for epoxy-coated deformed bar or wire. The Code also allows full mechanical splices.
6. Ties for columns must have a minimum diameter of $\frac{3}{8}$ in. to enclose longitudinal bars of no. 10 size or smaller and a minimum diameter of $\frac{1}{2}$ in. for larger bar diameters (ACI Code, Section 7.10.5).
7. Spacing of ties shall not exceed the smallest of 48 times the tie diameter, 16 times the longitudinal bar diameter, or the least dimension of the column. Table 10.1 gives spacings for no. 3 and no. 4 ties. The Code does not give restrictions on the size of columns to allow wider utilization of reinforced concrete columns in smaller sizes.

Table 10.1 Maximum Spacings of Ties

Column Least Side or Diameter (in.)	Spacings of Ties (in.) for Bar					
	No. 6	No. 7	No. 8	No. 9	No. 10	No. 11
12	12	12	12	12	12	12
14	12	14	14	14	14	14
16	12	14	16	16	16	16
18	12	14	16	18	18	18
20	12	14	16	18	18	20
22–40	12	14	16	18	18	22
Ties	No. 3	No. 3	No. 3	No. 3	No. 3	No. 4

10.5 SPIRAL REINFORCEMENT

Spiral reinforcement in compression members prevents a sudden crushing of concrete and buckling of longitudinal steel bars. It has the advantage of producing a tough column that undergoes gradual and ductile failure. The minimum spiral ratio required by the ACI Code is meant to provide an additional compressive capacity to compensate for the spalling of the column shell. The strength contribution of the shell is

$$P_u(\text{shell}) = 0.85 f_c'(A_g - A_{ch}) \tag{10.3}$$

where A_g is the gross concrete area and A_{ch} is the core area (Fig. 10.4).

In spirally reinforced columns, spiral steel is at least twice as effective as longitudinal bars; therefore, the strength contribution of spiral equals $2\rho_s A_{ch} f_{yt}$, where ρ_s is the ratio of volume of spiral reinforcement to total volume of core.

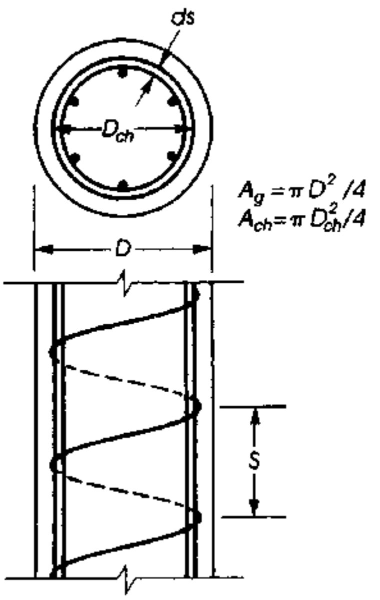


Figure 10.4 Dimensions of a column spiral.

Table 10.2 Spirals for Circular Columns ($f_y = 60$ ksi)

Column Diameter (in.)	$f'_c = 4$ ksi No. 3 Spirals Spacing (in.)	$f'_c = 5$ ksi No. 3 and no. 4 Spirals Spiral No.	Spacing (in.)	$f'_c = 6$ ksi No. 4 Spirals Spacing (in.)
12	2.0	4	2.75	2.25
14	2.0	4	3.00	2.25
16	2.0	4	3.00	2.50
18	2.0	4	3.00	2.50
20	2.0	4	3.00	2.50
22	2.0	4	3.00	2.50
24	2.0	3	1.75	2.50
26 to 40	2.25	3	1.75	2.75

If the strength of the column shell is equated to the spiral strength contribution, then

$$0.85 f'_c (A_g - A_{ch}) = 2\rho_s A_{ch} f_{yt} \tag{10.4}$$

$$\rho_s = 0.425 \left(\frac{A_g}{A_{ch}} - 1 \right) \frac{f'_c}{f_{yt}}$$

The ACI Code adopted a minimum ratio of ρ_s according to the following equation:

$$\text{Minimum}\, \rho_s = 0.45 \left(\frac{A_g}{A_{ch}} - 1 \right) \frac{f'_c}{f_{yt}} \tag{10.2}$$

The design relationship of spirals may be obtained as follows (Fig. 10.4):

$$\rho_s = \frac{\text{volume of spiral in one loop}}{\text{volume of core for a spacing } S}$$

$$= \frac{a_s \pi (D_{ch} - d_s)}{\left(\frac{\pi}{4} D_{ch}^2\right) S} = \frac{4 a_s (D_{ch} - d_s)}{D_{ch}^2 S} \tag{10.5}$$

where

a_s = area of spiral reinforcement
D_{ch} =diameter of the core measured to the outside diameter of spiral
D = diameter of the column
d_s = diameter of the spiral
S = spacing of the spiral

Table 10.2 gives spiral spacings for no. 3 and no. 4 spirals with $f_y = 60$ ksi.

10.6 DESIGN EQUATIONS

The nominal load strength of an axially loaded column was given in Eq. 10.1. Because a perfect axially loaded column does not exist, some eccentricity occurs on the column section, thus reducing its load capacity, P_o. To take that into consideration, the ACI Code specifies that the maximum nominal load, P_o, should be multiplied by a factor equal to 0.8 for tied columns and 0.85 for spirally reinforced columns. Introducing the strength reduction factor, the axial load strength of columns according to the ACI Code, Section 10.3.6, are as follows:

$$P_u = \phi P_n = \phi(0.80)[0.85f_c'(A_g - A_{st}) + A_{st}f_y] \tag{10.6}$$

for tied columns and

$$P_u = \phi P_n = \phi(0.85)[0.85f_c'(A_g - A_{st}) + A_{st}f_y] \tag{10.7}$$

for spiral columns, where

A_g = gross concrete area
A_{st} = total steel compressive area
ϕ = 0.65 for tied columns and 0.70 for spirally reinforced columns

Equations 10.8 and 10.9 may be written as follows:

$$P_u = \phi P_n = \phi K[0.85f_c'A_g + A_{st}(f_y - 0.85f_c')] \tag{10.8}$$

where $\phi = 0.65$ and $K = 0.8$ for tied columns and $\phi = 0.75$ and $K = 0.85$ for spiral columns.

If the gross steel ratio is $\rho_g = A_{st}/A_g$, or $A_{st} = \rho_g A_g$, then Eq. 10.8 may be written as follows:

$$P_u = \phi P_n = \phi K A_g[0.85f_c' + \rho_g(f_y - 0.85f_c')] \tag{10.9}$$

Equation 10.8 can be used to calculate the axial load strength of the column, whereas Eq. 10.9 is used when the external factored load is given and it is required to calculate the size of the column section, A_g, based on an assumed steel ratio, ρ_g, between a minimum of 1% and a maximum of 8%.

It is a common practice to use grade 60 reinforcing steel bars in columns with a concrete compressive strength of 4 ksi or greater to produce relatively small concrete column sections.

10.7 AXIAL TENSION

Concrete will not crack as long as stresses are below its tensile strength; in this case, both concrete and steel resist the tensile stresses, but when the tension force exceeds the tensile strength of concrete (about one-tenth of the compressive strength), cracks develop across the section, and the entire tension force is resisted by steel. The nominal load that the member can carry is that due to tension steel only:

$$T_n = A_{st}f_y \tag{10.10}$$

$$T_u = \phi A_{st}f_y \tag{10.11}$$

where $\phi = 0.9$ for axial tension.

Tie rods in arches and similar structures are subjected to axial tension. Under working loads, the concrete cracks and the steel bars carry the whole tension force. The concrete acts as a fire and corrosion protector. Special provisions must be taken for water structures, as in the case of water tanks. In such designs, the concrete is not allowed to crack under the tension caused by the fluid pressure.

10.8 LONG COLUMNS

The equations developed in this chapter for the strength of axially loaded members are for short columns. In the case of long columns, the load capacity of the column is reduced by a reduction factor.

A long column is one with a high slenderness ratio, h/r, where h is the effective height of the column and r is the radius of gyration. The design of long columns is explained in detail in Chapter 12.

Example 10.1

Determine the allowable design axial load on a 12-in. square, short tied column reinforced with four no. 9 bars. Ties are no. 3 spaced at 12 in. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Using Eq. 10.9,

$$P_u = \phi P_n = \phi K[0.85 f'_c A_g + A_{st}(f_y - 0.85 f'_c)]$$

For a tied column, $\phi = 0.65$, $K = 0.8$, and $A_{st} = 4.0\,\text{in.}^2$

$$P_u = \phi P_u = 0.65(0.8)[0.85(4)(12 \times 12) + 4(60 - 0.85 \times 4)] = 372\text{ K}$$

2. Check steel percentage: $\rho_g = \frac{4}{144} = 0.02778 = 2.778\%$. This is less than 8% and greater than 1%.
3. Check tie spacings: Minimum tie diameter is no. 3. Spacing is the smallest of the 48-tie diameter, 16-bar diameter, or least column side. $S_1 = 48(\frac{3}{8}) = 18$ in., $S_2 = 16(\frac{9}{8}) = 18$ in., $S_3 = 12.0$ in. Ties are adequate (Table 10.1).

Example 10.2

Design a square tied column to support an axial dead load of 400 K and a live load of 232 K using $f'_c = 5$ksi, $f_y = 60$ ksi and a steel ratio of about 5%. Design the necessary ties.

Solution

1. Calculate $P_u = 1.2P_D + 1.6P_L = 1.2(400) + 1.6(232) = 851$ K. Using Eq. 10.10, $P_u = 851 = 0.65(0.8)A_g[0.85 \times 5 + 0.05(60 - 0.85 \times 5)]$, $A_g = 232.5\,\text{in.}^2$, and column side $= 15.25$ in., so use 16 in. (Actual $A_g = 256\,\text{in.}^2$)
2. Because a larger section is adopted, the steel percentage may be reduced by using $A_g = 256\,\text{in.}^2$ in Eq. 10.8:

$$851 = 0.65(0.8)[0.85 \times 5 \times 256 + A_{st}(60 - 0.85 \times 5)]$$

$$A_{st} = 9.84\text{ in.}^2$$

Use eight no. 11 bars ($A_{st} = 12.50\,\text{in.}^2$). See Fig. 10.5.

3. Design of ties (by calculation or from Table 10.1): Choose no. 3 ties with spacings equal to the least of $S_1 = 16(\frac{11}{8}) = 22$ in., $S_2 = 48(\frac{3}{8}) = 18$ in., or $S_3 =$ column side $= 16$ in. Use no. 3 ties spaced at 16 in. Clear distance between bars is 4.23 in., which is less than 6 in. Therefore, no additional ties are required.

Example 10.3

Repeat Example 10.2 using a rectangular section that has a width of $b = 14$ in.

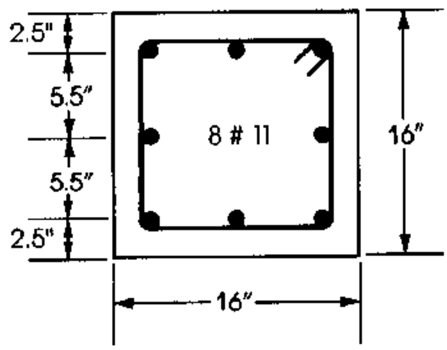


Figure 10.5 Example 10.2.

Solution

1. $P_u = 851$ K and calculated $A_g = 232.5\,\text{in.}^2$ For $b = 14$ in., $h = 232.5/14 = 16.6$ in. Choose a column 14×18 in.; actual $A_g = 252\,\text{in.}^2$.
2. $P_u = 851 = 0.65(0.8)[0.85 \times 5 \times 252 + A_{st}(60 - 0.85 \times 5)]$

$$A_{st} = 10.14\ \text{in.}^2$$

Use eight no. 10 bars. ($A_{st} = 10.16\,\text{in.}^2$)

3. Design of ties: Choose no. 3 ties, $S_1 = 20$ in., $S_2 = 18$ in., and $S_3 = 14$ in. (least side). Use no. 3 ties spaced at 14 in. Clear distance between bars in the long direction is $(18 - 5)/2 -$ bar diameter of $1.27 = 5.23$ in. < 6 in. No additional ties are needed. Clear distance in the short direction is $(14 - 5)/2 - 1.27 = 3.23$ in. < 6 in.

Example 10.4

Design a circular spiral column to support an axial dead load of 475 K and a live load of 250 K using $f'_c = 4$ ksi, $f_y = 60$ ksi, and a steel ratio of about 3%. Also, design the necessary spirals.

Solution

1. Calculate $P_u = 1.2P_D + 1.6P_L = 1.2(475) + 1.6(250) = 970$ K. Using Eq. 10.10 and spiral columns,

$$P_u = 970 = 0.75(0.85)A_g[0.85 \times 4 + 0.03(60 - 0.85 \times 4)]$$

$A_g = 299\,\text{in.}^2$ and column diameter $= 19.5$ in., so use 20 in. Actual $A_g = 314.2\,\text{in.}^2$

2. Calculate A_{st} needed from Eq. 10.8:

$$P_u = 970 = 0.75(0.85)[0.85 \times 4 \times 314.2 + A_{st}(60 - 0.85 \times 4)]$$

$$A_{st} = 8\ \text{in.}^3$$

Use eight no. 10 bars. ($A_{st} = 10.16\,\text{in.}^2$)

3. Design of spirals: The diameter of core is $20 - 2(1.5) = 17$ in. The area of core is

$$A_{ch} = \frac{\pi}{4}(17)^2 \quad A_g = \frac{\pi}{4}(20)^2$$

$$\text{Minimum } \rho_s = 0.45\left(\frac{A_g}{A_{ch}} - 1\right)\frac{f'_c}{f_{yt}} = 0.45\left(\frac{20^2}{17^2} - 1\right)\left(\frac{4}{60}\right) = 0.01152$$

Assume no. 3 spiral, $a_s = 0.11$ in.^{2}, and $d_s = 0.375$ in.

$$\rho_s = 0.01152 = \frac{4a_s(D_{ch} - d_b)}{SD_{ch}^2} = \frac{4(0.11)(17 - 0.375)}{S(17)^2}$$

Spacing s is equal to 2.2 in; use no. 3 spiral at $s = 2$ in. (as shown in Table 10.2).

Example 10.5

Design a rectangular tied short column to carry a factored axial load of 1765 kN. Use $f_c' = 30$ MPa, $f_y = 400$ MPa, column width $(b) = 300$ mm, and a steel ratio of about 2%.

Solution SI Units

1. Using Eq. 10.9,

$$P_u = 0.8\phi A_g[0.85 f_c' + \rho_g(f_y - 0.85 f_c')]$$

   Assuming a steel percentage of 2%,

$$1765 \times 10^3 = 0.8 \times 0.65 A_g[0.85 \times 30 + 0.02(400 - 0.85 \times 30)]$$

$$A_g = 102{,}887 \text{ mm}^2$$

   For $b = 300$ mm, the other side of the rectangular column is 343 mm. Therefore, use a section of 300 by 350 mm ($A_g = 105{,}000$ mm^2).

2. $A_s = 0.02 \times 102{,}887 = 2057$ mm^2. Choose six bars, 22 mm in diameter ($A_s = 2280$ mm^2).
3. Check the axial load strength of the section using Eq. 10.6:

$$\phi P_n = 0.8\phi[0.85 f_c'(A_g - A_{st}) + A_{st} f_y]$$

$$= 0.8 \times 0.65[0.85 \times 30(105{,}000 - 2280) + 2280 \times 400] \times 10^{-3}$$

$$= 1836 \text{ kN}$$

   This meets the required P_u of 1765 kN.

4. Choose ties 10 mm in diameter. Spacing is the least of (1) $16 \times 22 = 352$ mm, (2) $48 \times 10 = 480$ mm, or (3) 300 mm. Choose 10-mm ties spaced at 300 mm.

SUMMARY

Sections 10.1–10.4

Columns may be tied or spirally reinforced.

$$\phi = 0.65 \text{ for tied columns}$$

$$\phi = 0.75 \text{ for spirally reinforced columns}$$

ρ_g must be $\leq 8\%$ and $\geq 1\%$.

Section 10.5

Minimum ratio of spirals is

$$\rho_s = 0.45\left(\frac{A_g}{A_{ch}} - 1\right)\frac{f_c'}{f_{yt}} \tag{10.2}$$

$$\rho_s = \frac{4a_s(D_{ch} - d_s)}{D_{ch}^2 S} \tag{10.5}$$

The minimum diameter of spirals is $\frac{3}{8}$ in., and their clear spacings should be not more than 3 in. or less than 1 in.

Section 10.6

For tied columns,

$$P_u = \phi P_n = 0.8\phi[0.85 f_c'(A_g - A_{st}) + A_{st} f_y] \tag{10.6}$$

or

$$P_u = \phi P_n = 0.8\phi A_g[0.85 f_c' + \rho_g(f_y - 0.85 f_c')]$$

For spiral columns,

$$P_u = \phi P_n = 0.85\phi[0.85 f_c'(A_g - A_{st}) + A_{st} f_y] \tag{10.7}$$

or

$$P_u = \phi P_n = 0.85\phi A_g[0.85 f_c' + \rho_g(f_y - 0.85 f_c')]$$

where $\rho_g = A_{st}/A_g$.

Section 10.7

1. For axial tension,

$$T_u = \phi A_{st} f_y \quad (\phi = 0.9) \tag{10.11}$$

2. Arrangements of vertical bars and ties in columns are shown in Fig. 10.3.

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PROBLEMS

10.1 For each problem, determine the allowable design load-bearing strength ($0.8\phi P_o$) for each of the following short rectangular columns according to the ACI Code limitations. Assume $f_y = 60$ ksi and properly tied columns (b = width of column, in., and h = total depth, in.).

Number	f'_c(ksi)	b (in.)	h (in.)	Bars	Answer ($\phi\ kP_o$) K
(a)	4	16	16	8 no. 9	688
(b)	4	20	20	16 no. 11	1442
(c)	4	12	12	8 no. 8	439
(d)	4	12	24	12 no. 10	955
(e)	5	14	14	10 no. 9	722
(f)	5	16	16	4 no. 10	712
(g)	5	14	26	12 no. 10	1244
(h)	5	18	32	8 no. 11	1634
(i)	6	16	16	8 no. 10	968
(j)	6	12	20	6 no. 10	852

10.2 For each problem, determine the allowable design load-bearing strength of each of the following short, spirally reinforced circular columns according to the ACI Code limitations. Assume $f_y = 60$ ksi and the spirals are adequate (D = diameter of column, in.).

Number	f'_c(ksi)	D (in.)	Bars	Answer ($\phi\ kP_o$) K
(a)	4	14	8 no. 9	581
(b)	4	16	6 no. 10	663
(c)	5	18	8 no. 10	980
(d)	5	20	12 no. 10	1300
(e)	6	15	8 no. 9	797

10.3 For each problem, design a short square, rectangular, or circular column, as indicated, for each set of axial loads given, according to ACI limitations. Also, design the necessary ties or spirals and draw sketches of the column sections showing all bar arrangements. Use $f_y = 60$ ksi and a steel ratio close to the ρ_g given (P_D = dead load, P_L = live load, b = width of a rectangular column, and = $\rho_g = A_{st}/A_g$).

Number	f'_c(ksi)	P_D (K)	P_L (K)	ρ_g%	Section	One Solution
(a)	4	200	200	4	Square	14 × 14, 8 no. 9
(b)	4	750	400	3.5	Square	24 × 24, 16 no. 10
(c)	4	220	165	7	Square	12 × 12, 8 no. 10
(d)	5	330	230	3	Square	16 × 16, 8 no. 9
(e)	4	190	170	2	Rectangular, b = 12 in.	12 × 18, 6 no. 8
(f)	4	280	315	4.5	Rectangular, b = 14 in.	14 × 20, 10 no. 10
(g)	4	210	150	3	Rectangular, b = 12 in.	12 × 16, 6 no. 9
(h)	5	690	460	2	Rectangular, b = 18 in.	18 × 32, 8 no. 10
(i)	4	350	130	4	Circular—spiral	16, 7 no. 9
(j)	4	475	220	3.25	Circular—spiral	20, 7 no. 10
(k)	4	400	260	5	Circular—spiral	18, 9 no. 10
(l)	5	285	200	4.25	Circular—spiral	15, 6 no. 10

For SI units, use 1 psi = 0.0069 MPa, 1 K = 4.45 kN, and 1 in. = 25.4 mm.

CHAPTER 11

MEMBERS IN COMPRESSION AND BENDING

Residential building, Minneapolis, Minnesota.

11.1 INTRODUCTION

Vertical members that are part of a building frame are subjected to combined axial loads and bending moments. These forces develop due to external loads, such as dead, live, and wind loads. The forces are determined by manual calculations or computer applications that are based on the principles of statics and structural analysis. For example, Fig. 11.1 shows a two-hinged portal frame that carries a uniform factored load on *BC*. The bending moment is drawn on the tension side of the frame for clarification. Columns *AB* and *CD* are subjected to an axial compressive force and a bending moment. The ratio of the moment to the axial force is usually defined as the eccentricity, e, where $e = M_n/P_n$ (Fig. 11.1). The eccentricity, e, represents the distance from the plastic centroid of the section to the point of application of the load. The plastic centroid is obtained by determining the location of the resultant force produced by the steel and the concrete, assuming that both are stressed in compression to f_y and 0.85 f'_c, respectively. For symmetrical sections, the plastic centroid coincides with the centroid of the section. For nonsymmetrical sections, the plastic centroid is determined by taking moments about an arbitrary axis, as explained in Example 11.1.

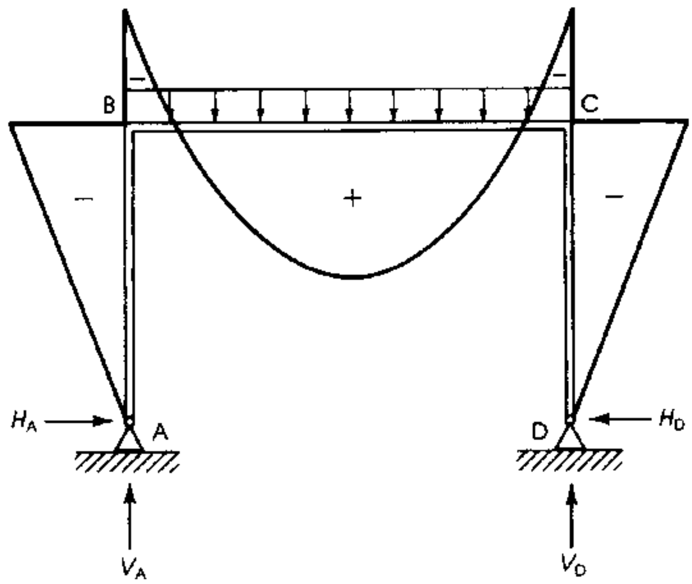


Figure 11.1 Two-hinged portal frame with bending moment diagram drawn on the tension side.

Example 11.1

Determine the plastic centroid of the section shown in Fig. 11.2. Given: $f_c' = 4$ ksi and $f_y = 60$ ksi.

Solution

1. It is assumed that the concrete is stressed in compression to 0.85 f_c':

$$F_c = \text{force in concrete} = (0.85\,f_c')\,A_g$$
$$= (0.85 \times 4) \times 14 \times 20 = 952 \text{ K}$$

F_c is located at the centroid of the concrete section (at 10 in. from axis A-A).

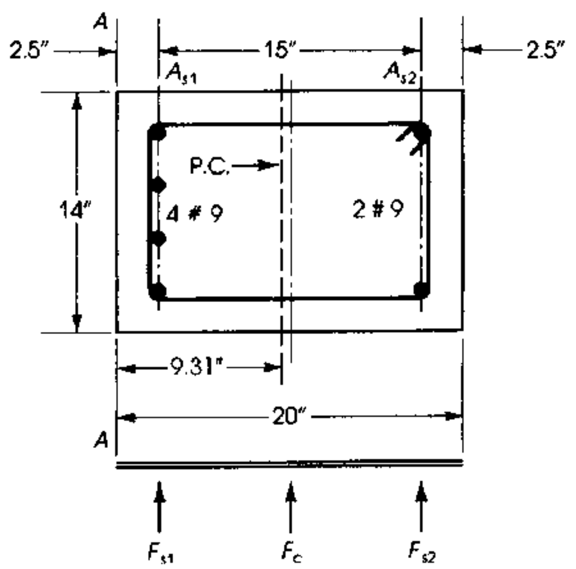


Figure 11.2 Example 11.1: Plastic centroid (P.C.) of section.

2. Forces in steel bars:

$$F_{s1} = A_{s1} f_y = 4 \times 60 = 240 \text{ K}$$

$$F_{s2} = A_{s2} f_y = 2 \times 60 = 120 \text{ K}$$

3. Take moments about A-A:

$$x = \frac{(952 \times 10) + (240 \times 2.5) + (120 \times 17.5)}{952 + 240 + 120} = 9.31 \text{ in.}$$

Therefore, the plastic centroid lies at 9.31 in. from axis A-A.

4. If $A_{s1} = A_{s2}$ (symmetrical section), then $x = 10$ in. from axis A-A.

11.2 DESIGN ASSUMPTIONS FOR COLUMNS

The design limitations for columns, according to the ACI Code, Section 10.2, are as follows:

1. Strains in concrete and steel are proportional to the distance from the neutral axis.
2. Equilibrium of forces and strain compatibility must be satisfied.
3. The maximum usable compressive strain in concrete is 0.003.
4. Strength of concrete in tension can be neglected.
5. The stress in the steel is $f_s = \varepsilon E_s \leq f_y$.
6. The concrete stress block may be taken as a rectangular shape with concrete stress of 0.85 f'_c that extends from the extreme compressive fibers a distance $a = \beta_1 c$, where c is the distance to the neutral axis and β_1 is 0.85 when $f'_c \leq 4000$ psi (30 MPa); β_1 decreases by 0.05 for each 1000 psi above 4000 psi (0.008 per 1 MPa above 30 MPa) but is not less than 0.65. (Refer to Fig. 3.6, Chapter 3.)

11.3 LOAD-MOMENT INTERACTION DIAGRAM

When a normal force is applied on a short reinforced concrete column, the following cases may arise, according to the location of the normal force with respect to the plastic centroid. Refer to Fig. 11.3*a* and 11.3*b*:

Axial compression (P_0). This is a theoretical case assuming that a large axial load is acting at the plastic centroid; $e = 0$ and $M_n = 0$. Failure of the column occurs by crushing of the concrete and yielding of steel bars. This is represented by P_0 on the curve of Fig. 11.3*a*.

1. *Maximum nominal axial load* $P_{n\,\max}$*:* This is the case of a normal force acting on the section with minimum eccentricity. According to the ACI Code, $P_{n\,\max} = 0.80 P_0$ for tied columns and $0.85 P_0$ for spirally reinforced columns, as explained in Chapter 10. In this case, failure occurs by crushing of the concrete and the yielding of steel bars.
2. *Compression failure:* This is the case of a large axial load acting at a small eccentricity. The range of this case varies from a maximum value of $P_n = P_{n\,\max}$ to a minimum value of $P_n = P_b$ (balanced load). Failure occurs by crushing of the concrete on the compression side with a strain of 0.003, whereas the stress in the steel bars (on the tension side) is less than the yield strength, f_y ($f_s < f_y$). In this case $P_n > P_b$ and $e < e_b$.

3. *Balanced condition* (P_b): A balanced condition is reached when the compression strain in the concrete reaches 0.003 and the strain in the tensile reinforcement reaches $\varepsilon_y = f_y/E_s$ simultaneously; failure of concrete occurs at the same time as the steel yields. The moment that accompanies this load is called the *balanced moment*, M_b, and the relevant balanced eccentricity is $e_b = M_b/P_b$.
4. *Tension failure:* This is the case of a small axial load with large eccentricity, that is, a large moment. Before failure, tension occurs in a large portion of the section, causing the tension steel bars to yield before actual crushing of the concrete. At failure, the strain in the tension steel is greater than the yield strain, ε_y, whereas the strain in the concrete reaches 0.003. The range of this case extends from the balanced to the case of pure flexure (Fig. 11.3). When tension controls, $P_n < P_b$ and $e > e_b$.
5. *Pure flexure:* The section in this case is subjected to a bending moment, M_n, whereas the axial load is $P_n = 0$. Failure occurs as in a beam subjected to bending moment only. The eccentricity is assumed to be at infinity. Note that radial lines from the origin represent constant ratios of $M_n/P_n = e =$ eccentricity of the load P_n from the plastic centroid.

Cases 1 and 2 were discussed in Chapter 10, and Case 6 was discussed in detail in Chapter 3. The other cases are discussed in this chapter.

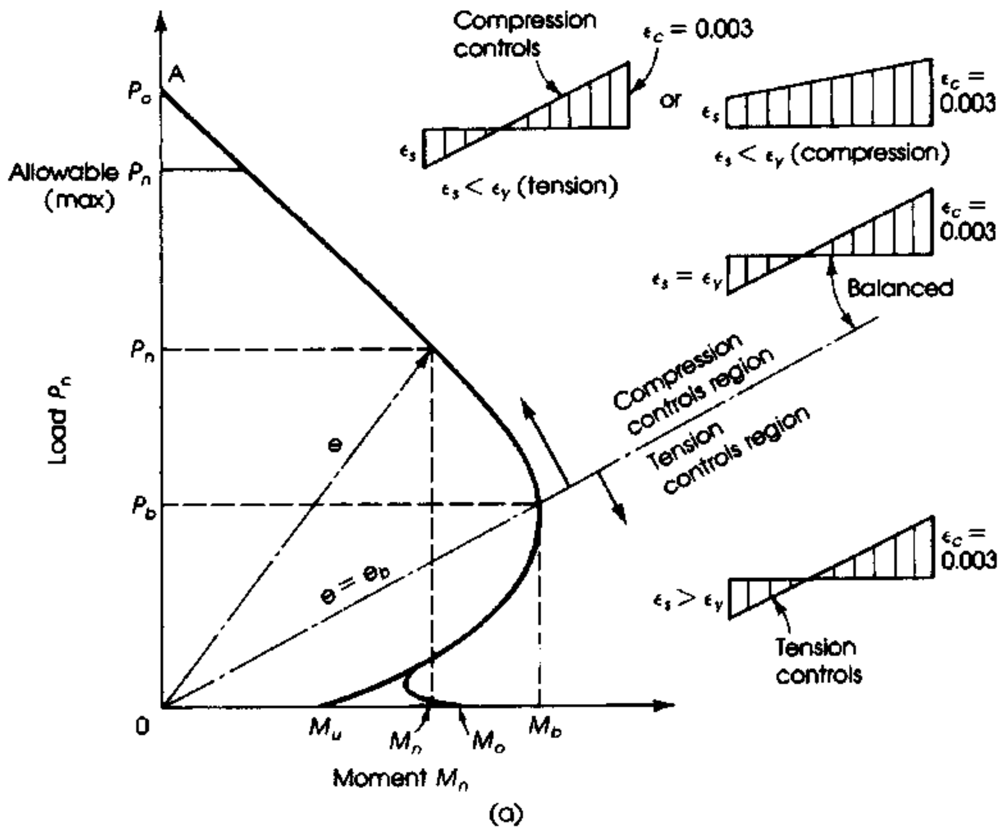
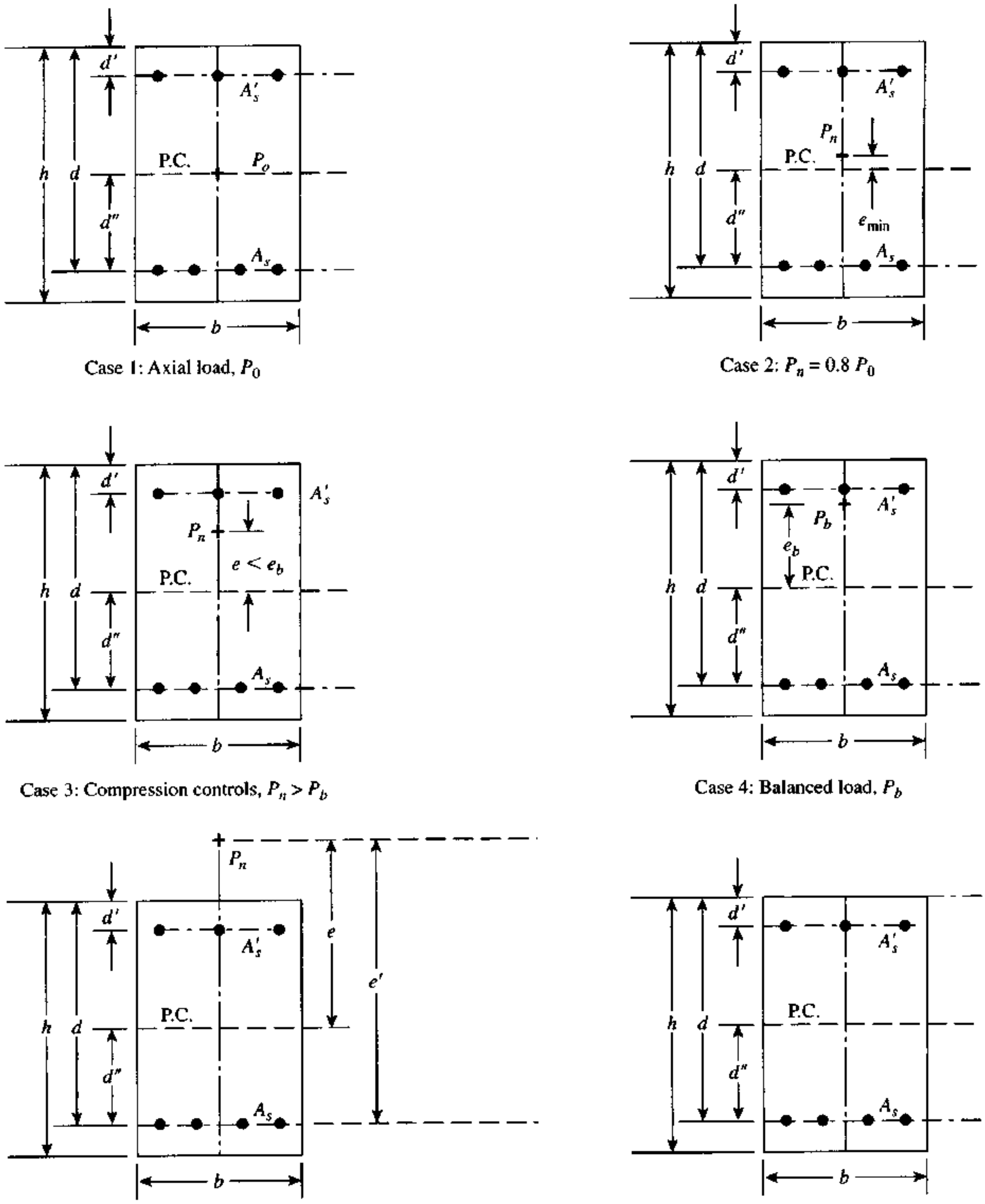


Figure 11.3 (*a*) Load-moment strength interaction diagram showing ranges of cases discussed in text, and (*b*) column sections showing the location of P_n for different load conditions.



Case 1: Axial load, P_0

Case 2: $P_n = 0.8\ P_0$

Case 3: Compression controls, $P_n > P_b$

Case 4: Balanced load, P_b

Case 5: Tension controls, $P_n < P_b$

Case 6: Pure moment, $P_n = 0$

(b)

Figure 11.3 (*continued*)

11.4 SAFETY PROVISIONS

The safety provisions for load factors were discussed earlier in Section 3.6. For columns, the safety provisions may be summarized as follows:

1. Load factors for gravity and wind loads are

$$U = 1.4D$$
$$U = 1.2D + 1.6L$$
$$U = 1.2D + 1.6L + 0.8W$$
$$U = 1.2D + 1.0L + 1.6W$$
$$U = 0.9D + 1.6W$$

The most critical factored load should be used.

2. The strength reduction factor, ϕ, to be used for columns may vary according to the following cases:

a. When $P_u = \phi P_n \geq 0.1 f_c' A_g$, ϕ is 0.65 for tied columns and 0.75 for spirally reinforced columns. This case occurs generally when compression failure is expected. A_g is the gross area of the concrete section.

b. The sections in which the net tensile strain, ε_t, at the extreme tension steel, at nominal strength, is between 0.005 and 0.002 (transition region) ϕ varies linearly between 0.90 and 0.65 (or 0.75), respectively (Fig. 11.4). Refer to Section 3.7. For spiral sections,

$$\phi = 0.75 + (\varepsilon_t - 0.002)(50) \quad \text{or} \quad \phi = 0.75 + 0.15\left[\frac{1}{c/d_t} - \frac{5}{3}\right] \tag{11.1}$$

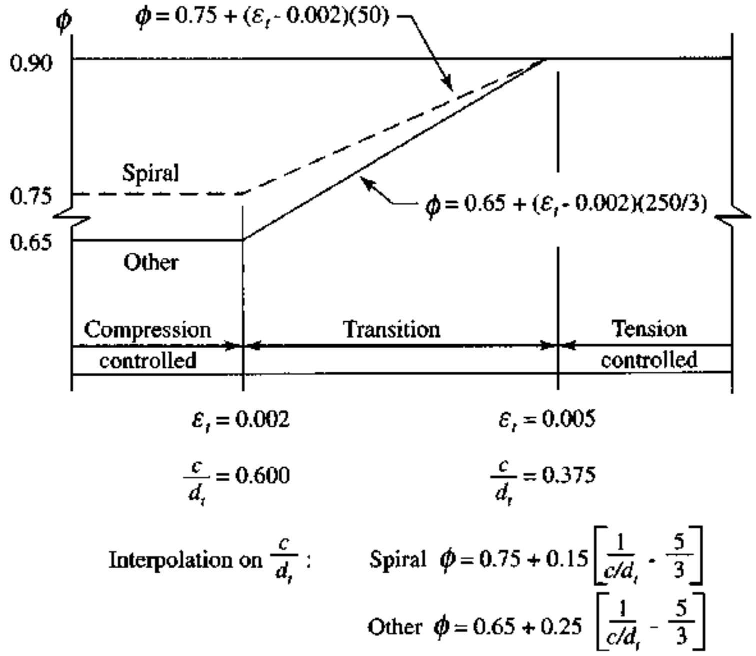


Figure 11.4 Variation in ϕ with NTS for grade 60 steel 7. Courtesy of ACI.

For spiral sections

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) \quad \text{or} \quad \phi = 0.65 + 0.25\left[\frac{1}{c/d_t} - \frac{5}{3}\right] \tag{11.2}$$

c. When $P_u = 0$, the case of pure flexure, then $\phi = 0.90$ for tension-controlled sections and varies between 0.90 and 0.65 (or 0.75) in the transition region.

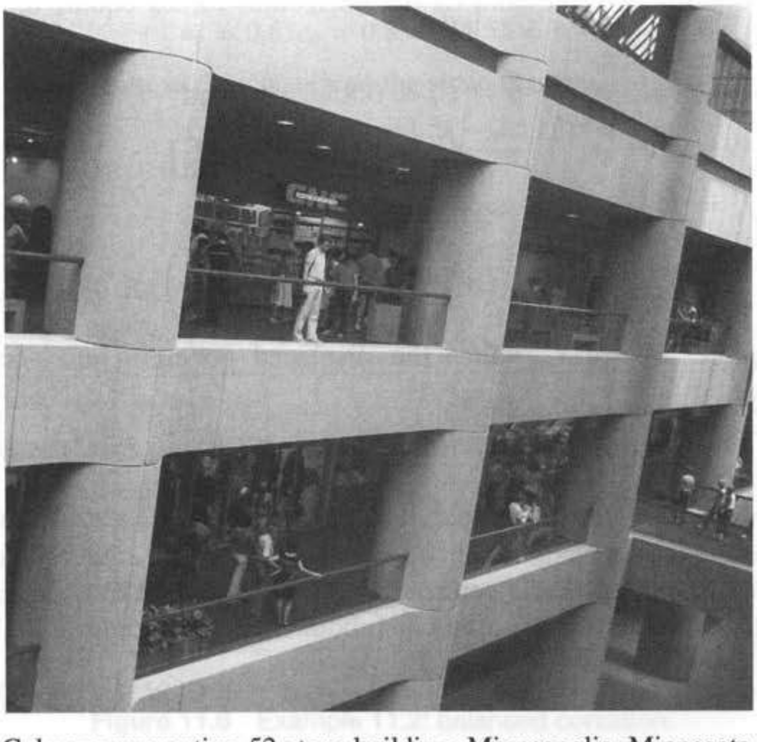
11.5 BALANCED CONDITION — RECTANGULAR SECTIONS

A balanced condition occurs in a column section when a load is applied on the section and produces, at nominal strength, a strain of 0.003 in the compressive fibers of concrete and a strain $\varepsilon_y = f_y/E_s$ in the tension steel bars simultaneously. This is a special case where the neutral axis can be determined from the strain diagram with known extreme values. When the applied eccentric load is greater than P_b, compression controls; if it is smaller than P_b, tension controls in the section.

The analysis of a balanced column section can be explained in steps (Fig. 11.5):

1. Let c equal the distance from the extreme compressive fibers to the neutral axis. From the strain diagram,

$$\frac{c_b(\text{balanced})}{d_t} = \frac{0.003}{0.003 + \dfrac{f_y}{E_s}} \qquad (\text{where } E_s = 29{,}000 \text{ ksi}) \tag{11.3}$$



Columns supporting 52-story building, Minneapolis, Minnesota.
(Columns are 96 × 64 in. with round ends.)

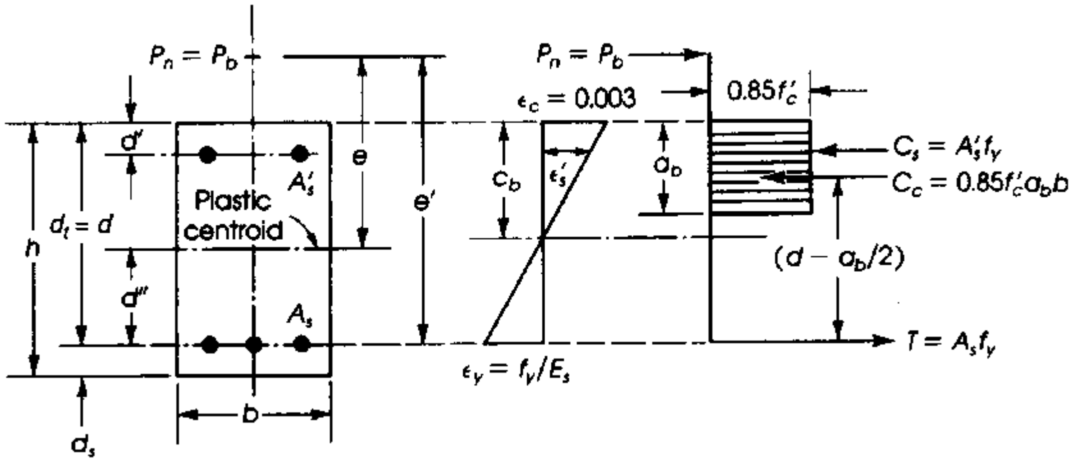


Figure 11.5 Balanced condition (rectangular section).

and

$$c_b = \frac{87d_t}{87 + f_y} \qquad \text{(where } f_y \text{ is in ksi)}$$

The depth of the equivalent compressive block is

$$a_b = \beta_1 c_b = \left(\frac{87}{87 + f_y}\right)\beta_1 d_t \tag{11.4}$$

where $\beta_1 = 0.85$ for $f'_c \leq 4000$ psi and decreases by 0.05 for each 1000-psi increase in f'_c.

2. From equilibrium, the sum of the horizontal forces equals 0: $P_b - C_c - C_s + T = 0$, where

$$C_c = 0.85 f'_c ab \quad \text{and} \quad T = A_s f_y \tag{11.5}$$

$$C_s = A'_s(f'_s - 0.85 f'_c)$$

(Use $f'_s = f_y$ if compression steel yields.)

$$f'_s = 87\left(\frac{c - d'}{c}\right) \leq f_y$$

The expression of C_s takes the displaced concrete into account. Therefore, Eq. 11.5 becomes

$$P_b = 0.85\, f'_c ab + A'_s(f'_s - 0.85\, f'_c) - A_s f_y \tag{11.6}$$

3. The eccentricity e_b is measured from the plastic centroid and e' is measured from the centroid of the tension steel: $e' = e + d''$ (in this case $e = eb + d''$), where d'' is the distance from the plastic centroid to the centroid of the tension steel. The value of e_b can be determined by taking moments about the plastic centroid.

$$P_b e_b = C_c\left(d - \frac{a}{2} - d''\right) + C_s(d - d' - d'') + Td'' \tag{11.7}$$

or

$$P_b e_b = M_b = 0.85\,f_c'ab\left(d - \frac{a}{2} - d''\right) + A_s'(f_y - 0.85\,f_c')(d - d' - d'') + A_s f_y d'' \tag{11.8}$$

The balanced eccentricity is

$$e_b = \frac{M_b}{P_b} \tag{11.9}$$

For nonrectangular sections, the same procedure applies, taking into consideration the actual area of concrete in compression.

The strength reduction factor, ϕ, for the balanced condition with $f_y = 60$ ksi, can be assumed $= 0.65$ (or 0.75). This is because $\varepsilon_s = \varepsilon_t = f_y/E_s = 0.00207$ (or 0.002), for which $\phi = 0.65$ (Fig. 11.4).

Example 11.2

Determine the balanced compressive force P_b; then determine e_b and M_b for the section shown in Fig. 11.6. Given: $f_c' = 4$ ksi and $f_y = 60$ ksi.

Solution

1. For a balanced condition, the strain in the concrete is 0.003 and the strain in the tension steel is

$$\varepsilon_y = \frac{f_y}{E_s} = \frac{60}{29{,}000} = 0.00207$$

2. Locate the neutral axis:

$$c_b = \frac{87}{87 + f_y} d_t = \frac{87}{87 + 60}(19.5) = 11.54 \text{ in.}$$

$$a_b = 0.85 c_b = 0.85 \times 11.54 = 9.81 \text{ in.}$$

3. Check if compression steel yields. From the strain diagram,

$$\frac{\varepsilon_s'}{0.003} = \frac{c - d'}{c} = \frac{11.54 - 2.5}{11.54} \qquad \varepsilon_s' = 0.00235$$

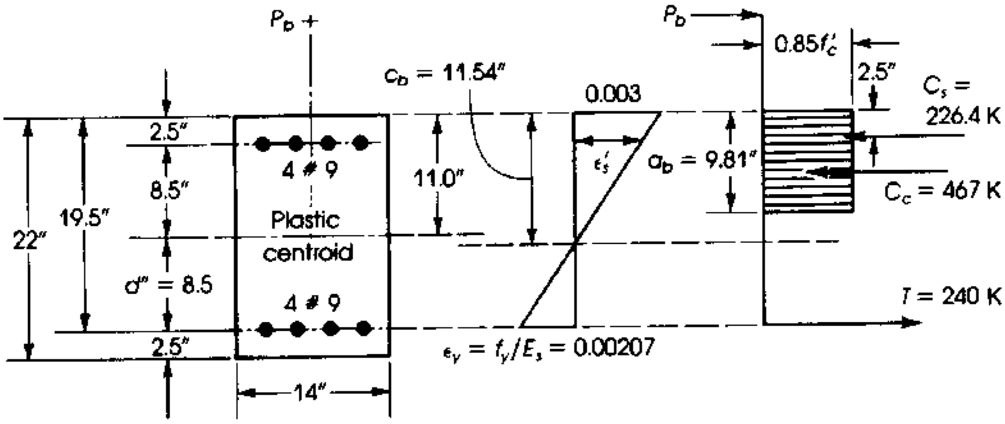


Figure 11.6 Example 11.2: balanced condition.

which exceeds ε_y of 0.00207; thus, compression steel yields. Or check that

$$f'_s = 87\left(\frac{c-d''}{c}\right) \le f_y$$

$$f'_s = \frac{87(11.54-2.5)}{11.54} = 68 \text{ ksi} > 60 \text{ ksi}$$

Then $f'_s = f_y = 60$ ksi.

4. Calculate the forces acting on the section:

$$C_c = 0.85\, f'_c ab = 0.85 \times 4 \times 9.81 \times 14 = 467 \text{ K}$$

$$T = A_s f_y = 4 \times 60 = 240 \text{ K}$$

$$C_s = A'_s(f_y - 0.85\, f'_c) = 4(60-3.4) = 226.4 \text{ K}$$

5. Calculate P_b and e_b:

$$P_b = C_c + C_s - T = 467 + 226.4 - 240 = 453.4 \text{ K}$$

From Eq. 11.7,

$$M_b = P_b e_b = C_c\left(d - \frac{a}{2} - d''\right) + C_s(d - d' - d'') + Td''$$

The plastic centroid is at the centroid of the section, and $d'' = 8.5$ in.

$$M_b = 453.4 e_b = 467\left(19.5 - \frac{9.81}{2} - 8.5\right) + 226.4(19.5 - 2.5 - 8.5) + 240 \times 8.5$$

$$= 6810.8 \text{ K·in.} = 567.6 \text{ K·ft}$$

$$e_b = \frac{M_b}{P_b} = \frac{6810.8}{453.4} = 15.0 \text{ in.}$$

6. For a balanced condition, $\phi = 0.65$, $\phi P_b = 294.7$ K, and $\phi M_b = 368.9$ K·ft.

11.6 COLUMN SECTIONS UNDER ECCENTRIC LOADING

For the two cases when compression or tension failure occurs, two basic equations of equilibrium can be used in the analysis of columns under eccentric loadings: (1) the sum of the horizontal or vertical forces = 0, and (2) the sum of moments about any axis = 0. Referring to Fig. 11.7, the following equations may be established.

1.
$$P_n - C_c - C_s + T = 0 \tag{11.10}$$

where

$C_c = 0.85\, f'_c ab$

$C_s = A'_s(f'_s - 0.85\, f'_c)$ (If compression steel yields, then $f'_s = f_y$.)

$T = A_s f_s$ (If tension steel yields, then $f_s = f_y$.)

2. Taking moments about A_s,

$$P_n e' - C_c\left(d - \frac{a}{2}\right) - C_s(d - d') = 0 \tag{11.11}$$

Reinforced concrete tied columns under construction. The two columns are separated by an expansion joint.

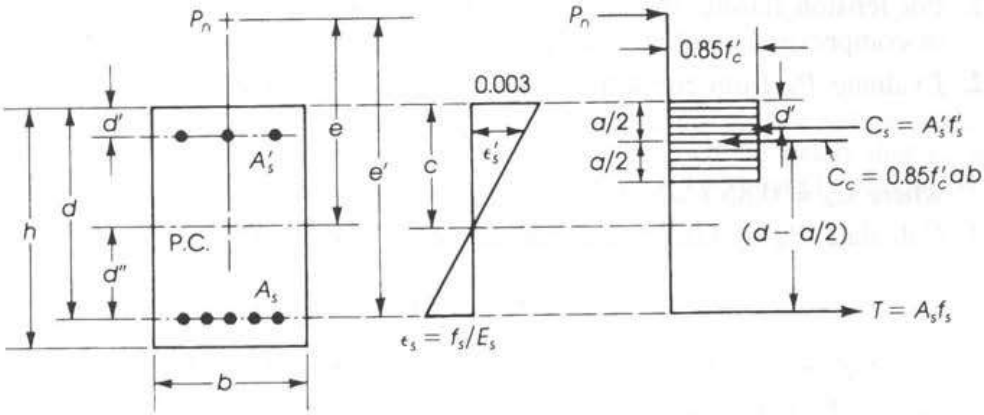


Figure 11.7 General case, rectangular section.

The quantity $e' = e + d''$, and $e' = (e + d - h/2)$ for symmetrical reinforcement (d'' is the distance from the plastic centroid to the centroid of the tension steel.)

$$P_n = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] \tag{11.12}$$

Taking moments about C_c,

$$P_n\left[e' - \left(d - \frac{a}{2}\right)\right] - T\left(d - \frac{a}{2}\right) - C_s\left(\frac{a}{2} - d'\right) = 0 \tag{11.13}$$

$$P_n = \frac{T\left(d - \frac{a}{2}\right) + C_s\left(\frac{a}{2} - d'\right)}{\left(e' + \frac{a}{2} - d\right)} \tag{11.14}$$

If $A_s = A'_s$ and $f_s = f'_s = f_y$, then

$$P_n = \frac{A_s f_y(d-d')}{\left(e' + \frac{a}{2} - d\right)} = \frac{A_s f_y(d-d')}{\left(e - \frac{h}{2} + \frac{a}{2}\right)} \tag{11.15}$$

$$A_s = A'_s = \frac{P_n\left(e - \frac{h}{2} + \frac{a}{2}\right)}{f_y(d-d')} \tag{11.16}$$

11.7 STRENGTH OF COLUMNS FOR TENSION FAILURE

When a column is subjected to an eccentric force with large eccentricity e, tension failure is expected. The column section fails due to the yielding of steel and crushing of concrete when the strain in the steel exceeds $\varepsilon_y(\varepsilon_y = f_y/E_s)$. In this case the nominal strength, P_n, will be less than P_b or the eccentricity, $e = M_n/P_n$, is greater than the balanced eccentricity, e_b. Because it is difficult in some cases to predict if tension or compression controls, it can be assumed (as a guide) a tension failure will occur when $e > d$. This assumption should be checked later.

The general equations of equilibrium, Eqs. 11.10 and 11.11, may be used to calculate the nominal strength of the column. This is illustrated in steps as follows:

1. For tension failure, the tension steel yields and its stress is $f_s = f_y$. Assume that stress in compression steel is $f'_s = f_y$.
2. Evaluate P_n from equilibrium conditions (Eq. 11.10):

$$P_n = C_c + C_s - T$$

   where $C_c = 0.85\, f'_c ab$, $C_s = A'_s(f_y - 0.85\, f'_c)$, and $T = A_s f_y$.
3. Calculate P_n by taking moments about A_s (Eq. 11.11):

$$P_n \cdot e' = C_c\left(d - \frac{a}{2}\right) + C_s(d - d')$$

   where $e' = e + d''$ and $e' = e + d - h/2$ when $A_s = A'_s$.
4. Equate P_n from steps 2 and 3:

$$C_c + C_s - T = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right]$$

   This is a second-degree equation in a. Substitute the values of C_c, C_s, and T and solve for a.
5. The second-degree equation, after the substitution of C_c, C_s, and T, is reduced to the following equation:

$$Aa^2 + Ba + C = 0$$

   where

$$A = 0.425\, f'_c b$$
$$B = 0.85\, f'_c b(e' - d) = 2A(e' - d)$$
$$C = A'_s(f'_s - 0.85\, f'_c)(e' - d + d') - A_s f_y e'$$

Solve for a to get

$$a = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Note that the value of $(f'_s - 0.85 f'_c)$ must be a positive value. If this value is negative, then let $(f'_s - 0.85 f'_c) = 0$.

6. Substitute a in the equation of step 2 to obtain P_n. The moment M_n can be calculated:

$$M_n = P_n \cdot e$$

7. Check if compression steel yields as assumed. If $\varepsilon'_s \geq \varepsilon_y$, then compression steel yields; otherwise, $f'_s = E_s \varepsilon'_s$. Repeat steps 2 through 5. Note that $\varepsilon'_s = [(c - d'/c]\,0.003$, $\varepsilon_y = f_y/E_s$ and $c = a\beta_1$.
8. Check that tension controls. Tension controls when $e > e_b$ or $P_n < P_b$. Example 11.3 illustrates this procedure.
9. The net tensile strain, ε_t, in this section, is normally greater than the limit strain of 0.002 for a compression-controlled section (Fig. 11.4). Consequently, the value of the strength reduction factor, ϕ, will vary between 0.65 (or 0.75) and 0.90. Equation 11.1 or 11.2 can be used to calculate ϕ.

Example 11.3

Determine the nominal compressive strength, P_n, for the section given in Example 11.2 if $e = 20$ in. (See Fig. 11.8.)

Solution

1. Because $e = 20$ in. is greater than $d = 19.5$ in., assume that tension failure condition controls (to be checked later). The strain in the tension steel, ε_s, will be greater than ε_y and its stress is f_y. Assume that compression steel yields $f'_s = f_y$, which should be checked later.
2. From the equation of equilibrium (Eq. 11.10),

$$P_n = C_c + C_s - T$$

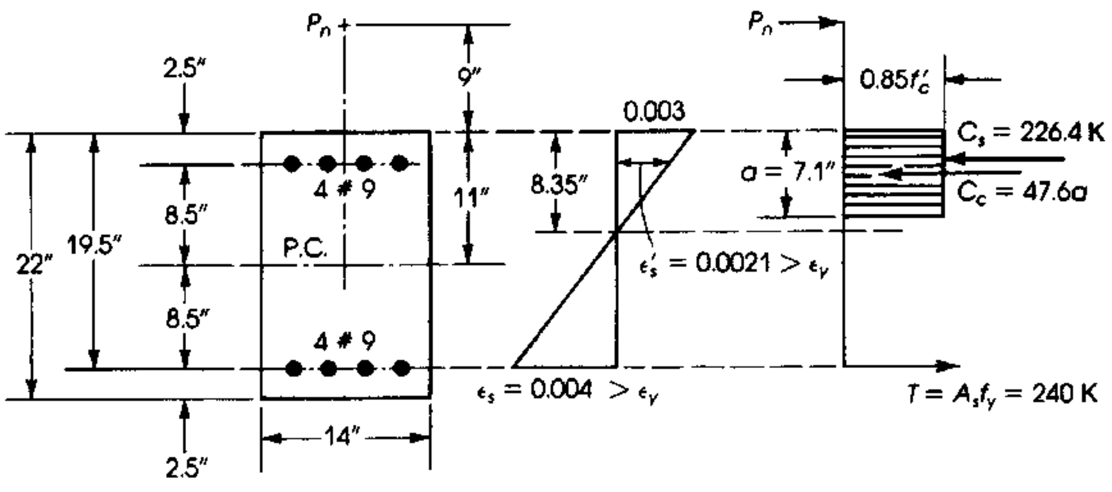


Figure 11.8 Example 11.3: tension failure.

where

$$C_c = 0.85\, f_c'ab = 0.85 \times 4 \times 14a = 47.6a$$

$$C_s = A_s'(f_y - 0.85\, f_c') = 4(60 - 0.85 \times 4) = 226.4 \text{ K}$$

$$T = A_s f_y = 4 \times 60 = 240 \text{ K}$$

$$P_n = 47.6a + 226.4 - 240 = (47.6a - 13.6) \qquad \text{(I)}$$

3. Taking moments about A_s (Eq. 11.12),

$$P_n = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right]$$

Note that for the plastic centroid at the center of the section, $d'' = 8.5$ in.

$$e' = e + d'' = 20 + 8.5 = 28.5 \text{ in.}$$

$$P_n = \frac{1}{28.5}\left[47.6a\left(19.5 - \frac{a}{2}\right) + 226.4 \times 17\right]$$

$$P_n = 32.56a - 0.835a^2 + 135.0$$

4. Equating Eqs. I and II,

$$P_n = (47.6a - 13.6) = 32.56a - 0.835a^2 + 135.0 \qquad \text{(II)}$$

or

$$a^2 + 18a - 178.0 = 0 \qquad a = 7.1 \text{ in.}$$

5. From Eq. I:

$$P_n = 47.6 \times 7.1 - 13.6 = 324.4 \text{ K}$$

$$M_n = P_n e = 324.4 \times \frac{20}{12} = 540.67 \text{ K·ft}$$

6. Check if compression steel has yielded:

$$c = \frac{a}{0.85} = \frac{7.1}{0.85} = 8.35 \text{ in.} \qquad \varepsilon_y = \frac{60}{29{,}000} = 0.00207$$

$$\varepsilon_s' = \frac{(8.35 - 2.5)}{8.35}(0.003) = 0.0021 > \varepsilon_y$$

Compression steel yields. Check strain in tension steel:

$$\varepsilon_s = \left(\frac{19.5 - 8.35}{8.35}\right) \times 0.003 = 0.004 > \varepsilon_y$$

If compression steel does not yield, use f_s' as calculated from $f_s' = \varepsilon_s' E_s$ and revise the calculations.

7. Calculate ϕ: Since $\varepsilon_t = 0.004$, the section is in the transition region.

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.817$$

$$\phi P_n = 0.817(324.4) = 264.9 \text{ K}$$

$$\phi M_n = 0.817(540.67) = 441.7 \text{ K·ft}$$

8. Because $e = 20\,\text{in.} > e_b = 15\,\text{in.}$ (Example 11.2), there is a tension failure condition.
9. The same results can be obtained using the values of A, B, and C given earlier.

$$Aa^2 + Ba + C = 0$$

$$A = 0.425\, f'_c b = 0.425(4)(14) = 23.8$$

$$B = 2A(e' - d) = 2(23.8)(28.5 - 19.5) = 428.4$$

$$C = 4(60 - 0.85 \times 4)(28.5 - 19.5 + 2.5) - 4(60)(28.5)$$

$$= -4236.4$$

Solve for a to get $a = 7.1\,\text{in.}$ and $P_n = 324.4\,\text{K}$.

11.8 STRENGTH OF COLUMNS FOR COMPRESSION FAILURE

If the compressive applied force, P_n, exceeds the balanced force, P_b, or the eccentricity, $e = M_n/P_n$, is less than e_b, compression failure is expected. In this case compression controls, and the strain in the concrete will reach 0.003, whereas the strain in the steel is less than ε_y (Fig. 11.9). A large part of the column will be in compression. The neutral axis moves toward the tension steel, increasing the compression area, and therefore the distance to the neutral axis c is greater than the balanced c_b (Fig. 11.9).

Because it is difficult to predict compression or tension failure whenever a section is given, compression failure can be assumed when $e < 2d/3$, which should be checked later. The nominal load strength, P_n, can be calculated using the principles of statics. The analysis of column sections for compression failure can be achieved using Eqs. 11.10 and 11.11 given earlier and one of the following solutions.

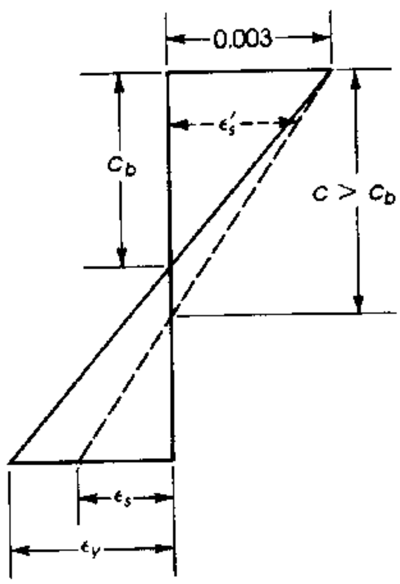


Figure 11.9 Strain diagram when compression controls. When $\varepsilon_s < \varepsilon_y$, $c > c_b$ and $\varepsilon'_s \geq \varepsilon_y$.

11.8.1 Trial Solution

This solution can be summarized as follows:

1. Calculate the distance to the neutral axis for a balanced section, c_b:

$$c_b = \left(\frac{87d_t}{87 + f_y}\right) \tag{11.17}$$

where f_y is in ksi.

2. Evaluate P_n using equilibrium conditions:

$$P_n = C_c + C_s - T \tag{11.18}$$

3. Evaluate P_n by taking moments about the tension steel, A_s:

$$P_n \cdot e' = C_c\left(d - \frac{a}{2}\right) + C_s(d - d') \tag{11.19}$$

where $e' = e + d - h/2$ when $A_s = A'_s$ or $e' = e + d''$ in general, $C_c = 0.85 f'_c ab$, $C_s = A'_s(f'_s - 0.85 f'_c)$, and $T = A_s f_s$.

4. Assume a value for c such that $c > c_b$ (calculated in step 1). Calculate $a = \beta_1 c$. Assume $f'_s = f_y$.
5. Calculate f_s based on the assumed c:

$$f_s = \varepsilon_s E_s = 87\left(\frac{d_t - c}{c}\right) \text{ ksi} \leq f_y$$

6. Substitute the preceding values in Eq. 11.10 to calculate P_{n1} and in Eq. 11.11 to calculate P_{n2}. If P_{n1} is close to P_{n2}, then choose the smaller or average of P_{n1} and P_{n2}. If P_{n1} is not close to P_{n2}, assume a new c or a and repeat the calculations starting from step 4 until P_{n1} is close to P_{n2}. (1% is quite reasonable.)
7. Check that compression steel yields by calculating $\varepsilon'_s = 0.003[(c - d')/c]$ and comparing it with $\varepsilon_y = f_y/E_s$. When $\varepsilon'_s \geq \varepsilon_y$, compression steel yields; otherwise, $f'_s = \varepsilon'_s E_s$ or, directly,

$$f'_s = 87\left(\frac{c - d'}{c}\right) \leq f_y \text{ ksi}$$

8. Check that $e < e_b$ or $P_n > P_b$ for compression failure. Example 11.4 illustrates the procedure.
9. The net tensile strain, ε_t, in the section is normally less than 0.002 for compression-controlled sections (Fig. 11.4). Consequently, the strength reduction factor (ϕ) = 0.65 (or 0.70 for spiral columns).

Example 11.4

Determine the nominal compressive strength, P_n, for the section given in Example 11.2 if $e = 10$ in. (See Fig. 11.10.)

Solution

1. Because $e = 10$ in. < (2/3)d = 13 in., assume compression failure. This assumption will be checked later. Calculate the distance to the neutral axis for a balanced section, c_b:

$$c_b = \frac{87}{87 + f_y} d_t = \frac{87}{87 + 60}(19.5) = 11.54 \text{ in.}$$

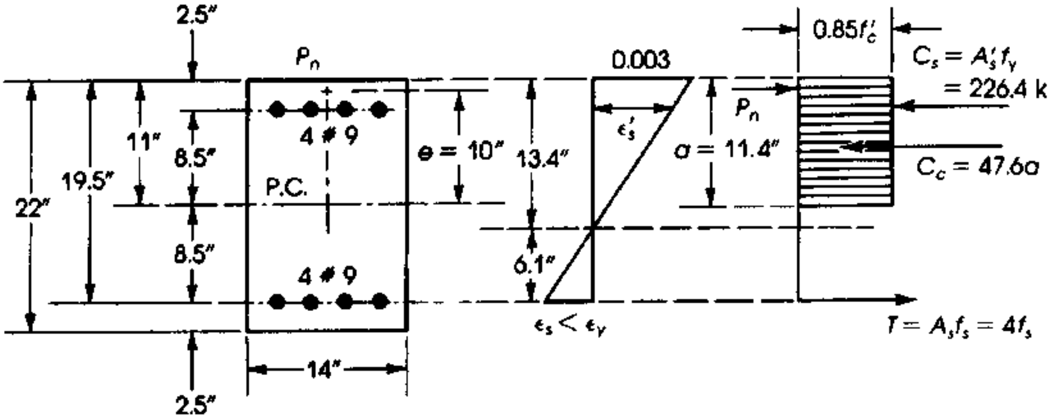


Figure 11.10 Example 11.4: compression controls.

2. From the equations of equilibrium,

$$P_n = C_c + C_s - T \tag{11.10}$$

where

$$C_c = 0.85\, f'_c ab = 0.85 \times 4 \times 14a = 47.6a$$

$$C_s = A'_s(f_y - 0.85\, f'_c) = 4(60 - 0.85 \times 4) = 226.4 \text{ K}$$

Assume compression steel yields. (This assumption will be checked later.)

$$T = A_s f_s = 4f_s \qquad (f_s < f_y) \tag{I}$$

$$P_n = 47.6a + 226.4 - 4f_s$$

3. Taking moments about A_s,

$$P_n = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] \tag{11.11}$$

The plastic centroid is at the center of the section and $d'' = 8.5$ in.

$$e' = e + d'' = 10 + 8.5 = 18.5 \text{ in.}$$

$$P_n = \frac{1}{18.5}\left[47.6a\left(19.5 - \frac{a}{2}\right) + 226.4(19.5 - 2.5)\right] \tag{II}$$

$$P_n = 50.17a - 1.29a^2 + 208$$

4. Assume $c = 13.45$ in., which exceeds c_b (11.54 in.).

$$a = 0.85 \times 13.45 = 11.43 \text{ in.}$$

Substitute $a = 11.43$ in Eq. II:

$$P_{n1} = 50.17 \times 11.43 - 1.29(11.43)^2 + 208 = 612.9 \text{ K}$$

5. Calculate f_s from the strain diagram when $c = 13.45$ in.

$$f_s = \left(\frac{19.5 - 13.45}{13.45}\right)87 = 39.13 \text{ ksi} \quad \varepsilon_s = \varepsilon_t = f_s/E_s = 0.00135$$

6. Substitute $a = 11.43$ in. and $f_s = 39.13$ ksi in Eq. I to calculate P_{n2}:

$$P_{n2} = 47.6(11.43) + 226.4 - 4(39.13) = 613.9 \text{ K}$$

which is very close to the calculated P_{n1} of 612.9 K (less than 1% difference).

$$M_n = P_n \cdot e = 612.9\left(\frac{10}{12}\right) = 510.8 \text{ K·ft}$$

7. Check if compression steel yields. From the strain diagram,

$$\varepsilon_s' = \frac{13.45 - 2.5}{13.45}(0.003) = 0.00244 > \varepsilon_y = 0.00207$$

Compression steel yields, as assumed.

8. $P_n = 612.9$ K is greater than $P_b = 453.4$ K, and $e = 10$ in. $< e_b = 15$ in., both calculated in the previous example, indicating that compression controls, as assumed. Note that it may take a few trials to get P_{n1} close to P_{n2}.
9. Calculate ϕ:

$$d_t = d = 19.5 \text{ in.} \qquad c = 13.45 \text{ in.}$$

ε_t (at the tension steel level) $= 0.003(d_t - c)/c$.

$$\varepsilon_t = 0.003(19.5 - 13.45)/13.45 = 0.00135$$

Since $\varepsilon_t < 0.002$, then $\phi = 0.65$.

$$\phi P_n = 0.65(612.9) = 398.4 \text{ K}$$

$$\phi M_n = 0.65(510.8) = 332 \text{ K·ft.}$$

11.8.2 Numerical Analysis Solution

The analysis of columns when compression controls can also be performed by reducing the calculations into one cubic equation in the form

$$Aa^3 + Ba^2 + Ca + D = 0$$

and then solving for a by a numerical method, or a can be obtained directly by using one of many inexpensive scientific calculators with built-in programs that are available. From the equations of equilibrium,

$$\begin{aligned} P_n &= C_c + C_s - T \\ &= (0.85 f_c'ab) + A_s'(f_y - 0.85 f_c') - A_s f_s \end{aligned} \tag{11.10}$$

Taking moments about the tension steel, A_s,

$$\begin{aligned} P_n &= \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] \\ &= \frac{1}{e'}\left[0.85 f_c'ab\left(d - \frac{a}{2}\right) + A_s'(f_y - 0.85 f_c')(d - d')\right] \end{aligned} \tag{11.11}$$

From the strain diagram,

$$\varepsilon_s = \left(\frac{d_t - c}{c}\right)(0.003) = \frac{\left(d - \dfrac{a}{\beta_1}\right)}{\dfrac{a}{\beta_1}}(0.003)$$

The stress in the tension steel is

$$f_s = \varepsilon_s E_s = 29{,}000\varepsilon_s = \frac{87}{a}(\beta_1 d - a)$$

Substituting this value of f_s in Eq. 11.10 and equating Eqs. 11.10 and 11.11 and simplifying gives

$$\left(\frac{0.85\,f_c'b}{2}\right)a^3 + [0.85f_c'b(e'-d)]a^2 + [A_s'(f_y - 0.85\,f_c')(e'-d+d') + 87A_se']a$$
$$-\,87\,A_se'\beta_1 d = 0$$

This is a cubic equation in terms of a:

$$Aa^3 + Ba^2 + Ca + D = 0$$

where

$$A = \frac{0.85\,f_c'b}{2}$$
$$B = 0.85\,f_c'b(e'-d)$$
$$C = A_s'(f_y - 0.85\,f_c')(e'-d+d') + 87\,A_se'$$
$$D = -87\,A_se'\beta_1 d$$

Once the values of A, B, C, and D are calculated, a can be determined by trial or directly by a scientific calculator. Also, the solution of the cubic equation can be obtained by using the well known Newton-Raphson method. This method is very powerful for finding a root of $f(x) = 0$. It involves a simple technique, and the solution converges rapidly by using the following steps:

1. Let $f(a) = Aa^3 + Ba^2 + Ca + D$, and calculate A, B, C, and D.
2. Calculate the first derivative of $f(a)$:

$$f'(a) = 3Aa^2 + 2Ba + C$$

3. Assume any initial value of a, say, a_0, and compute the next value:

$$a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}$$

4. Use the obtained value a_1 in the same way to get

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$$

5. Repeat the same steps to get the answer up to the desired accuracy. In the case of the analysis of columns when compression controls, the value a is greater than the balanced $a(a_b)$. Therefore, start with $a_0 = a_b$ and repeat twice to get reasonable results.

Example 11.5

Repeat Example 11.4 using numerical solution.

Solution

1. Calculate A, B, C, and D and determine $f(a)$.

$$A = 0.85 \times 4 \times \frac{14}{2} = 23.8$$

$$B = 0.85 \times 4 \times 14(18.5 - 19.5) = -47.6$$

$$C = 4(60 - 0.85 \times 4)(18.5 - 19.5 + 2.5) + 87 \times 4 \times 18.5$$
$$= 6777.6$$
$$D = -87 \times 4 \times 18.5 \times (0.85 \times 19.5) = -106{,}710$$
$$f(a) = 23.8a^3 - 47.6a^2 + 6777.6a - 106{,}710$$

2. Calculate the first derivative:

$$f'(a) = 71.4a^2 - 95.2a + 6777.6$$

3. Let $a_0 = a_b = 9.81$ in. For a balanced section, $c_b = 11.54$ in. and $a_b = 9.81$ in.

$$a_1 = 9.81 - \frac{f(9.81)}{f'(9.81)} = 9.81 - \frac{-22{,}334}{12{,}715} = 11.566 \text{ in.}$$

4. Calculate a_2:

$$a_2 = 11.566 - \frac{f(11.566)}{f'(11.566)} = 11.566 - \frac{2136}{15{,}228} = 11.43 \text{ in.}$$

This value of a is similar to that obtained earlier in Example 11.3. Substitute the value of a in Eq. 11.10 or 11.11 to get $P_n = 612.9$ K.

11.8.3 Approximate Solution

An approximate equation was suggested by Whitney to estimate the nominal compressive strength of short columns when compression controls, as follows [15]:

$$P_n = \frac{bhf'_c}{\frac{3he}{d^2} + 1.18} + \frac{A'_s f_y}{\frac{e}{(d - d')} + 0.5} \tag{11.17}$$

This equation can be used only when the reinforcement is symmetrically placed in single layers parallel to the axis of bending.

A second approximate equation was suggested by Hsu [16]:

$$\frac{P_n - P_b}{P_o - P_b} + \left(\frac{M_n}{M_b}\right)^{1.5} = 1.0 \tag{11.18}$$

where

P_n = nominal axial strength of the column section
P_b, M_b = nominal load and moment of the balanced section
M_n = nominal bending moment = $P_n \cdot e$
P_o = nominal axial load at $e = 0$
= $0.85 f'_c(A_g - A_{st}) + A_{st} f_y$
A_g = gross area of the section = bh
A_{st} = total area of nonprestressed longitudinal reinforcement

Example 11.6

Determine the nominal compressive strength, P_n, for the section given in Example 11.4 by Eqs. 11.17 and 11.18 using the same eccentricity, $e = 10$ in., and compare results.

Solution

1. Solution by Whitney equation (Eq. 11.29):
   a. Properties of the section shown in Fig. 11.10 are $b = 14$ in., $h = 22$ in., $d = 19.5$ in., $d' = 2.5$ in., $A'_s = 4.0$ in.^{2}, and $(d - d') = 17$ in.

b. Apply the Whitney equation:

$$P_n = \frac{14 \times 22 \times 4}{(3 \times 22 \times 10)/(19.5)^2 + 1.18} = \frac{4 \times 60}{(\frac{10}{17}) + 0.5} = 643 \text{ K}$$

$$\phi P_n = 0.65 P_n = 418 \text{ K}$$

c. P_n calculated by the Whitney equation is not a conservative value in this example, and the value of $P_n = 643$ K is greater than the more accurate value of 612.9 K calculated by statics in Example 11.4.

2. Solution by Hsu equation (Eq. 11.18):

a. For a balanced condition, $P_b = 453.4$ K and $M_b = 6810.8$ K·in. (Example 11.2).

b. $P_0 = 0.85 f'_c(A_g - A_{st}) + A_{st} f$
$= 0.85(4)(14 \times 22 - 8) + 8(60) = 1500$ K

c. $\dfrac{P_n - 453.4}{1500 - 453.4} + \left(\dfrac{10 P_n}{6810.8}\right)^{1.5} = 1$

Multiply by 1000 and solve for P_n.

$$0.9555 P_n + 0.05626 P_n^{1.5} = 1433.2 \text{ K}$$

By trial, $P_n = 611$ K, which is very close to 612.9 K, as calculated by statics.

11.9 INTERACTION DIAGRAM EXAMPLE

In Example 11.2, the balanced loads P_b, M_b, and e_b were calculated for the section shown in Fig. 11.6 ($e_b = 15$ in.). Also, in Examples 11.3 and 11.4, the load capacity of the same section was calculated for the case when $e = 20$ in. (tension failure) and when $e = 10$ in. (compression failure). These values are shown in Table 11.1.

To plot the load–moment interaction diagram, different values of ϕP_n and ϕM_n were calculated for various e values that varied between $e = 0$ and $e =$ maximum for the case of

Table 11.1 Summary of the Load Strength of the Column Section in the Previous Examples

e (in.)	a (in.)	ϕ	P_n (K)	ϕP_n (K)	ϕM_n (K·ft)	Notes
0	—	0.65	1500	975	0.0	ϕP_{n0}
2.25	19.39	0.65	1200	780	146.3	0.8 ϕP_{n0}
4	16.82	0.65	1018	661.7	220.6	Compression
6	14.19	0.65	843.3	548.1	274.0	Compression
10*	11.43	0.65	612.9	398.4	332.0	Compression
12	10.63	0.65	538.0	349.7	349.7	Compression
15*	9.81	0.65	453.4	294.7	368.9	Balanced
20*	7.10	0.81	324.4	263.4	439.0	Transition
30	5.06	0.90	189.4	170.5	426.2	Tension
50	4.01	0.90	100.6	90.5	377.2	Tension
80	3.59	0.90	58.8	52.9	352.0	Tension
P.M.	3.08	0.90	0.0	0.0	352.0	Tension
P.M.	3.08	0.65	0.0	0.0	254.2	P.M. (X)

*= values calculated in Examples 11.2,11.3, and 11.4.
P.M. = pure moment.
X = Not applicable, for comparison only.

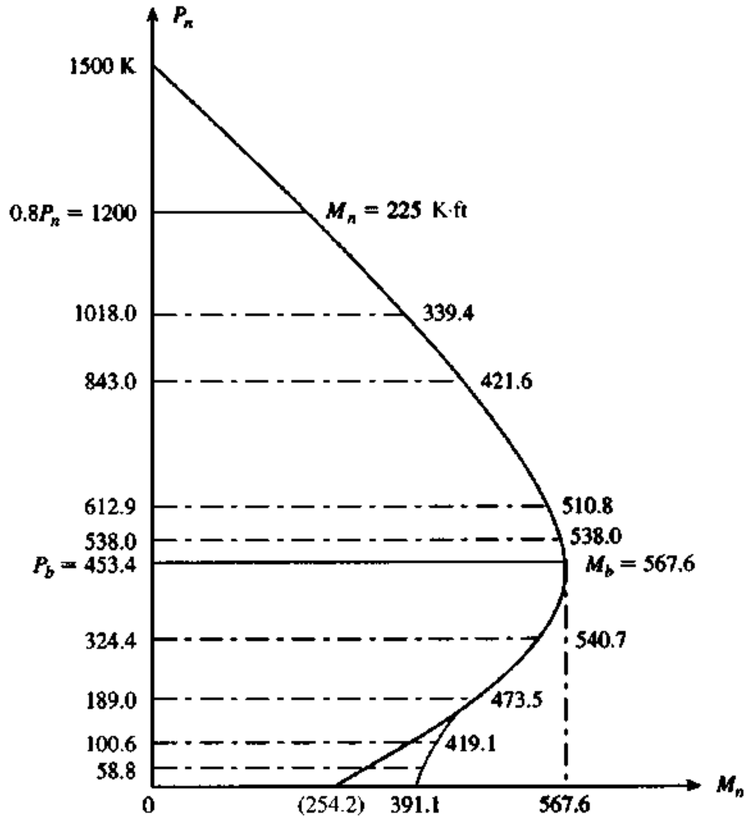


Figure 11.11 Interaction diagram of the column section shown in Fig. 11.10.

pure moment when $P_n = 0$. These values are shown in Table 11.1. The interaction diagram is shown in Fig. 11.11. The load $\phi P_{n0} = 975\,\text{K}$ represents the theoretical axial load when $e = 0$, whereas $0.8\ \phi P_{n0} = 780\,\text{K}$ represents the maximum axial load allowed by the ACI Code based on minimum eccentricity. Note that for compression failure, $e < e_b$ and $P_n > P_b$, and for tension failure, $e > e_b$ and $P_n < P_b$. The last two cases in the table represent the pure moment (P.M.) or beam-action case for $\phi = 0.9$ and $\phi = 0.65$ ($M_n = 391$ K·ft). To be consistent with the design of beams due to bending moments, the ACI Code allows the use of $\phi = 0.9$ with pure moment, so $\phi M_n = 352$ K·ft instead of 254.2 K·ft. Also note that ϕ varies between 0.65 and 0.9 according to Eq. 11.2 for tied columns. Note that $M_n = 391.1$ K·ft.

11.10 RECTANGULAR COLUMNS WITH SIDE BARS

In some column sections, the steel reinforcement bars are distributed around the four sides of the column section. The side bars are those placed on the sides along the depth of the section in addition to the tension and compression steel, A_s and A'_s, and can be denoted by A_{ss} (Fig. 11.12). In this case the same procedure explained earlier can be applied, taking into consideration the strain variation along the depth of the section and the relative force in each side bar either in the compression or tension zone of the section. These are added to those of C_c, C_s, and T to determine P_n Equation 11.10 becomes

$$P_n = C_c + \sum C_s - \Sigma T \tag{11.10a}$$

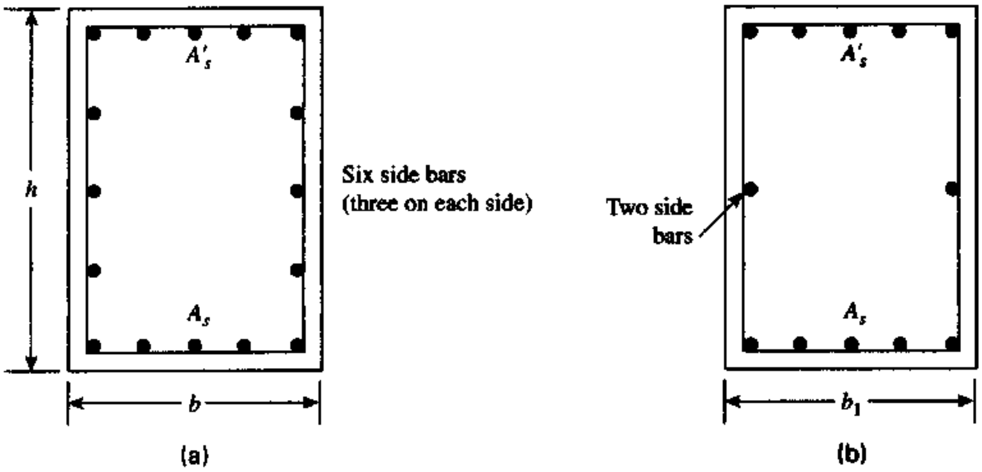


Figure 11.12 Side bars in rectangular sections: (*a*) six side bars and (*b*) two side bars (may be neglected).

Example 11.7 explains this analysis. Note that if the side bars are located near the neutral axis (Fig. 11.12*b*), the strains—and, consequently, the forces—in these bars are very small and can be neglected. Those bars close to A_s and A'_s have appreciable force and increase the load capacity of the section.

Example 11.7

Determine the balanced load, P_b moment, M_b, and e_b for the section shown in Fig. 11.13. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

The balanced section is similar to Example 11.2. Given: $b = h = 22$ in., $d = 19.5$ in., $d' = 2.5$ in., $A_s = A'_s = 6.35$ in^2. (five no. 10 bars), and six no. 10 side bars (three on each side).

1. Calculate the distance to the neutral axis:

$$c_b = \left(\frac{87}{87 + f_y}\right) d_t = \left(\frac{87}{87 + 60}\right) 19.5 = 11.54 \text{ in.}$$

$$a_b = 0.85(11.54) = 9.81 \text{ in.}$$

2. Calculate the forces in concrete and steel bars; refer to Fig. 11.13*a*. In the compression zone, $C_c = 0.85 f'_c ab = 0.85(4)(9.81)(22) = 733.8$ K.

$$f'_s = 87\left(\frac{c - d'}{c}\right) = 87\left(\frac{11.54 - 2.5}{11.54}\right) = 68.15 \text{ ksi} > 60 \text{ ksi}$$

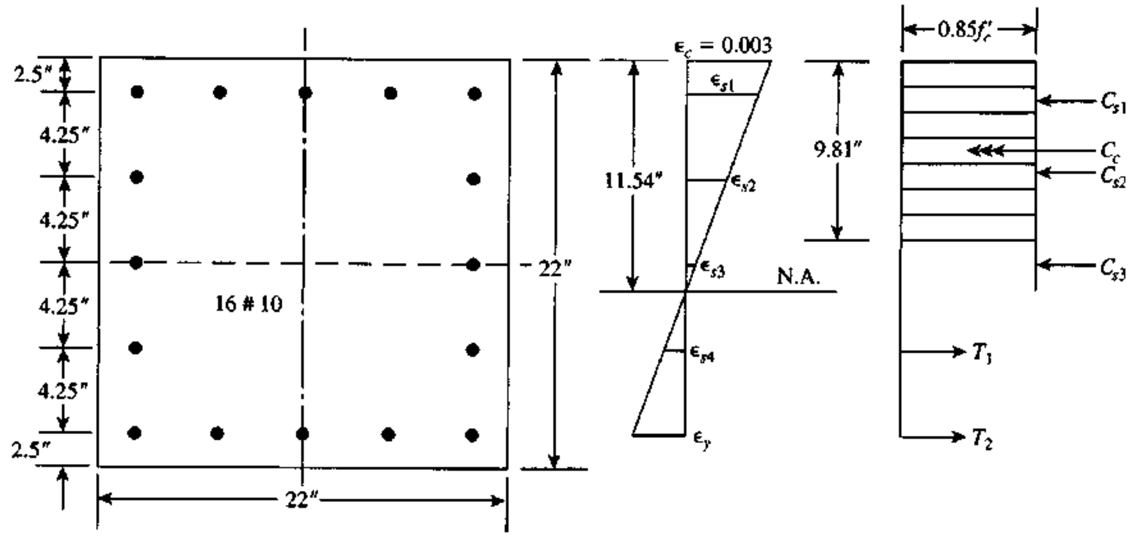
Then $f'_s = 60$ ksi.

$$C_{s1} = A'_s(f_y - 0.85 f'_c) = 6.35(60 - 0.85 \times 4) = 359.4 \text{ K}$$

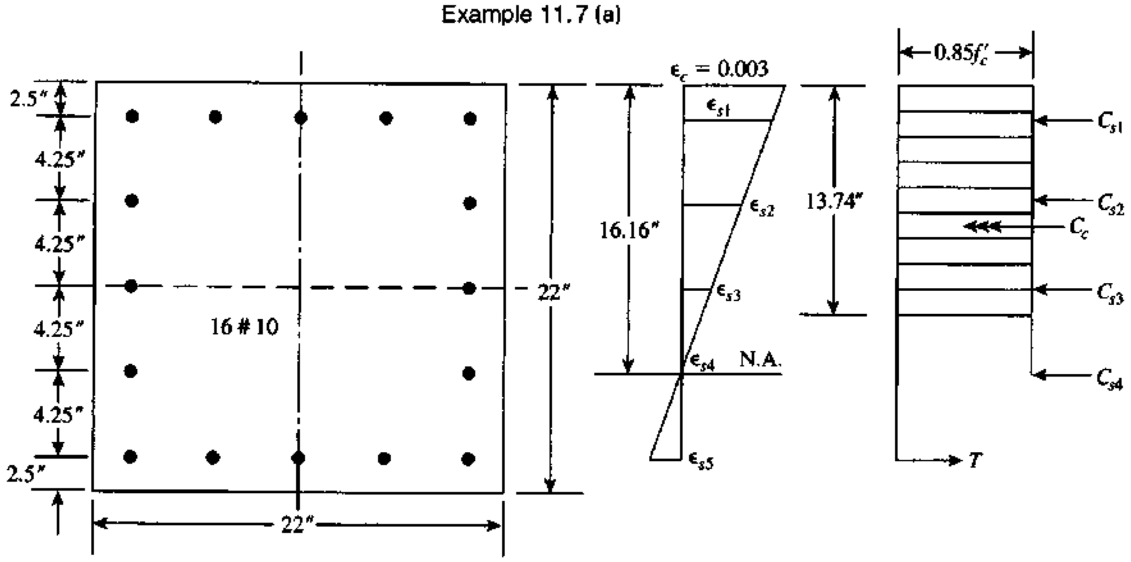
$$f_{s2} = 87\left(\frac{11.54 - 2.5 - 4.25}{11.54}\right) = 36.11 \text{ ksi}$$

$$C_{s2} = 2(1.27)(36.11 - 0.85 \times 4) = 83.1 \text{ K}$$

Similarly, $f_{s3} = 4.07$ ksi and $C_{s3} = 2(1.27)(4.07 - 0.85 \times 4) = 1.7$ K.



Example 11.7 (a)



Example 11.8 (b)

Figure 11.13 Example 11.7: (*a*) balanced section. Example 11.8: (*b*) for compression failure, $e = 6$ in.

In the tension zone,

$$\varepsilon_{s4} = 964.50 \times 10^{-6} \qquad f_{s4} = 28 \text{ ksi}$$

$$T_1 = 2(1.27)(28) = 71 \text{ K}$$

$$T_2 = A_s f_y = 6.35(60) = 381 \text{ K}$$

3. Calculate $P_b = C_c + \Sigma C_s - \Sigma T$.

$$P_b = 733.8 + (359.4 + 83.1 + 1.7) - (71 + 381)$$

$$= 726 \text{ K}$$

4. Taking moments about the plastic centroid,

$$M_b = 733.8(6.095) + 359.4(8.5) + 83.1(4.25) + 71(4.25) + 381(8.5)$$

$$= 11{,}421 \text{ K·in.} = 952 \text{ K·ft}$$

$$e_b = \frac{M_b}{P_b} = 15.735 \text{ in.}$$

5. Determine ϕ: For a balanced section, $\varepsilon_t = \varepsilon_y = 0.002$, $\phi = 0.65$,

$$\phi P_b = 0.65\, P_b = 472 \text{ K, and } \phi M_b = 0.65\, M_b = 618.8 \text{ K·ft.}$$

Example 11.8

Repeat the previous example when $e = 6.0$ in.

Solution

1. Because $e = 6$ in. $< e_b = 15.735$ in., this is a compression failure condition. Assume $c = 16.16$ in. (by trial) and $a = 0.85(16.16) = 13.74$ in. (Fig. 11.13*b*).
2. Calculate the forces in concrete and steel bars:

$$C_c = 0.85(4)(13.74)(22) = 1027.75 \text{ K}$$

In a similar approach to the balanced case, $f_{s1} = 60$ ksi and $C_{s1} = 359.41$.

$$f_{s2} = 50.66 \text{ ksi} \quad C_{s2} = 120.0 \text{ K}$$

$$f_{s3} = 27.78 \text{ ksi} \quad C_{s3} = 61.92 \text{ K}$$

$$f_{s4} = 4.9 \text{ ksi} \quad C_{s4} = 3.81 \text{ K}$$

$$f_{s5} = 18 \text{ ksi} \quad T = 6.35(18) = 114.2 \text{ K}$$

3. Calculate $P_n = C_c + \Sigma C_s - \Sigma T = 1458.7$ K.

$$M_n = P_n \cdot e = 729.35 \text{ K·ft} \quad (e = 6 \text{ in.})$$

4. Check P_n by taking moments about A_s,

$$P_n = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_{s1}(d - d') + C_{s2}(d - d' - s) + C_{s3}(d - d' - 2s) + C_{s4}(d - d' - 3s)\right]$$

$$e' = e + d - \frac{h}{2} = 6 + 19.5 - \frac{22}{2} = 14.5 \text{ in.}$$

s = distance between side bars

= 4.25 in. (s = constant in this example.)

$$P_n = \frac{1}{14.5}\left[1027.75\left(19.5 - \frac{13.74}{2}\right) + 359.41(17) + 120(17 - 4.25) + 61.92(17 - 8.5) + 3.81(17 - 12.75)\right] = 1459 \text{ K}$$

5. Calculate ϕ:

$$d_t = d = 19.5 \text{ in.} \quad c = 16.16 \text{ in.}$$

ε_t (at the tension steel level) $= 0.003(d_t - c)/c$

$$\varepsilon_t = 0.003(19.5 - 16.16)/16.16 = 0.00062$$

Since $\varepsilon_t < 0.002$, then $\phi = 0.65$.

$$\phi P_n = 0.65(1459) = 948.3 \text{ K}$$

$$\phi M_n = 0.65(729.5) = 474 \text{ K·ft}$$

Note: If side bars are neglected, then

$$P_b = 733.8 + 359.4 - 381 = 712.2 \text{ K}$$

$$P_n \text{ (at } e = 6 \text{ in.)} = 1027.75 + 359.4 - 114.2 = 1273 \text{ K}$$

If side bars are considered, the increase in P_b is about 2% and that in P_n is about 14.6%.

11.11 LOAD CAPACITY OF CIRCULAR COLUMNS

11.11.1 Balanced Condition

The values of the balanced load P_b and the balanced moment M_b for circular sections can be determined using the equations of equilibrium, as was done in the case of rectangular sections. The bars in a circular section are arranged in such a way that their distance from the axis of plastic centroid varies, depending on the number of bars in the section. The main problem is to find the depth of the compressive block a and the stresses in the reinforcing bars. The following example explains the analysis of circular sections under balanced conditions. A similar procedure can be adopted to analyze sections when tension or compression controls.

Example 11.9

Determine the balanced load P_b and the balanced moment M_b for the 16-in. diameter circular spiral column reinforced with eight no. 9 bars shown in Fig. 11.14. Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Because the reinforcement bars are symmetrical about the axis A-A passing through the center of the circle, the plastic centroid lies on that axis.
2. Determine the location of the neutral axis:

$$d_t = 13.1 \text{ in.} \qquad \varepsilon_y = \frac{f_y}{E_s} \qquad (E_s = 29{,}000 \text{ ksi})$$

$$\frac{c_b}{d_t} = \frac{0.003}{0.003 + \varepsilon_y} = \frac{0.003}{0.003 + \frac{f_y}{E_s}} = \frac{87}{87 + f_y}$$

$$c_b = \frac{87}{87 + 60}(13.1) = 7.75 \text{ in.}$$

$$a_b = 0.85 \times 7.75 = 6.59 \text{ in.}$$

3. Calculate the properties of a circular segment (Fig. 11.15):

$$\text{Area of segment} = r^2(\alpha - \sin\alpha \cos\alpha) \qquad (11.19)$$

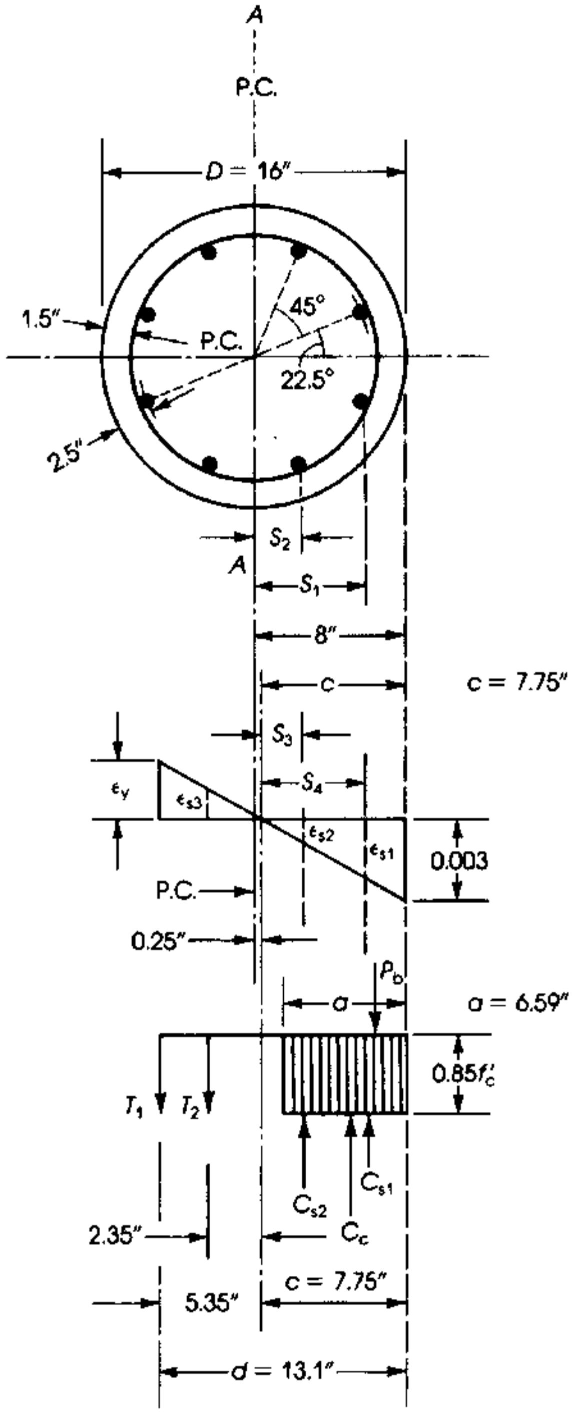


Figure 11.14 Example 11.9: eight no. 9 bars

$$S = 8 - 2.5 = 5.5 \text{ in.}$$
$$S_1 = S\cos 22.5^\circ = 5.1 \text{ in.}$$
$$S_2 = S\cos 67.5^\circ = 2.1 \text{ in.}$$
$$d = 8 + 5.1 = 13.1 \text{ in.}$$
$$S_3 = 1.85 \text{ in.}$$
$$S_4 = 4.85 \text{ in.}$$

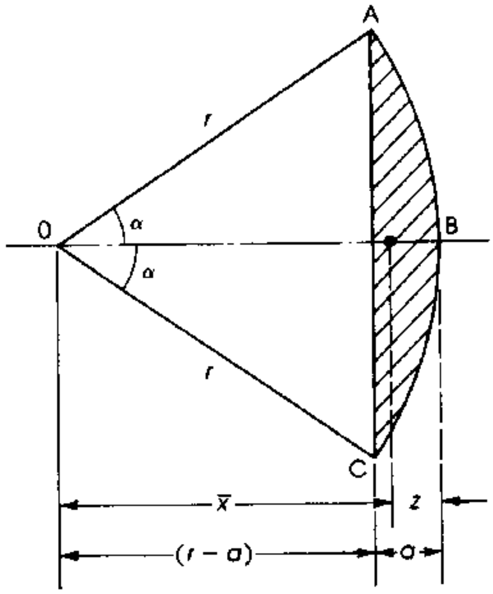


Figure 11.15 Example 11.9: Properties of circular segments.

Location of centroid $\bar{x}$ (from the circle center 0):

$$\bar{x} = \frac{2}{3}\frac{(r \sin^3 \alpha)}{(\alpha - \sin \alpha \cos \alpha)} \tag{11.20}$$

$$Z = r - \bar{x} \tag{11.21}$$

$$r \cos \alpha = (r - a) \quad \text{or} \quad \cos \alpha = \left(1 - \frac{a}{r}\right) \tag{11.22}$$

$$\cos \alpha = \left(1 - \frac{6.59}{8}\right) = 0.176$$

and $\alpha = 79.85°$, $\sin \alpha = 0.984$, and $\alpha = 1.394$ rad.

$$\text{Area for segment} = (8)^2(1.394 - 0.984 \times 0.176)$$

$$= 78.12 \text{ in.}^2$$

$$\bar{x} = \left(\frac{2}{3}\right)\frac{8(0.984)^3}{(1.394 - 0.984 \times 0.176)} = 4.16 \text{ in.}$$

$$Z = r - \bar{x} = 8 - 4.16 = 3.84 \text{ in.}$$

4. Calculate the compressive force C_c:

$$C_c = 0.85 f'_c \times \text{area of segment}$$

$$= 0.85 \times 4 \times 78.12 = 265.6 \text{ K}$$

It acts at 4.16 in. from the center of the column.

5. Calculate the strains, stresses, and forces in the tension and the compression steel. Determine the strains from the strain diagram. For T_1,

$$\varepsilon = e_y = 0.00207 \qquad f_s = f_y = 60 \text{ ksi}$$

$$T_1 = 2 \times 60 = 120 \text{ K}$$

For T_2,

$$\varepsilon_{s3} = \frac{2.35}{5.35}\varepsilon_y = \frac{2.35}{5.35} \times 0.00207 = 0.00091$$

$$f_{s3} = 0.00091 \times 29{,}000 = 26.4 \text{ ksi}$$

$$T_2 = 26.4 \times 2 = 52.8 \text{ K}$$

For C_{s1},

$$\varepsilon_{s1} = \frac{4.85}{7.75} \times 0.003 = 0.00188$$

$$f_{s1} = 0.00188 \times 29{,}000 = 54.5 \text{ ksi} < 60 \text{ ksi}$$

$$C_{s1} = 2(54.5 - 3.4) = 102.2 \text{ K}$$

For C_{s2},

$$\varepsilon_{s2} = \frac{1.85}{7.75} \times 0.003 = 0.000716$$

$$f_{s2} = 0.000716 \times 29{,}000 = 20.8 \text{ ksi}$$

$$C_{s2} = 2(20.8 - 3.4) = 34.8 \text{ K}$$

The stresses in the compression steel have been reduced to take into account the concrete displaced by the steel bars.

6. The balanced force is $P_b = C_c + \Sigma C_s - \Sigma T$ ($\phi = 0.75$).

$$P_b = 265.6 + (102.2 + 34.8) - (120 + 52.8) = 230 \text{ K}$$

For a balanced section,

$$\varepsilon_t = 0.002 \quad \text{and} \quad \phi = 0.65$$

$$\phi P_b = 149.5 \text{ K}$$

7. Take moments about the plastic centroid (axis A-A through the center of the section) for all forces:

$$M_b = P_b e_b = [C_c \times 4.16 + C_{s1} \times 5.1 + C_{s2} \times 2.1 + T_1 \times 5.1 + T_2 \times 2.1]$$

$$= 2422.1 \text{ K·in.} = 201.9 \text{ K·ft}$$

$$\phi M_b = 131.2 \text{ K·ft}$$

$$e_b = \frac{2422.1}{230} = 10.5 \text{ in.}$$

11.11.2 Strength of Circular Columns for Compression Failure

A circular column section under eccentric load can be analyzed in similar steps as the balanced section. This is achieved by assuming a value for $c > c_b$ or $a > a_b$ and calculating the forces in concrete and steel at different locations to determine P_{n1} $P_{n1} = C_c + \Sigma C_s - \Sigma T$. Also, M_n can be calculated by taking moments about the plastic centroid (center of the section) and determining $P_{n2} = M_n/e$. If they are not close enough, within about 1%, assume a new c or a and repeat the calculations. (See also Section 11.8.) Compression controls when $e < e_b$ or $P_n > P_b$.

For example, if it is required to determine the load capacity of the column section of Example 11.9 when $e = 6$ in., P_n can be determined in steps similar to those of Example 11.9:

1. Because $e = 6$ in. is less than $e_b = 10.5$ in., compression failure condition occurs.
2. Assume $c = 9.0$ in. (by trial) $> c_b = 7.75$ in. and $a = 7.65$ in.
3. Calculate $\bar{x} = 3.585$ in., $Z = 4.415$ in., and the area of concrete segment $= 94.93\text{ in.}^2$

4–5. Calculate forces: and $C_c = 322.7$ K, $C_{s1} = 110.7$ K, $C_{s2} = 53.1$ K, $T_1 = 21.6$ K, and $T_2 = 78.9$ K.

6. Calculate $P_{n1} = C_c + \Sigma C_s - \Sigma T = 386$ K.
7. Taking moments about the center of the column (plastic centroid): $M_n = 191$ K·ft, $P_{n2} = M_n/6 = 382$ K, which is close to P_{n1} (the difference is about 1%). Therefore, $P_n = 382$ K. Note that if the column is spirally reinforced, $\phi = 0.70$.

An approximate equation for estimating P_n in a circular section when compression controls was suggested by Whitney [15]:

$$P_n = \frac{A_g f'_c}{\left[\dfrac{9.6he}{(0.8h + 0.67D_s)^2} + 1.18\right]} + \frac{A_{st} f_y}{\left(\dfrac{3e}{D_s} + 1\right)} \tag{11.23}$$

where

A_g = gross area of the section
h = diameter of section
D_s = diameter measured through the centroid of the bar arrangement
A_{st} = total vertical steel area
e = eccentricity measured from the plastic centroid

Example 11.10

Calculate the nominal compressive strength P_n for the section of Example 11.9 using the Whitney equation if the eccentricity is $e = 6$ in.

Solution

1. $e = 6$ in. is less than $e_b = 10.5$ in., calculated earlier; thus, compression controls.
2. Using the Whitney equation,

$$A_g = \frac{\pi}{4}h^2 = \frac{\pi}{4}(16)^2 = 20.1\text{ in.}^2$$

$$h = 16\text{ in.} \quad D_s = 16 - 5 = 11.0\text{ in.} \quad A_{st} = 8 \times 1 = 8\text{ in.}^2$$

$$P_n = \frac{(201.1 \times 4)}{\left[\dfrac{(9.6 \times 16 \times 6)}{(0.8 \times 16 + 0.67 \times 11)^2} + 1.18\right]} + \frac{8 \times 60}{\left(\dfrac{3 \times 6}{11}\right) + 1}$$

$$= 415.5\text{ K}$$

3. $M_n = P_n e = 415.5 \times \frac{6}{12} = 207.8$ K·ft. The value of P_n here is greater than $P_n = 382$ K calculated earlier by statics.

11.11.3 Strength of Circular Columns for Tension Failure

Tension failure occurs in circular columns when the load is applied at an eccentricity $e > e_b$, or $P_n < P_b$. In this case, the column section can be analyzed in steps similar to those of the balanced section and Example 11.8. This is achieved by assuming $c < c_b$ or $a < a_b$ and then following the steps explained in Section 11.11.1. Note that because the steel bars are uniformly distributed along the perimeter of the circular section, the tension steel A_s provided could be relatively low, and the load capacity becomes relatively small. Therefore, it is advisable to avoid the use of circular columns for tension failure cases.

11.12 ANALYSIS AND DESIGN OF COLUMNS USING CHARTS

The analysis of column sections explained earlier is based on the principles of statics. For preliminary analysis or design of columns, special charts or tables may be used either to determine ϕP_n and ϕM_n for a given section or determine the steel requirement for a given load P_u and moment M_u. These charts and tables are published by the American Concrete Institute (ACI) [7], the Concrete Reinforcing Steel Institute (CRSI), and the Portland Cement Association (PCA). Final design of columns must be based on statics by using manual calculations or computer programs. The use of the ACI charts is illustrated in the following examples. The charts are given in Figs. 11.16 and 11.17 [7]. These data are limited to the column sections shown on the top right corner of the charts.

Example 11.11

Determine the necessary reinforcement for a short tied column shown in Fig. 11.18*a* to support a factored load of 483 K and a factored moment of 322 K·ft. The column section has a width of 14 in. and a total depth, h, of 20 in. Use $f'_c = 4$ ksi, $f_y = 60$ ksi.

Solution

1. The eccentricity $e = M_u/P_u = 322 \times 12/483 = 8$ in. Let $d = 20 - 2.5 = 17.5$ in., $\gamma h = 20 - 5 = 15$ in., and $\gamma = 15/20 = 0.75$.
2. Since $e = 8$ in. $< d$, assume compression-controlled section with $\phi = 0.65$.

$$P_n = 483/0.65 = 743 \text{ K} \quad \text{and} \quad M_n = 322/0.65 = 495.4 \text{ K·ft.}$$

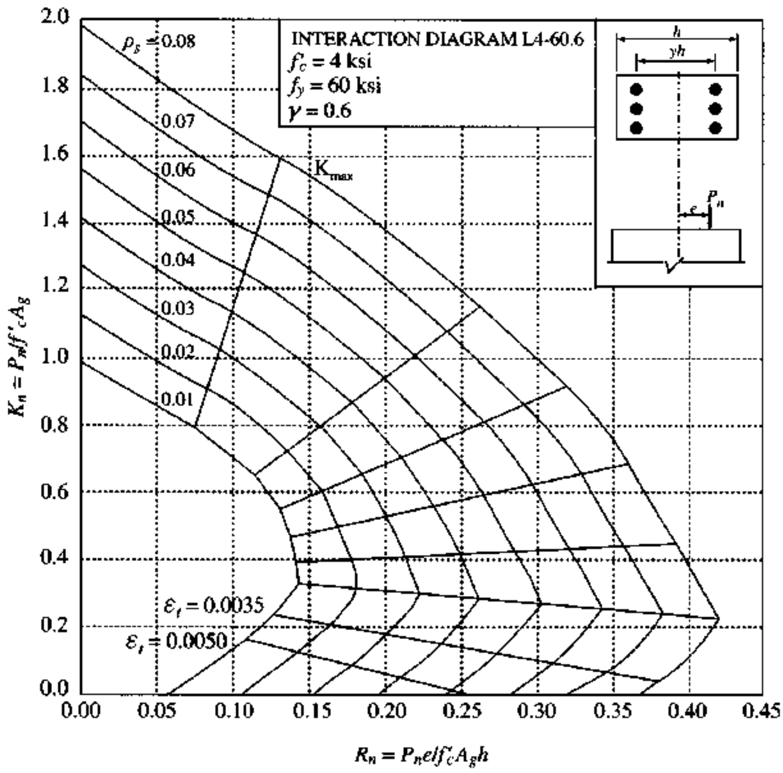
$$K_n = \frac{743}{(4 \times 14 \times 20)} = 0.663$$

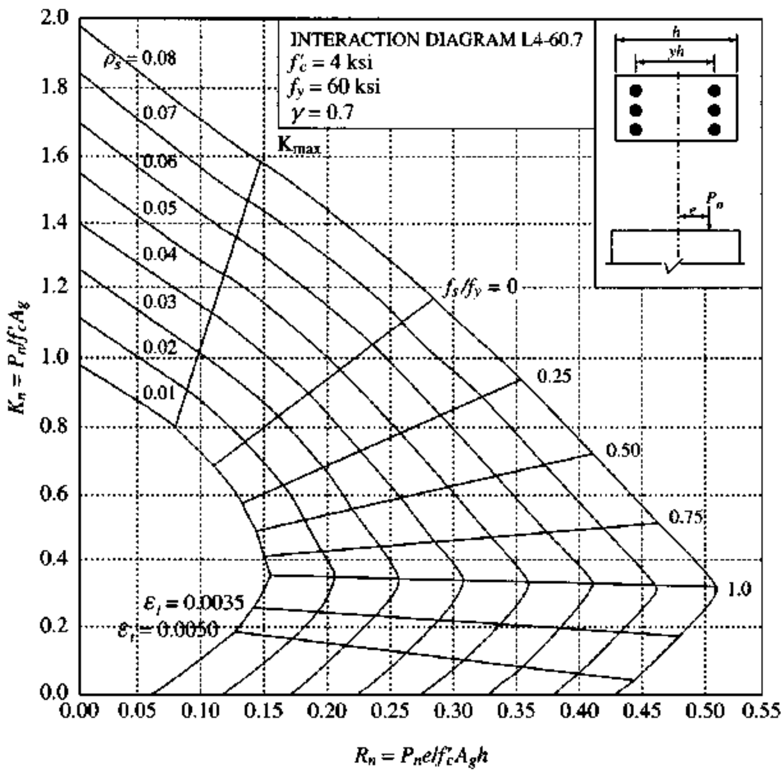
$$R_n = K_n \left(\frac{e}{h}\right) = 0.663 \left(\frac{8}{20}\right) = 0.265$$

3. From the charts of Fig. 11.16, for $\gamma = 0.7$, $\rho = 0.034$. Also, for $\gamma = 0.8$, $\rho = 0.039$. By interpolation, for $\gamma = 0.75$, $\rho = 0.0365$.

$$A_s = 0.0365\,(14 \times 20) = 10.22 \text{ in.}^2$$

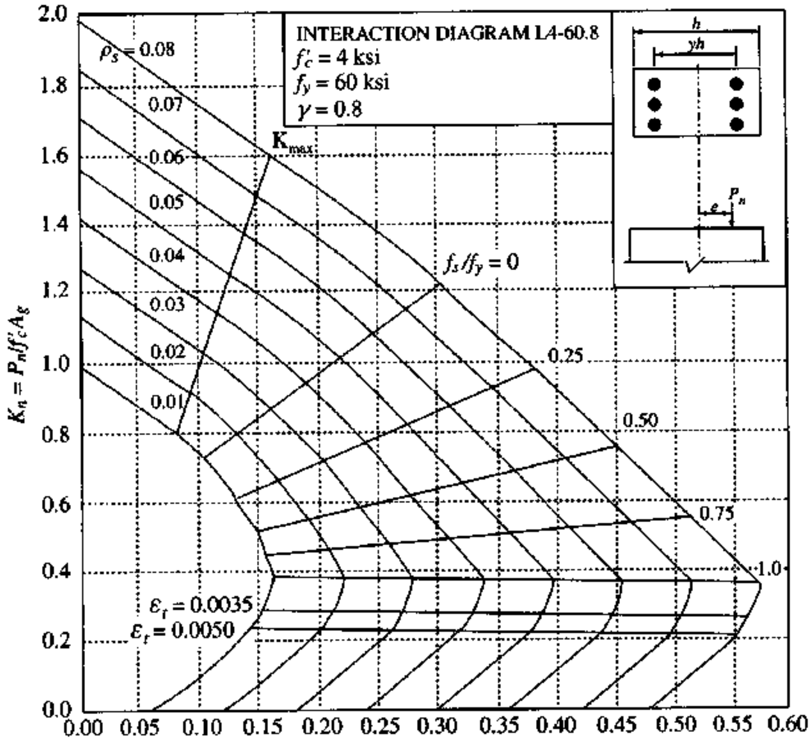
Use eight no. 10 bars ($A_s = 10.16 \text{ in.}_2$), four on each short side. Use no. 3 ties spaced at 14 in. (Fig. 11.18*a*).



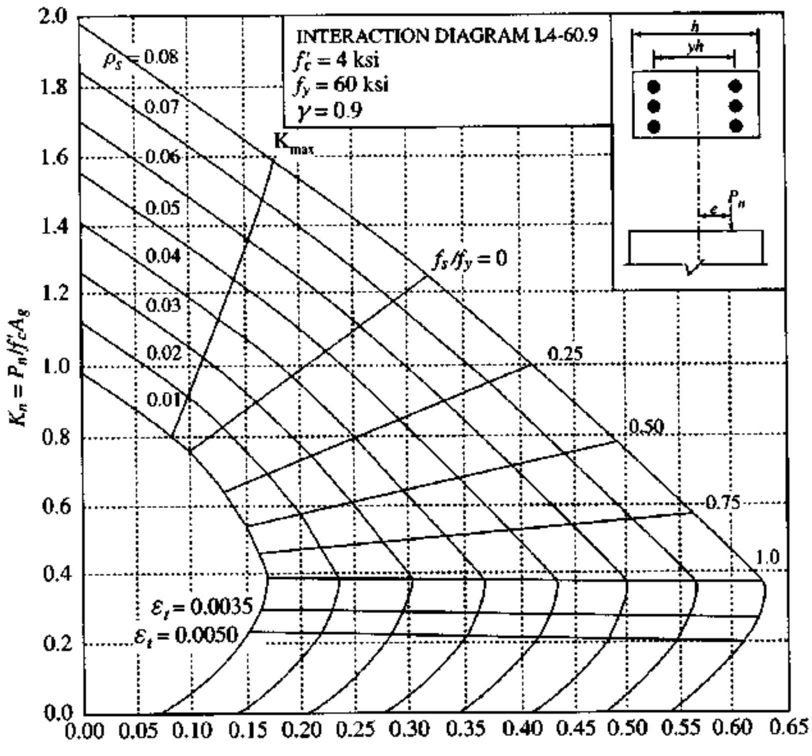


(b)

Figure 11.16 Load-moment strength interaction diagram for rectangular columns where f'_c = 4 ksi, f_y = 60 ksi, and (a) γ = 0.60, (b) γ = 0.70, (c) γ = 0.80, and (d) γ = 0.90. Courtesy of American Concrete Institute [7].

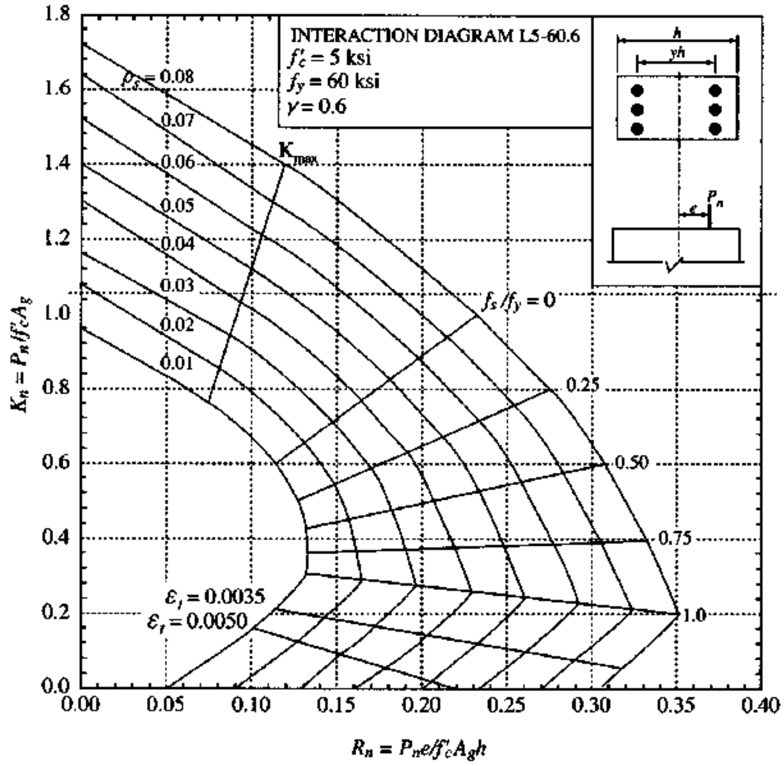


(c)

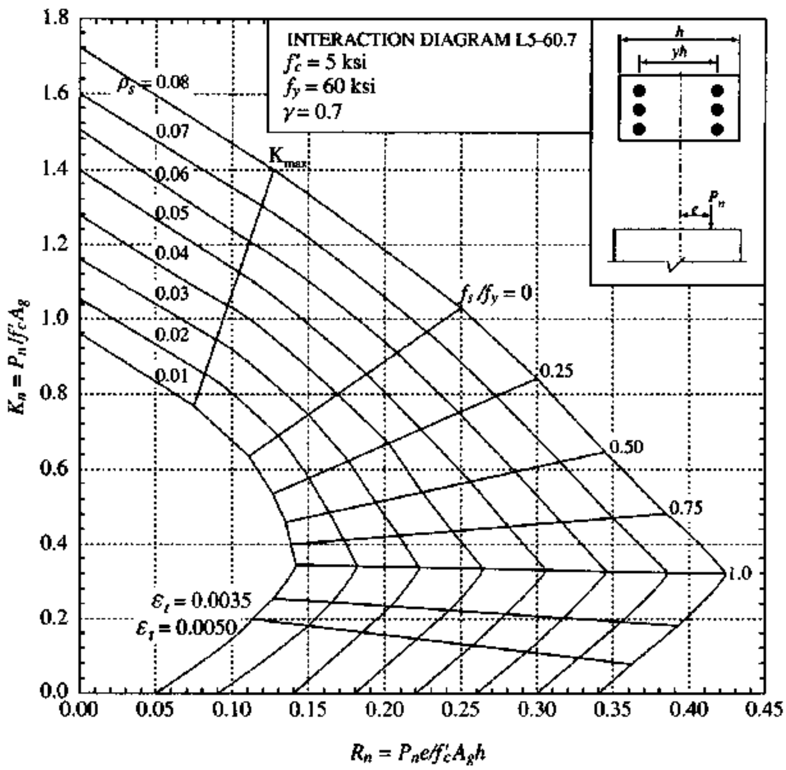


(d)

Figure 11.16 (*continued*)

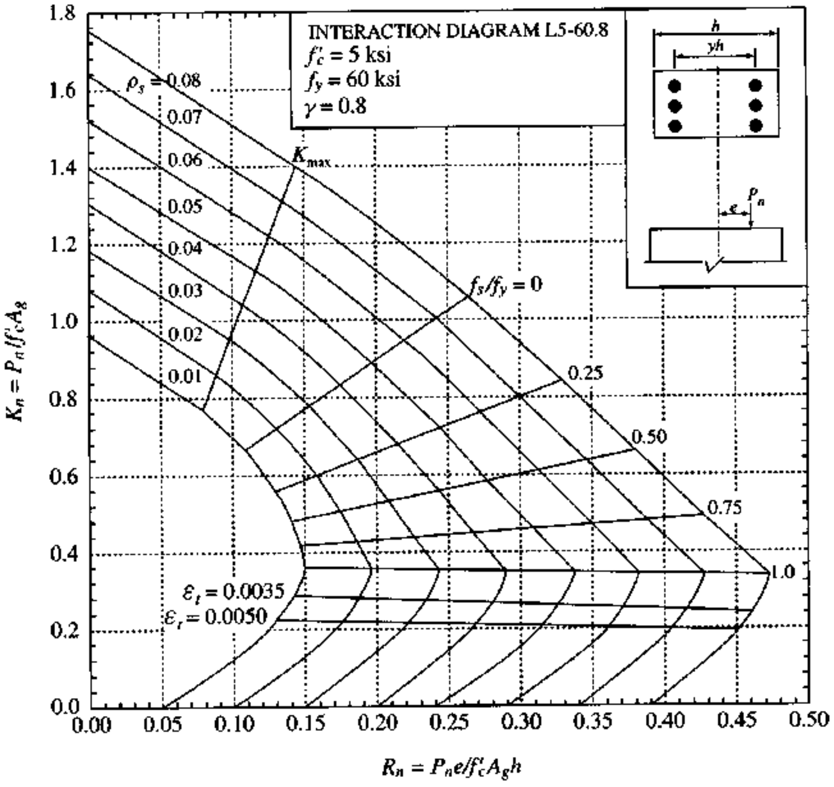


(*a*)

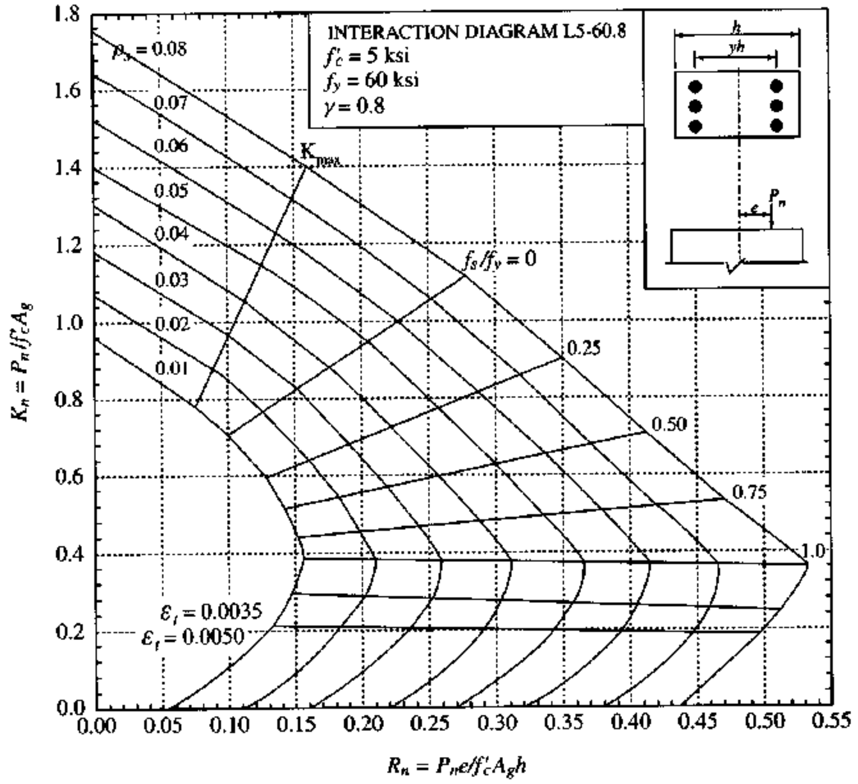


(*b*)

Figure 11.17 Load–moment strength interaction diagram for rectangular columns where $f'_c = 5$ ksi, $f_y = 60$ ksi, and (*a*) $\gamma = 0.60$, (*b*) $\gamma = 0.70$, (*c*) $\gamma = 0.80$, and (*d*) $\gamma = 0.90$. Courtesy of American Concrete Institute [7].



(c)



(d)

Figure 11.17 *(continued)*

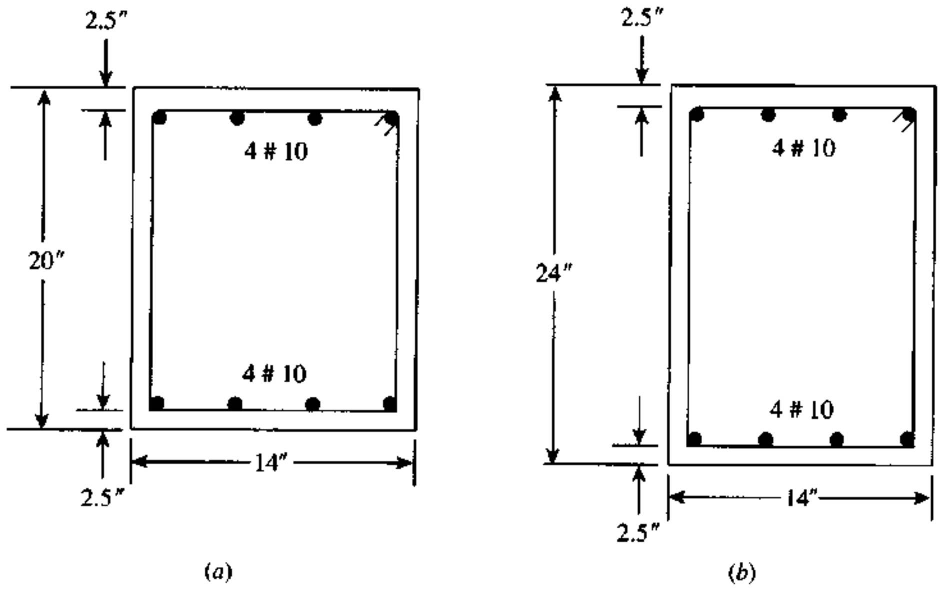


Figure 11.18 Column sections of (*a*) Example 11.11 and (*b*) Example 11.12.

Example 11.12

Use the charts to determine the column strength, ϕP_n, of the short column shown in Fig. 11.18*b* acting at an eccentricity $e = 12$ in. Use $f'_c = 5$ ksi and $f_y = 60$ ksi.

Solution

1. Properties of the section: $H = 24$ in., $\gamma h = 24 - 5 = 19$ in. (distance between tension and compression steel). $\gamma = 19/24 = 0.79$, and $\rho = 8(1.27)/(14 \times 24) = 0.03$.
2. Since $e < d$, assume compression-controlled section. Let $\varepsilon_t = 0.002$, $f_s/f_y = 1.0$ and $\phi = 0.65$. From the charts of Fig. 11.17, get $K_n = 0.36 = P_n/(5 \times 14 \times 24)$. Then $P_n = 605$ K.
3. Check assumption for compression-controlled section: For $K_n = 0.36$, $R_n = K_n\ (e/h) = 0.36\ (12/24) = 0.18$. From charts, get $\rho = 0.018 < 0.03$. Therefore, $P_n > 605$ K (to use $\rho = 0.003$).
4. Second trial: Let $\varepsilon_t = 0.0015$, $f_s = 0.0015\ (29{,}000) = 43.5$ ksi.

$$f_s/f_y = 43.5/60 = 0.725 \quad \rho = 0.03 \quad K_n = 0.44$$

$$0.44 = P_n/(5 \times 14 \times 24) \quad P_n = 740 \text{ K}$$

5. Check assumption: For $K_n = 0.44$, $R_n = 0.44\ (12/24) = 0.22$. From charts, $\rho = 0.03$ as given. Therefore, $P_n = 740$ K.

$$\phi P_n = 0.65(740) = 480 \text{ K} \quad \text{and} \quad \phi M_n = 0.65(740) = 480 \text{ K·ft}$$

By analysis, $\phi P_n = 485$ K(which is close to 480 K·ft).

11.13 DESIGN OF COLUMNS UNDER ECCENTRIC LOADING

In the previous sections, the analysis, behavior, and the load–interaction diagram of columns subjected to an axial load and bending moment were discussed. The design of columns is more complicated, because the external load and moment, P_u and M_u, are given and it is required to determine many unknowns, such as b, h, A_s, and A'_s, within the ACI Code limitations. It is a common practice to assume a column section first and then determine the amount of reinforcement needed. If the designer needs to change the steel reinforcement calculated, then the cross-section may be adjusted accordingly. The following examples illustrate the design of columns.

11.13.1 Design of Columns for Compression Failure

For compression failure, it is preferable to use $A_s = A'_s$ for rectangular sections. The eccentricity, e, is equal to M_u/P_u. Based on the magnitude of e, two cases may develop.

1. When e is relatively very small (say, $e \leq 4$ in.), a minimum eccentricity case may develop that can be treated by using Eq. 10.8, as explained in the examples of Chapter 10. Alternatively, the designer may proceed as in Case 2. This loading case occurs in the design of the lower-floor columns in a multistory building, where the moment, M_u, develops from one floor system and the load, P_u, develops from all floor loads above the column section.
2. The compression failure zone represents the range from the axial to the balanced load, as shown in Figs. 11.3 and 11.11. In this case, a cross-section (bh) may be assumed and then the steel reinforcement is calculated for the given P_u and M_u. The steps can be summarized as follows:
   a. Assume a square or rectangular section (bh); then determine d, d', and $e = M_u/P_u$.
   b. Assuming $A_s = A'_s$, calculate A'_s from Eq. 11.17 using the dimensions of the assumed section, and $\phi = 0.65$ for tied columns. Let $A_s = A'_s$ and then choose adequate bars. Determine the actual areas used for A_s and A'_s. Alternatively, use the ACI charts.
   c. Check that $\rho_g = (A_s + A'_s)/bh$ is less than or equal to 8% and greater or equal to 1%. If ρ_g is small, reduce the assumed section, but increase the section if less steel is required.
   d. Check the adequacy of the final section by calculating ϕP_n from statics; as explained in the previous examples, ϕP_n should be greater than or equal to P_u.
   e. Determine the necessary ties.

A simple approximate formula for determining the initial size of the column bh or the total steel ratio ρ_g is

$$P_n = K_c bh^2 \quad \text{or} \quad P_u = \phi P_n = \phi K_c bh^2 \tag{11.24}$$

where K_c has the values shown in Table 11.2 and plotted in Fig. 11.19 for $f_y = 60$ ksi and $A_s = A'_s$. Units for K_c are in lb/in.^{3}

The values of K_c shown in Table 11.2 are approximate and easy to use, because K_c increases by 0.02 for each increase of 1 ksi in f'_c. For the same section, as the eccentricity,

Table 11.2 Values of K_c ($f_y = 60$ ksi)

	K_c		
ρ_g (%)	$f'_c = 4$ ksi	$f'_c = 5$ ksi	$f'_c = 6$ ksi
1%	0.090	0.110	0.130
4%	0.137	0.157	0.177
8%	0.200	0.220	0.240

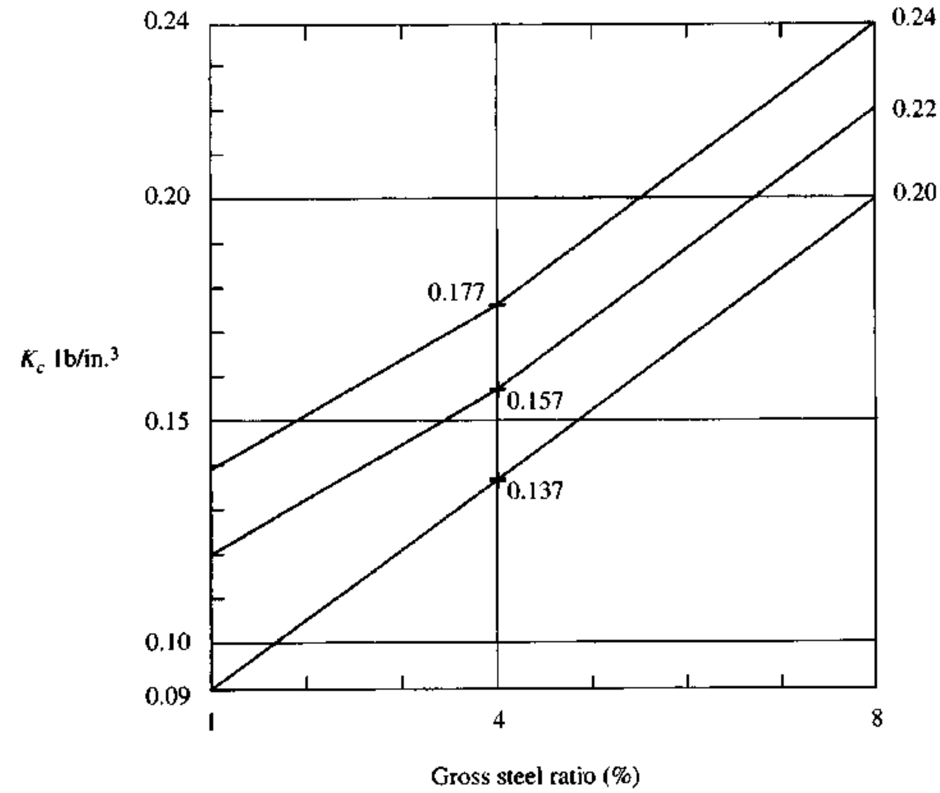


Figure 11.19 Values of K_c versus ρ_g (%).

$e = M_u/P_u$, increases, P_n decreases, and, consequently, K_c decreases. Thus, K_c values represent a load P_n on the interaction diagram between 0.8 P_{n_0} and P_b as shown in Fig. 11.3 or 11.11.

Linear interpolation can be used. For example, $K_c = 0.1685$ for $\rho_g = 6\%$ and $f'_c = 4$ ksi. The steps in designing a column section can be summarized as follows:

1. Assume an initial size of the column section bh.
2. Calculate $K_c = P_u/(\phi bh^2)$.
3. Determine ρ_g from Table 11.2 for the given f'_c.
4. Determine $A_s = A'_s = \rho_g bh/2$ and choose bars and ties.
5. Determine ϕP_n of the final section by statics (accurate solution). The value of ϕP_n should be greater than or equal to P_u. If not, adjust bh or ρ_g.

Alternatively, if a specific steel ratio is desired, say $\rho_g = 6\%$, then proceed as follows:

1. Assume ρ_g as required and then calculate $e = M_u/P_u$.
2. Based on the given f'_c and ρ_g, determine K_c from Table 11.2.
3. Calculate $bh^2 = P_u/\phi K_c$; then choose b and h. Repeat steps 4 and 5. It should be checked that ρ_g is less than or equal to 8% and greater than or equal to 1%. Also, check that c calculated by statics is greater than $c_b = 87d_t/(87 + f_y)$ for compression failure to control.

Example 11.13

Determine the tension and compression reinforcement for a 16 × 24-in. rectangular tied column to support $P_u = 780$ K and $M_u = 390$ K-ft. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Calculate $e = M_u/P_u = 420(12)/840 = 6.0$ in. We have $h = 24$ in; let $d = 21.5$ in. and $d' = 2.5$ in. Because e is less than $\frac{2}{3}d = 14.38$ in., assume compression failure.
2. Assume $A_s = A'_s$ and use Eq. 11.17 to determine the initial value of $A'_s P_n = P_u/\phi = 780/0.65 = 1200$ K.

$$P_n = \frac{bhf'_c}{\left(\frac{3he}{d^2}\right) + 1.18} + \left[\frac{A'_s f_y}{\left(\frac{e}{d-d'}\right) + 0.5}\right] \tag{11.17}$$

For $P_n = 1200$ K, $e = 6$ in., $d = 21.5$ in., $d' = 2.5$ in., and $h = 24$ in., calculate $A'_s = 6.44$ in.$^2 = A_s$. Choose five no. 10 bars ($A_s = 6.35$ in.^{2}) for A_s and A'_s (Fig. 11.20).

3. $\rho_g = 2(6.35)/(16 \times 24) = 0.033$, which is less than 0.08 and >0.01.
4. Check the section by statics following the steps of Example 11.4 to get

$$a = 16.64 \text{ in.} \quad c = 19.58 \text{ in.} \quad C_c = 905.2 \text{ K}$$

$$C_s = 6.35(60 - 0.85 \times 4) = 359.4 \text{ K}$$

$$f_s = 87\left(\frac{d-c}{c}\right) = 8.55 \text{ ksi}$$

$$T = A_s f_s = 6.35(8.55) = 54.3 \text{ K}$$

$$P_n = C_c + C_s - T = 1210.3 \text{ K} > 1200 \text{ K}$$

Note that if $\phi P_n < P_u$, increase A_s and A'_s, for example, to six no. 10 bars, and check the section again.

5. Check P_n based on moments about A_s (Eq. 11.12) to get $P_n = 1210$K.
6. For a balanced section,

$$c_b = \left(\frac{87}{87 + f_y}\right) d_t = \left(\frac{87}{147}\right) 21.5 = 12.7 \text{ in.}$$

Because $c = 19.58$ in. > $c_b = 12.7$ in., this is a compression failure case, as assumed.

7. Use no. 3 ties spaced at 16 in. (Refer to Chapter 10.)

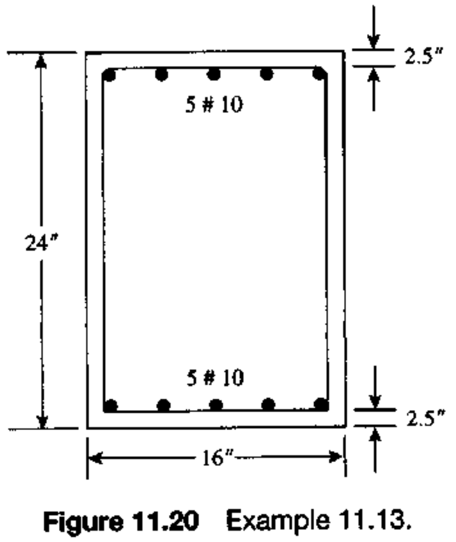


Figure 11.20 Example 11.13.

Example 11.14

Repeat Example 11.13 using Eq. 11.24.

Solution

1. The column section is given: 16×24 in.
2. Determine K_c from Eq. 11.24:

$$K_c = \frac{P_u}{\phi bh^2} = \frac{780}{0.65 \times 16 \times 24^2} = 0.13 \text{ lb/in.}^3$$

3. From Table 11.2 or Fig. 11.19, for $K_c = 0.13$, $f_c' = 4$ ksi, by interpolation, get $\rho_g = 3.5\%$.
4. Calculate $A_s = A_s' = \rho bh/2 = 0.035(16)(24)/2 = 6.77$ in.^{2} Choose five no. 10 bars ($A_s = 6.35$ in.^{2}) for the first trial.
5. Determine ϕP_n using steps 4–7 in Example 11.13. $\phi P_n = 1210.3$ K $> P_n = 1200$ K, so the section is adequate.
6. If the section is not adequate, or $\phi P_n < P_u$, increase A_s and A_s' and check again to get closer values.

Example 11.15

Design a rectangular column section to support $P_u = 696$ K and $M_u = 465$ K·ft with a total steel ratio ρ_g of about 4%. Use $f_c' = 4$ ksi, $f_y = 60$ ksi, and $b = 18$ in.

Solution

1. Calculate $e = M_u/P_u = 465(12)/696 = 8$ in. Assume compression failure ($\phi = 0.65$) (to be checked later) and $A_s = A_s'$.
2. For $\rho_s = 4\%$ and $f_c' = 4$ ksi, $K_c = 0.137$ (Table 11.2).
3. Calculate bh^2 from Eq. 11.24: $P_u = \phi K_c bh^2$, or $696 = 0.65(0.137)(18)h^2$. Thus, $h = 20.84$ in. Let $h = 22$ in.
4. Calculate $A_s = A_s' = 0.04(18 \times 22)/2 = 7.92$ in.^{2} Choose five no. 11 bars ($A_s = 7.8$ in.^{2}) in one row for A_s and A_s' (Fig. 11.21). Choose no. 4 ties spaced at 18 in.

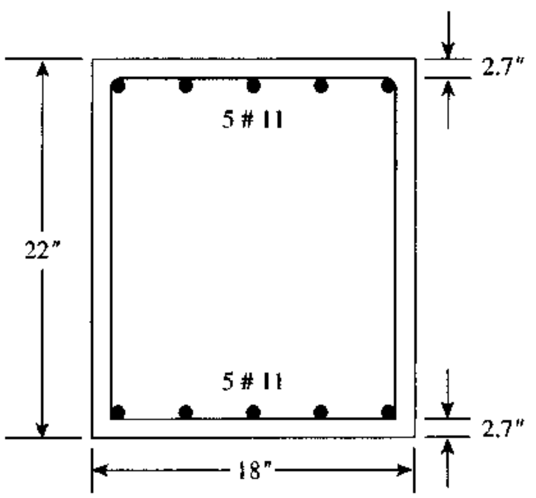


Figure 11.21 Example 11.15.

5. Check the final section by analysis, similar to Example 11.4, to get $a = 13.15$ in., $c = 15.47$ in., $C_c = 0.85\ f'_c ab = 804.8$ K, $f'_s = 60$ ksi, $C_s = A'_s(f_y - 0.85f'_c) = 441.5$ K, $f_s = 87[(d - c)/c] = 21.24$ ksi, and $T = A_s f_s = 168$ K. Also, $P_n = C_c + C_s - T = 1078.3$ K and $\phi P_n = 0.65P_n = 701\text{ K} > 696\text{ K}$. The section is adequate.
6. For a balanced section,

$$c_b = \left(\frac{87}{87 + f_y}\right) \qquad d_t = \left(\frac{87}{147}\right) 19.3 = 11.42 \text{ in.} < c = 15.47 \qquad (d = 19.3 \text{ in.})$$

Therefore, this is a compression failure case, as assumed.

11.13.2 Design of Columns for Tension Failure

Tension failure occurs when $P_n < P_b$ or the eccentricity $e > e_b$, as explained in Section 11.7. In the design of columns, P_u and M_u are given, and it is required to determine the column size and its reinforcement. It may be assumed (as a guide) that tension controls when the ratio of M_u (K·ft) to P_u (kips) is greater than 1.75 for sections of $h < 24$ in. and 2.0 for $h \geq 24$ in. In this case, a section may be assumed, and then A_s and A'_s are determined. The ACI charts may be used to determine ρ_g for a given section with $A_s = A'_s$. Note that ϕ varies between 0.65 (0.75) and 0.9, as explained in Section 11.4.

When tension controls, the tension steel yields, whereas the compression steel may or may not yield. Assuming initially $f'_s = f_y$ and $A_s = A'_s$, Eq. 11.16 (Section 11.6) may be used to determine the initial values of A_s and A'_s:

$$A_s = A'_s = \frac{P_n\left(e - \frac{h}{2} + \frac{a}{2}\right)}{f_y(d - d')} \tag{11.16}$$

Because a is not known yet, assume $a = 0.4d$ and $P_u = \phi P_n$; then

$$A_s = A'_s = \frac{P_u(e - 0.5h + 0.2d)}{\phi f_y(d - d')} \tag{11.25}$$

The final column section should be checked by statics to prove that $\phi P_n \geq P_u$. Example 11.16 explains this approach.

When the load P_u is very small relative to M_u, the section dimensions may be determined due to M_u only, assuming $P_u = 0$. The final section should be checked by statics. This case occurs in single- or two-story building frames used mainly for exhibition halls or similar structures. In this case, A'_s may be assumed to be less than A_s. A detailed design of a one-story, two-hinged frame exhibition hall is given in Chapter 16.

Example 11.16

Determine the necessary reinforcement for a 16 × 22-in. rectangular tied column to support a factored load $P_u = 257$ K and a factored moment $M_u = 643$ K·ft. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Calculate $e = M_u/P_u = 643(12)/257 = 30$ in; let $d = 22 - 2.5 = 19.5$ in. Because $M_u/P_u = 500/200 = 2.5 > 1.75$, or because $e > d$, assume tension failure case, $\phi = 0.9$ (to be checked later).

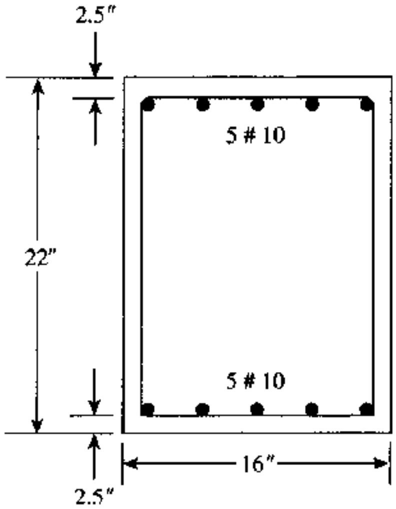


Figure 11.22 Example 11.16.

2. Assume $A_s = A'_s$ and $f'_s = f_y$ and use Eq. 11.25 to determine A_s and A'_s. Let $P_u = 257.0\,\text{K}$, $e = 30\,\text{in.}$, $h = 22\,\text{in.}$, $d = 19.5\,\text{in.}$, and $d' = 2.5\,\text{in.}$

$$A_s = A'_s = \frac{257(30 - 0.5 \times 22 + 0.2 \times 19.5)}{0.9(60)(17.0)} = 6.41\ \text{in.}^2$$

Choose five no. 10 bars (6.35 in.^{2}) in one row for each of A_s and A'_s (Fig. 11.22).

3. Check $\rho_g = 2(6.35)(16 \times 22) = 0.036$, which is less than 0.08 and greater than 0.01.
4. Check the chosen section by statics similar to Example 11.3.
   a. Determine the value of a using the general equation $Aa^2 + Ba + C = 0$ with $e' = e + d - h/2 = 38.5\,\text{in.}$, $A = 0.425\ f'_c b = 27.2$, $B = 2A(e' - d) = 1033.6$, $C = A'_s(f_y - 0.85 f'_c)(e' - d + d') - A_s f_y e' = -6941.2$. Solve to get $a = 5.82$ in. and $c = a/0.85 = 6.85$.
   b. Check f'_s:

$$f'_s = 87\left(\frac{c - d'}{c}\right) = 87\left(\frac{6.85 - 2.5}{6.85}\right) = 55.26\ \text{ksi}$$

   Let $f'_s = 57$ ksi.
   c. Recalculate a:

$$C = A'_s(f'_s - 0.85 f'_c)(e' - d + d') - A_s f_y e' = -7351$$

   Solve now for a to get $a = 6.13$ and $c = 7.21$ in.
   d. Check f'_s:

$$f'_s = 87\left(\frac{c - 2.5}{c}\right) = 56.83\ \text{ksi}$$

   Calculate

$$C_c = 0.85(4)(6.13)(16) = 333.5\ \text{K},\ C_s = A'_s(f'_s - 0.85 f'_c) = 6.35(57 - 0.85 \times 4)$$
$$= 340.4\ \text{K},\ T = A_s f_y = 6.35(60) = 381\ \text{K}.$$

   e. $P_n = C_c + C_s - T = 292.9$ K.
5. Determine ϕ: $\varepsilon_t = [(d_t - c)/c]\ 0.003 = 0.00511$. Because $\varepsilon_t = 0.00511 > 0.005$, $\phi = 0.9$.
6. $\phi P_n = 0.9(292.9) = 263.6\,\text{K} > 257\,\text{K}$, the section is adequate.

11.14 BIAXIAL BENDING

The analysis and design of columns under eccentric loading was discussed earlier in this chapter, considering a uniaxial case. This means that the load P_n was acting along the y-axis (Fig. 11.23), causing a combination of axial load P_n and a moment about the x-axis equal to $M_{nx} = P_n e_y$ or acting along the x-axis (Fig. 11.24) with an eccentricity e_x, causing a combination of an axial load P_n and a moment $M_{nx} = P_n e_x$.

If the load P_n is acting anywhere such that its distance from the x-axis is e_y and its distance from the y-axis is e_x, then the column section will be subjected to a combination of forces: an axial load P_n a moment about the x-axis $= M_{nx} = P_n e_y$ and a moment about the y-axis $= M_{nx} = P_n e_x$ (Fig. 11.25). The column section in this case is said to be subjected to *biaxial bending*. The analysis and design of columns under this combination of forces is not simple when the principles of statics are used. The neutral axis is at an angle with respect to both axes, and lengthy calculations are needed to determine the location of the neutral axis, strains, concrete compression area, and internal forces and their point of application. Therefore, it was necessary

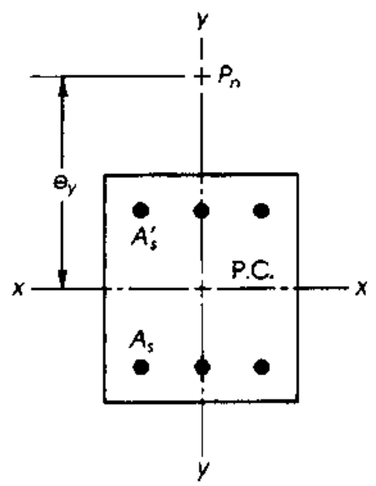


Figure 11.23 Uniaxial bending with load P_n along the y-axis with eccentricity e_y.

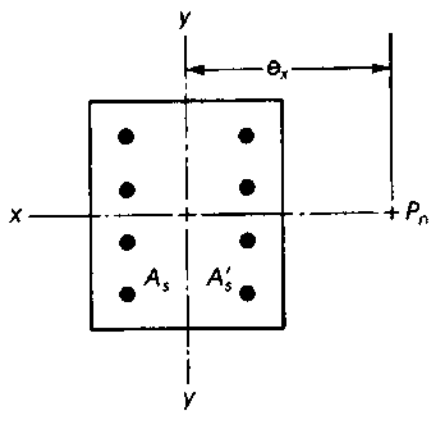


Figure 11.24 Uniaxial bending with load P_n along the x-axis, with eccentricity e_x.

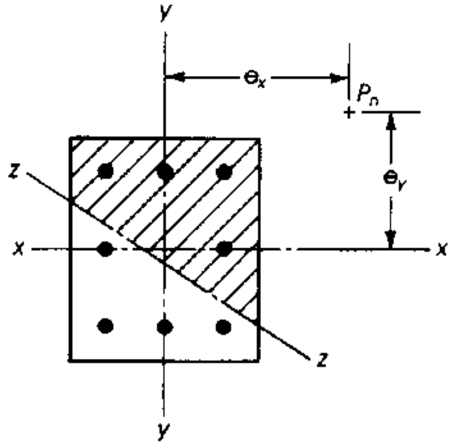


Figure 11.25 Biaxial bending.

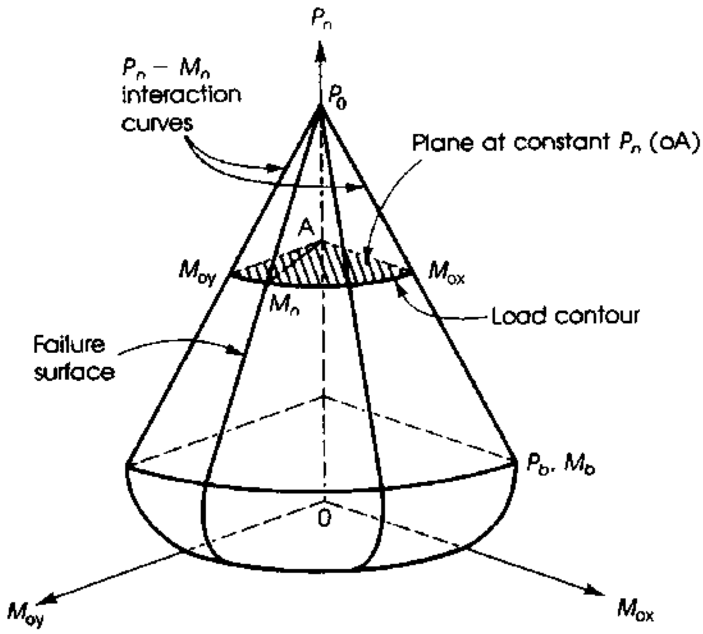


Figure 11.26 Biaxial interaction surface.

to develop practical solutions to estimate the strength of columns under axial load and biaxial bending. The formulas developed relate the response of the column in biaxial bending to its uniaxial strength about each major axis.

The biaxial bending strength of an axially loaded column can be represented by a three-dimensional interaction curve, as shown in Fig. 11.26. The surface is formed by a series of uniaxial interaction curves drawn radially from the P_n-axis. The curve M_{ox} represents the interaction curve in uniaxial bending about the x-axis, and the curve M_{oy} represents the curve in uniaxial bending about the y-axis. The plane at constant axial load P_n shown in Fig. 11.26 represents the contour of the bending moment M_n about any axis.

Different shapes of columns may be used to resist axial loads and biaxial bending. Circular, square, or rectangular column cross-sections may be used with equal or unequal bending capacities in the x- and y-directions.

11.15 CIRCULAR COLUMNS WITH UNIFORM REINFORCEMENT UNDER BIAXIAL BENDING

Circular columns with reinforcement distributed uniformly about the perimeter of the section have almost the same moment capacity in all directions. If a circular column is subjected to biaxial bending about the x- and y-axes, the equivalent uniaxial M_u moment can be calculated using the following equations:

$$M_u = \sqrt{(M_{ux})^2 + (M_{uy})^2} = P_u \cdot e \tag{11.26}$$

and

$$e = \sqrt{(e_x)^2 + (e_y)^2} = \frac{M_u}{P_u} \tag{11.27}$$

where

$M_{ux} = P_u e_y$ = factored moment about the x-axis

$M_{uy} = P_u e_x$ = factored moment about the y-axis

$M_u = P_u e$ = equivalent uniaxial factored moment of the section due to M_{ux} and M_{uy}

In circular columns, a minimum of six bars should be used, and these should be uniformly distributed in the section.

Example 11.17: Circular Column

Determine the load capacity P_n of a 20-in.-diameter column reinforced with 10 no. 10 bars when $e_x = 4$ in. and $e_y = 6$ in. Use $f'_c = 4$ ksi and f_y and 60 ksi.

Solution

1. Calculate the eccentricity that is equivalent to uniaxial loading by using Eq. 11.41.

$$e(\text{for uniaxial loading}) = \sqrt{e_x^2 + e_y^2} = \sqrt{(4)^2 + (6)^2} = 7.211 \text{ in.}$$

2. Determine the load capacity of the column based on $e = 7.211$ in. Proceed as in Example 11.9:

$$d = 17.12 \text{ in.} \qquad a = 9.81 \text{ in.} \qquad c = 11.54 \text{ in. (by trial)}$$

$$C_c = 521.2 \text{ K} \qquad \sum C_s = 269.8 \text{ K} \qquad \sum T = 132.1 \text{ K}$$

$$P_n = C_c + \sum C_s - \sum T = 650 \text{ K}$$

3. For a balanced condition,

$$c_b = \left(\frac{87}{87 + f_y}\right) d_t = \left(\frac{87}{147}\right) 17.12 = 10.13 \text{ in.}$$

$$c = 11.54 \text{ in.} > c_b, \text{ which is a compression failure case.}$$

11.16 SQUARE AND RECTANGULAR COLUMNS UNDER BIAXIAL BENDING

11.16.1 Bresler Reciprocal Method

Square or rectangular columns with unequal bending moments about their major axes will require a different amount of reinforcement in each direction. An approximate method of analysis of such sections was developed by Boris Bresler and is called the Bresler reciprocal method [9,12]. According to this method, the load capacity of the column under biaxial bending can be determined by using the following expression:

$$\frac{1}{P_u} = \frac{1}{P_{ux}} + \frac{1}{P_{uy}} - \frac{1}{P_{u_0}} \tag{11.28}$$

or

$$\frac{1}{P_n} = \frac{1}{P_{nx}} + \frac{1}{P_{ny}} - \frac{1}{P_{n_0}} \tag{11.29}$$

where

P_u = factored load under biaxial bending
P_{ux} = factored uniaxial load when the load acts at an eccentricity e_y and $e_x = 0$
P_{uy} = factored uniaxial load when the load acts at an eccentricity e_x and $e_y = 0$
P_{u_0} = factored axial load when $e_x = e_y = 0$

$$P_n = \frac{P_u}{\phi} \quad P_{nx} = \frac{P_{ux}}{\phi} \quad P_{ny} = \frac{P_{uy}}{\phi} \quad P_{n_0} = \frac{P_{u_0}}{\phi}$$

The uniaxial load strengths P_{nx}, P_{ny}, and P_{n_0} can be calculated according to the equations and method given earlier in this chapter. After that, they are substituted into Eq. 11.29 to calculate P_n.

The Bresler equation is valid for all cases when P_n is equal to or greater than $0.10P_{n_0}$. When P_n is less than $0.10P_{n_0}$, the axial force may be neglected and the section can be designed as a member subjected to pure biaxial bending according to the following equations:

$$\frac{M_{ux}}{M_x} + \frac{M_{uy}}{M_y} \le 1.0 \tag{11.30}$$

or

$$\frac{M_{nx}}{M_{ox}} + \frac{M_{ny}}{M_{oy}} \le 1.0 \tag{11.31}$$

where

$M_{ux} = P_u e_y$ = design moment about the x-axis
$M_{uy} = P_u e_x$ = design moment about the y-axis
M_x and M_y = uniaxial moment strengths about the x-and y-axes

$$M_{nx} = \frac{M_{ux}}{\phi} \qquad M_{ny} = \frac{M_{uy}}{\phi} \qquad M_{ox} = \frac{M_x}{\phi} \qquad M_{oy} = \frac{M_y}{\phi}$$

The Bresler equation is not recommended when the section is subjected to axial tension loads.

11.16.2 Bresler Load Contour Method

In this method, the failure surface shown in Fig. 11.26 is cut at a constant value of P_n, giving the related values of M_{nx} and M_{ny}. The general nondimension expression for the load contour method is

$$\left(\frac{M_{nx}}{M_{ox}}\right)^{\alpha 1} + \left(\frac{M_{ny}}{M_{oy}}\right)^{\alpha 2} = 1.0 \tag{11.32}$$

Bresler indicated that the exponent α can have the same value in both terms of this expression ($\alpha_1 = \alpha_2$). Furthermore, he indicated that the value of α varies between 1.15 and 1.55 and can be assumed to be 1.5 for rectangular sections. For square sections, α varies between 1.5 and 2.0, and an average value of $\alpha = 1.75$ may be used for practical designs. When the reinforcement is uniformly distributed around the four faces in square columns, α may be assumed to be 1.5.

$$\left(\frac{M_{nx}}{M_{ox}}\right)^{1.5} + \left(\frac{M_{ny}}{M_{oy}}\right)^{1.5} = 1.0 \tag{11.33}$$

The British Code assumed $\alpha = 1.0$, 1.33, 1.67, and 2.0 when the ratio $P_u/1.1P_{u0}$ is equal to 0.2, 0.4, 0.6, and ≥ 0.8, respectively.

11.17 PARME LOAD CONTOUR METHOD

The load contour approach, proposed by the Portland Cement Association (PCA), is an extension of the method developed by Bresler. In this approach, which is also called the *Parme method* [11], a point B on the load contour (of a horizontal plane at a constant P_n shown in Fig. 11.26) is defined such that the biaxial moment capacities M_{nx} and M_{ny} are in the same ratio as the uniaxial moment capacities M_{ox} and M_{oy}; that is,

$$\frac{M_{nx}}{M_{ny}} = \frac{M_{ox}}{M_{oy}} \quad \text{or} \quad \frac{M_{nx}}{M_{ox}} = \frac{M_{ny}}{M_{oy}} = \beta$$

The ratio β is shown in Fig. 11.27 and represents that constant portion of the uniaxial moment capacities that may be permitted to act simultaneously on the column section.

For practical design, the load contour shown in Fig. 11.27 may be approximated by two straight lines, AB and BC. The slope of line AB is $(1 - \beta)/\beta$, and the slope of line BC is $\beta/(1 - \beta)$. Therefore, when

$$\frac{M_{ny}}{M_{oy}} > \frac{M_{nx}}{M_{ox}}$$

then

$$\frac{M_{ny}}{M_{oy}} + \frac{M_{nx}}{M_{ox}}\left(\frac{1-\beta}{\beta}\right) = 1 \tag{11.34}$$

and when

$$\frac{M_{ny}}{M_{oy}} < \frac{M_{nx}}{M_{ox}}$$

then

$$\frac{M_{nx}}{M_{ox}} + \frac{M_{ny}}{M_{oy}}\left(\frac{1-\beta}{\beta}\right) = 1 \tag{11.35}$$

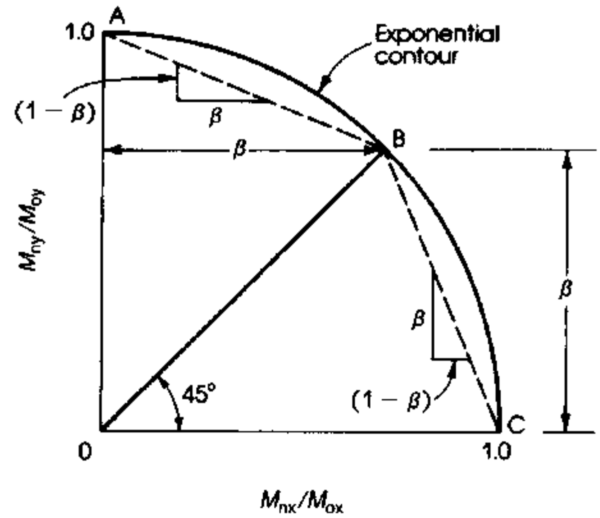


Figure 11.27 Nondimensional load contour at constant P_n (straight-line approximation).

The actual value of β depends on the ratio P_n/P_o as well as the material and properties of the cross-section. For lightly loaded columns, β will vary from 0.55 to 0.7. An average value of $\beta = 0.65$ can be used for design purposes.

When uniformly distributed reinforcement is adopted along all faces of rectangular columns, the ratio M_{oy}/M_{ox} is approximately b/h, where b and h are the width and total depth of the rectangular section, respectively. Substituting this ratio in Eqs. 11.34 and 11.35,

$$M_{ny} + M_{nx}\left(\frac{b}{h}\right)\left(\frac{1-\beta}{\beta}\right) \approx M_{oy} \tag{11.36}$$

and

$$M_{nx} + M_{ny}\left(\frac{h}{b}\right)\left(\frac{1-\beta}{\beta}\right) \approx M_{ox} \tag{11.37}$$

For $\beta = 0.65$ and $h/b = 1.5$,

$$M_{oy} \approx M_{ny} + 0.36M_{nx} \tag{11.38}$$

and

$$M_{ox} \approx M_{nx} + 0.80M_{ny} \tag{11.39}$$

From this presentation, it can be seen that direct explicit equations for the design of columns under axial load and biaxial bending are not available. Therefore, the designer should have enough experience to make an initial estimate of the section using the values of P_n, M_{nx} and M_{ny} and the uniaxial equations and then check the adequacy of the column section using the equations for biaxial bending or by computer.

Example 11.18

The section of a short tied column is 16 × 24 in. and is reinforced with eight no. 10 bars distributed as shown in Fig. 11.28. Determine the design load on the section ϕP_n if it acts at $e_x = 8$ in. and $e_y = 12$ in. Use $f'_c = 5$ ksi, $f_y = 60$ ksi, and the Bresler reciprocal equation.

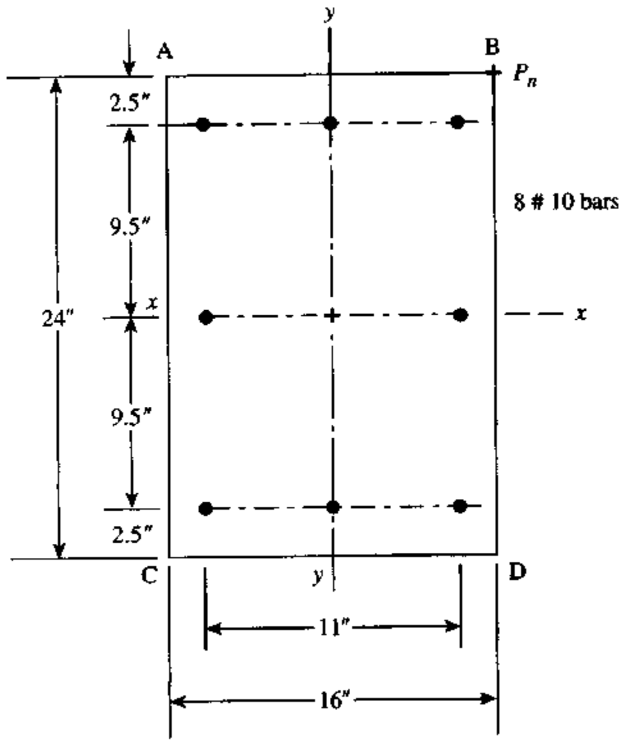


Figure 11.28 Example 11.18: biaxial load, Bresler method: $P_n = 421.5$ K.

Solution

1. Determine the uniaxial load capacity P_{nx} about the x-axis when $e_y = 12$ in. In this case, $b = 16$ in., $h = 24$ in., $d = 21.5$ in., $d' = 2.5$ in., and $A_s = A_s' = 3.81$ in.^{2} The solution will be performed using statics following the steps of Examples 11.2 and 11.4 for balanced and compression-control conditions.

   a. For the balanced condition,

$$c_b = \left(\frac{87}{87 + f_y}\right) \qquad d = \left(\frac{87}{147}\right) 21.5 = 12.72 \text{ in.}$$

$$a_b = 0.80(12.72) = 10.18 \text{ in.} \qquad (\beta_1 = 0.8 \text{ when} f_c' = 5 \text{ ksi})$$

$$C_c = 0.85 f_c' ab = 692.3 \text{ K} \qquad f_s' = 87 \left(\frac{c - d'}{c}\right) = 69.9 \text{ ksi}$$

   Then $f_s' = 60$ ksi.

$$C_s = A_s'(f_y - 0.85 f_c') = 212.4 \text{ K} \qquad T = A_s f_y = 228.6 \text{ K}$$

$$P_{ox} = C_c + C_s - T = 676.1 \text{ K}$$

$$\phi P_{bx} = 0.65 P_{bx} = 439.5 \text{ K} \qquad (\phi = 0.65 \text{ for } \varepsilon_t = 0.002)$$

   b. For $e_y = 12$ in. $< d = 21.5$ in., assume compression failure and follow the steps of Example 11.4 to get $a = 10.65$ in. and $c = a/0.8 = 13.31$ in. $> C_b = 12.72$ in. Thus, compression controls. Check

$$f_s' = 87 \left(\frac{c - d'}{c}\right) = 70 \text{ ksi} > f_y$$

Therefore, $f_s' = 60$ ksi. Check

$$f_s = 87\left(\frac{d-c}{c}\right) = 53.53 \text{ ksi} < 60 \text{ ksi}$$

Calculate forces: $C_c = 0.85 f_c' ab = 724.2$ K, $C_s = A_s'(f_y - 0.85 f_c') = 212.4$ K, $T = A_s f_s = 203.95$ K, $P_{nx} = C_c + C_s - T = 732.6$ K. $P_{nx} > P_{bx}$, so this is a compression failure case as assumed.

$$\varepsilon_t = \left(\frac{d-c}{c}\right) 0.003 = 0.00185$$

$$\varepsilon_t < 0.002 \quad \phi = 0.65$$

$$P_{ux} = \phi P_{nx} = 476.2 \text{ K}$$

c. Take moments about A_s using Eq. 11.11,

$$d'' = 9.5 \text{ in.} \qquad e' = 21.5 \text{ in.}$$

$$P_{nx} = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] = 732.5 \text{ K}$$

2. Determine the uniaxial load capacity P_{ny} about the y-axis when $e_x = 8$ in. In this case, $b = 24$ in., $h = 16$ in., $d = 13.5$ in., $d' = 2.5$ in., and $A_s = A_s' = 3.81$ in.^{2} The solution will be performed using statics, as explained in step 1.

a. Balanced condition:

$$c_b = \left(\frac{87}{87 + f_y}\right) d = \left(\frac{87}{147}\right) 13.5 = 7.99 \text{ in.} \qquad a_b = 0.8(7.99) = 6.39 \text{ in.}$$

$$C_c = 0.85 f_c' ab = 651.8 \text{ K} \qquad f_s' = 87\left(\frac{c-d'}{c}\right) = 59.8 \text{ ksi}$$

$$C_s = A_s'(f_s' - 0.85 f_s') = 211.6 \text{ K} \qquad T = A_s f_y = 228.6 \text{ K}$$

In a balanced load, $P_{by} = C_c + C_s - T = 634.8$ K, $\phi P_{by} = 0.65\ P_{by} = 444.4$ K.

b. For $e_x = 8$ in., assume compression failure case and follow the steps of Example 11.4 to get $a = 6.65$ in. and $c = a/0.8 = 8.31$ in. $> c_b$ (compression failure). Check

$$f_s' = 87\left(\frac{c-d'}{c}\right) = 60.8 \text{ ksi}$$

Therefore, $f_s' = 60$ ksi. Check

$$f_s = 87\left(\frac{d-c}{c}\right) = 54.3 \text{ ksi}$$

Calculate forces: $C_c = 0.85 f_c' ab = 678.3$ K, $C_s = A_s'(60 - 0.85 f_c') = 212.4$ K, $T = A_s f_s = 206.9$ K, $P_{ny} = C_c + C_s - T = 683.3$ K, and $\phi P_{ny} = P_{uy} = 0.65\ P_{ny} = 444.5$ K. Because $P_{ny} > P_{by}$, compression failure occurs, as assumed.

$$\varepsilon_t = \left(\frac{d-c}{c}\right) 0.003 = 0.00187$$

$$\varepsilon_t < 0.002 \quad \phi = 0.65$$

$$P_{uy} = \phi P_{ny} = 444.5 \text{ K}$$

c. Take moments about A_s using Eq. 11.11:

$$d'' = 5.5 \text{ in.} \quad e' = 13.5 \text{ in.}$$

$$P_{ny} = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] = 684 \text{ K}$$

3. Determine the theoretical axial load P_{n_0}:

$$P_{n_0} = 0.85 f_c' A_g + A_{st}(f_y - 0.85 f_c')$$

$$= 0.85(5)(16 \times 24) + 10.16(60 - 0.85 \times 5) = 2198.4 \text{ K} \quad \phi P_{n0} = 0.65 P_{n0} = 1429 \text{ K}$$

4. Using the Bresler equation (Eq. 11.42), multiply by 100:

$$\frac{100}{P_u} = \frac{100}{476.2} + \frac{100}{444.5} - \frac{100}{1429} = 0.365$$

$$P_u = 274 \text{ K} \quad \text{and} \quad P_n = \frac{P_u}{0.65} = 421.5 \text{ K}$$

Notes:

1. Approximate equations or the ACI charts may be used to calculate P_{nx} and P_{ny}. However, since the Bresler equation is an approximate solution, it is preferable to use accurate procedures, as was done in this example, to calculate P_{nx} and P_{ny}. Many approximations in the solution will produce inaccurate results. Computer programs based on statics are available and may be used with proper checking of the output.
2. In Example 11.18, the areas of the corner bars were used twice, once to calculate P_{nx} and once to calculate P_{ny}. The results obtained are consistent with similar solutions. A conservative solution is to use half of the corner bars in each direction, giving $A_s = A_s' = 2(1.27) = 2.54 \text{ in.}^2$, which will reduce the values of P_{nx} and P_{ny}.

Example 11.19

Determine the nominal design load, P_n, for the column section of the previous example using the Parme load contour method; see Fig. 11.29.

Solution

1. Assume $\beta = 0.65$. The uniaxial load capacities in the direction of x- and y-axes were calculated in Example 11.18:

$$P_{ux} = 476.2 \text{ K} \quad P_{uy} = 444.5 \text{ K} \quad P_{nx} = 732.6 \text{ K} \quad P_{ny} = 683.8 \text{ K}$$

2. The moment capacity of the section about the x-axis is

$$M_{ox} = P_{nx} \cdot e_y = 732.6 \times 12$$

The moment capacity of the section about the y-axis is

$$M_{oy} = P_{ny} e_x = 683.8 \times 8 \text{ K·in.}$$

3. Let the nominal load capacity be P_n. The nominal design moment on the section about the x-axis is

$$M_{nx} = P_n e_y = P_n \times 12 \text{ K·in.}$$

and that about the y-axis is

$$M_{ny} = P_n e_x = 8 P_n$$

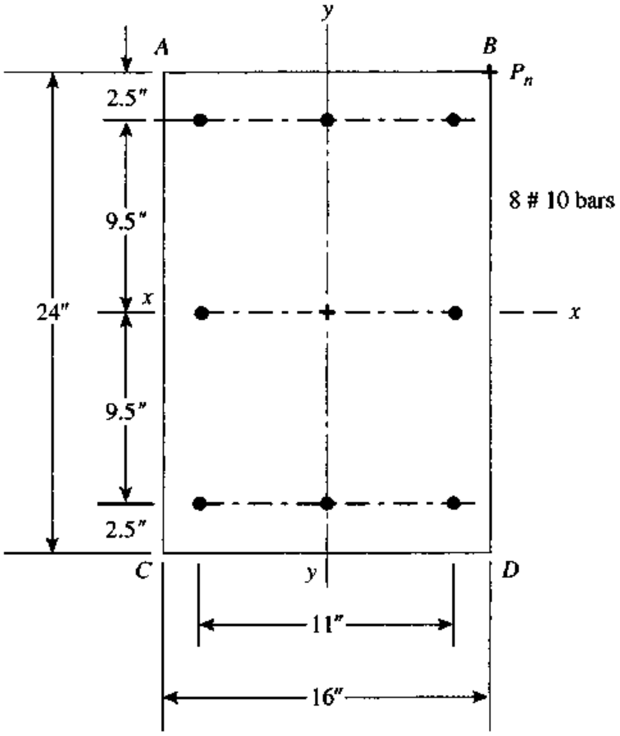


Figure 11.29 Example 11.19: biaxial load, PCA method: $P_n = 455$ K.

4. Check if $M_{ny}/M_{oy} > M_{nx}/M_{ox}$:

$$\frac{8P_n}{683.8 \times 8} > \frac{12P_n}{732.6 \times 12} \quad \text{or} \quad 1.463 \times 10^{-3} P_n > 1.365 \times 10^{-3} P_n$$

Then $M_{ny}/M_{oy} > M_{nx}/M_{ox}$. Therefore, use Eq. 11.48.

5. $$\frac{8P_n}{683.8 \times 8} + \frac{12P_n}{732.6 \times 12}\left(\frac{1 - 0.65}{0.65}\right) = 1$$

Multiply by 1000 to simplify calculations.

$$1.463P_n + 0.735P_n = 1000 \quad P_n = 455 \text{ K}$$

$$P_u = \phi P_n = 295.75 \text{ K} \quad (\phi = 0.65)$$

Note that P_u is greater than the value of 274 K obtained by the Bresler reciprocal method (Eq. 11.42) in the previous example by about 8%.

11.18 EQUATION OF FAILURE SURFACE

A general equation for the analysis and design of reinforced concrete short and tied rectangular columns was suggested by Hsu [16]. The equation is supposed to represent the failure surface and interaction diagrams of columns subjected to combined biaxial bending and axial load, as shown in Fig. 11.26. The axial load can be compressive or a tensile force. The equation is presented as follows:

$$\left(\frac{P_n - P_b}{P_o - P_b}\right) + \left(\frac{M_{nx}}{M_{bx}}\right)^{1.5} + \left(\frac{M_{ny}}{M_{by}}\right)^{1.5} = 1.0 \qquad (11.40)$$

where

P_n = nominal axial strength (positive if compression and negative if tension) for a given eccentricity

P_o = nominal axial load (positive if compression and negative if tension) at zero eccentricity

P_b = nominal axial compressive load at balanced strain condition

M_{nx}, M_{ny} = nominal bending moments about the x- and y-axes, respectively

M_{bx}, M_{by} = nominal balanced bending moments about the x- and y-axes, respectively, at balanced strain conditions

To use Eq. 11.4, all terms must have a positive sign. The value of P_o was given earlier (Eq. 10.1):

$$P_o = 0.85 f'_c (A_g - A_{st}) + A_{st} \cdot f_y \qquad (11.41)$$

The nominal balanced load, P_b, and the nominal balanced moment, $M_b = P_b e_b$, were given in Eq. 11.6 and 11.7, respectively, for sections with tension and compression reinforcement only. For other sections, these values can be obtained by using the principles of statics.

Note that the equation of failure surface can also be used for uniaxial bending representing the interaction diagram. In this case, the third term will be omitted when $e_x = 0$, and the second term will be omitted when $e_y = 0$.

When $e_x = 0$ (moment about the x-axis only),

$$\left(\frac{P_n - P_b}{P_o - P_b}\right) + \left(\frac{M_{nx}}{M_{bx}}\right)^{1.5} = 1.0 \qquad (11.42)$$

(This is Eq. 11.18, given earlier.) When $e_y = 0$ (moment about the y-axis only),

$$\left(\frac{P_n - P_b}{P_o - P_b}\right) + \left(\frac{M_{ny}}{M_{by}}\right)^{1.5} = 1.0 \qquad (11.43)$$

Applying Eq. 11.4 to Examples 11.2 and 11.4, $P_b = 453.4$ K, $M_{bx} = 6810.8$ K·in., $e_y = 10$ in., and $P_o = 0.85(4)(14 \times 22 - 8) + 8(60) = 1500$ K.

$$\frac{P_n - 453.4}{1500 - 453.4} + \left(\frac{10P_n}{6810.8}\right)^{1.5} = 1.0$$

Multiply by 1000 and solve for P_n:

$$(0.9555P_n - 433.2) + 0.05626P_n^{1.5} = 1000$$

$$0.9555P_n + 0.05626P_n^{1.5} = 1433.2$$

$P_n = 611$ K, which is close to that obtained by analysis.

Example 11.20

Determine the nominal design load, P_n, for the column section of Example 11.18 using the equation of failure surface.

Solution

1. Compute

$$\begin{aligned} P_o &= 0.85 f'_c(A_g - A_{st}) + A_{st} f_y \\ &= 0.85(5)(16 \times 24 - 10.16) + (10.16 \times 60) \\ &= 2198.4 \text{ K} \end{aligned}$$

2. Compute P_b and M_b using Eqs. 11.6 and 11.8 about the x- and y-axes, respectively.
   a. About the x-axis,

$$a_{bx} = \frac{87 d_t}{87 + f_y} = \frac{87(21.5)}{87 + 60} = 12.72 \text{ in.}$$

$$a_{bx} = 0.8(12.72) = 10.18 \text{ in.}$$

$$f'_s = 87\left(\frac{c - d'}{c}\right) = 69.9 \text{ ksi} \quad f'_s = 60 \text{ ksi}$$

$$d''_x = 9.5 \text{ in} \quad A_s = A'_s = 3.81 \text{ in.}^2$$

$$\begin{aligned} P_{bx} &= 0.85 f'_c a_x b + A'_s(f_y - 0.85 f'_c) - A_s f_y \\ &= 0.85(5)(10.18)(16) + 3.81(60 - 0.85 \times 5) - 3.81(60) \\ &= 676.1 \text{ K} \end{aligned}$$

$$\begin{aligned} M_{bx} &= 0.85(5)(10.18)(16)\left(21.5 - \frac{10.18}{2} - 9.5\right) \\ &\quad + 3.81(60 - 0.85 \times 5) \times (21.5 - 2.5 - 9.5) + 3.81(60)(9.5) \\ &= 8973 \text{ K·in.} = 747.8 \text{ K·ft} \end{aligned}$$

   b. About the y-axis: $d_t = 13.5$ in., $d''_y = 5.5$ in., $A_s = A'_s = 3.81$ in.^{2}

$$c_{by} = \frac{87(13.5)}{87 + 60} = 7.99 \text{ in.}$$

$$a_{by} = 0.8(7.99) = 6.39 \text{ in.} \quad f'_s = 87\left(\frac{c - d'}{c}\right) = 59.8 \text{ ksi}$$

$$\begin{aligned} P_{by} &= 0.85(5)(6.39)(24) + 3.81(59.8 - 0.85 \times 5) - 3.81(60) \\ &= 634.8 \text{ K} \end{aligned}$$

$$\begin{aligned} M_{by} &= 0.85(5)(6.39)(24)\left(13.5 - \frac{6.39}{2} 5.5\right) \\ &\quad + 3.81(59.8 - 0.85 \times 5)(13.5 - 2.5 - 5.5) + 3.81(60)(5.5) \\ &= 5557.3 \text{ K} \cdot \text{in.} = 463 \text{ K} \cdot \text{ft} \end{aligned}$$

3. Compute the nominal balanced load for biaxial bending, P_{bb}:

$$\tan\alpha = \frac{M_{ny}}{M_{nx}} = \frac{P_n \cdot e_x}{P_n \cdot e_y} = \frac{e_x}{e_y} = \frac{8}{12} \quad \alpha = 33.7^\circ$$

$$\frac{P_{bx} - P_{by}}{90^\circ} = \frac{\Delta P_b}{90^\circ - \alpha^\circ} \quad \text{or} \quad \frac{676.1 - 634.8}{90} = \frac{\Delta P_b}{90 - 33.7}$$

$$\Delta P_b = 25.8 \text{ K}$$

$$P_{bb} = P_{by} + \Delta P_b = 634.8 + 25.8 = 660.6 \text{ K}$$

4. Compute P_n from the equation of failure surface:

$$\frac{P_n - 660.6}{2198.4 - 660.6} + \left(\frac{P_n \times 12}{8973}\right)^{1.5} + \left(\frac{P_n \times 8}{5557.3}\right)^{1.5} = 1.0$$

Multiply by 1000 and solve for P_n:

$$(0.65P_n - 429.85) + 0.0489P_n^{1.5} + 0.0546P_n^{1.5} = 1000$$

$$0.65P_n + 0.1035P_n^{1.5} = 1429.85$$

By trial, $P_n = 487$ K. Because $P_n < P_{bb}$, it is a tension failure case for biaxial bending, and thus $P_o = -2198.4$ K (to keep the first term positive).

$$1000\left(\frac{P_n - 660.9}{-2198.4 - 660.9}\right) + 0.0489P_n^{1.5} + 0.0546P_n^{1.5} = 1000$$

$$0.35P_n + 0.1035P_n^{1.5} = 769.1$$

$$P_n = 429 \text{ K} \quad \text{and} \quad P_u = 0.65P_n = 278.8 \text{ K}$$

Note: The strength capacity, ϕP_n, of the same rectangular section was calculated using the Bresler reciprocal equation (Example 11.18), Parme method (Example 11.19), and Hsu method (Example 11.20) to get $\phi P_n = 421.5$ K, 455 K, and 429 K, respectively. The Parme method gave the highest value for this example.

11.19 SI EXAMPLE

Example 11.21

Determine the balanced compressive forces P_b, e_b, and M_b for the section shown in Fig. 11.30. Use $f'_c = 30$ MPa and $f_y = 400$ MPA ($b = 350$ mm, $d = 490$ mm).

Solution

1. For a balanced condition, the strain in the concrete is 0.003 and the strain in the tension steel is $\varepsilon_y = f_y/E_s = 400/200{,}000 = 0.002$, where $E_s = 200{,}000$ MPa.

$$A_s = A'_s = 4(700) = 2800 \text{ mm}^2$$

2. Locate the neutral axis depth, c_b:

$$c_b = \left(\frac{600}{600 + f_y}\right) d_t \quad (\text{where } f_y \text{ is in MPa})$$

$$= \left(\frac{600}{600 + 420}\right)(490) = 288 \text{ mm}$$

$$a_b = 0.85c_b = 0.85 \times 288 = 245 \text{ mm}$$

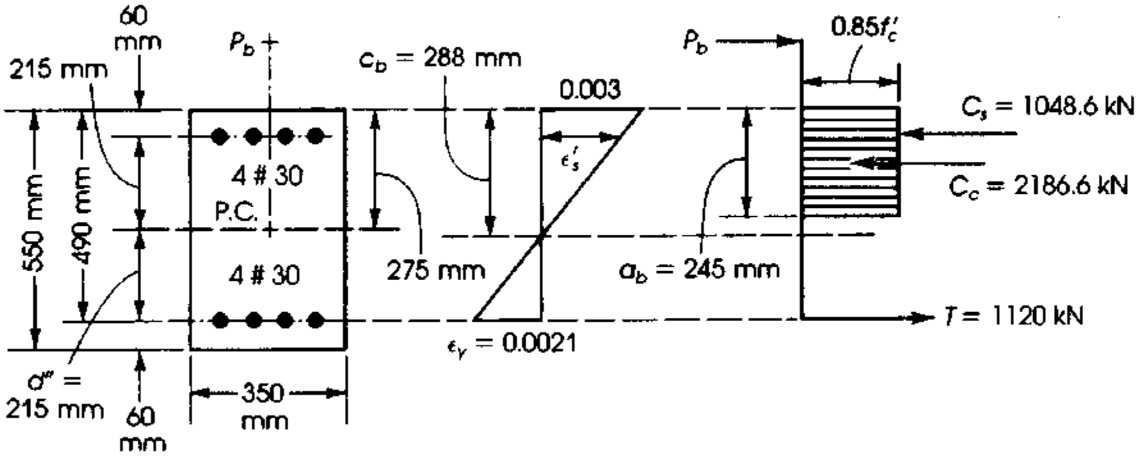


Figure 11.30 Example 11.21.

3. Check if compression steel yields. From the strain diagram,

$$\frac{\varepsilon_s'}{0.003} = \frac{c - d'}{c} = \frac{288 - 60}{288}$$
$$\varepsilon_s' = 0.00238 > \varepsilon_y$$

Therefore, compression steel yields.

4. Calculate the forces acting on the section:

$$C_c = 0.85 f_c' ab = \frac{0.85}{1000} \times 30 \times 245 \times 350 = 2186.6 \text{ kN}$$
$$T = A_s f_y = 2800 \times 0.400 \times 1120 \text{ kN}$$
$$C_s = A_s'(f_y - 0.85 f_c') = \frac{2800 \text{ mm}^2}{1000}(400 - 0.85 \times 30) = 1048.6 \text{ kN}$$

5. Calculate P_b and M_b:

$$P_b = C_c + C_s - T = 2115.2 \text{ kN}$$

From Eq. 11.10,

$$M_b = P_b e_b = C_c \left(d - \frac{a}{2} - d''\right) + C_s(d - d' - d'') + Td''$$

The plastic centroid is at the centroid of the section and $d'' = 215$ mm.

$$M_b = 2186.6\left(490 - \frac{245}{2} - 215\right) + 1048.6(490 - 60 - 215)$$
$$+ 1120 \times 215 - 799.7 \text{ kN} \cdot \text{m}$$
$$e_b = \frac{M_b}{P_b} = \frac{799.7}{2115.2} = 0.378 \text{ m} = 378 \text{ mm}$$

SUMMARY

Sections 11.1–11.3

1. The plastic centroid can be obtained by determining the location of the resultant force produced by the steel and the concrete, assuming both are stressed in compression to f_y and $0.85 f'_c$, respectively.
2. On a load–moment interaction diagram the following cases of analysis are developed:
   a. Axial compression, P_o
   b. Maximum nominal axial load, $P_{n\,\max} = 0.8\,P_o$ (for tied columns) and $P_{n\,\max} = 0.85 P_o$ (for spiral columns)
   c. Compression failure occurs when $P_n > P_b$ or $e < e_b$
   d. Balanced condition, P_b and M_b
   e. Tension failure occurs when $P_n < P_b$ or $e > e_b$
   f. Pure flexure

Section 11.4

1. For compression-controlled sections, $\phi = 0.65$, while for tension-controlled section, $\phi = 0.9$.
2. For the transition region,

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) \quad \text{(for tied columns)}$$

$$\phi = 0.75 + (\varepsilon_t - 0.002)(50) \quad \text{(for spiral columns)}$$

Section 11.5

For a balanced section,

$$c_b = \frac{87 d_t}{87 + f_y} \quad \text{and} \quad a_b = \beta_1 c_b$$

$$\beta_1 = 0.85 \text{ for} f'_c \leq 4 \text{ ksi}$$

$$P_b = C_c + C_s - T = 0.85 f'_c ab + A'_s(f_y - 0.85 f'_c) - A_s f_y$$

$$M_b = P_b e_b = C_c\left(d - \frac{a}{2} - d''\right) + T d'' + C_s(d - d' - d'')$$

$$e_b = \frac{M_b}{P_b}$$

Section 11.6

The equations for the general analysis of rectangular sections under eccentric forces are summarized.

Sections 11.7–11.8

Examples for the cases when tension and compression controls are given.

Sections 11.9–11.10

Examples are given for the interaction diagram and for the case when side bars are used.

Section 11.11

This section gives the load capacity of circular columns. The cases of a balanced section when compression controls are explained by examples.

Section 11.12

This section gives examples of the analysis and design of columns using charts.

Section 11.13

This section gives examples of the design of column sections.

Sections 11.14–11.18

Biaxial bending:

1. For circular columns with uniform reinforcement,

$$M_u = \sqrt{(M_{ux})^2 + (M_{uy})^2} \quad e = \sqrt{(e_x)^2 + (e_y)^2}$$

2. For square and rectangular sections,

$$\frac{1}{P_n} = \frac{1}{P_{nx}} + \frac{1}{P_{ny}} - \frac{1}{P_{n0}}$$

$$\frac{M_{nx}}{M_{ox}} + \frac{M_{ny}}{M_{oy}} \leq 1.0$$

3. In the Bresler load contour method,

$$\left(\frac{M_{nx}}{M_{ox}}\right)^{1.5} + \left(\frac{M_{ny}}{M_{oy}}\right)^{1.5} = 1.0$$

4. In the PCA load contour method,

$$M_{ny} + M_{nx}\left(\frac{b}{h}\right)\left(\frac{1-\beta}{\beta}\right) = M_{oy}$$

$$M_{nx} + M_{ny}\left(\frac{h}{b}\right)\left(\frac{1-\beta}{\beta}\right) = M_{ox}$$

5. Equations of failure surface method are given with applications.

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PROBLEMS

Note: For all problems, use $f_y = 60$ ksi, $d' = 2.5$ in., and $A_s = A'_s$ where applicable. Slight variation in answers are expected.

11.1 (Rectangular sections: balanced condition) For the rectangular column sections given in Table 11.3, determine the balanced compressive load, P_b, the balanced moment, M_b, and the balanced eccentricity, e_b, for each assigned problem. (Answers are given in Table 11.3.) ($\phi = 0.65$.)

11.2 (Rectangular sections: compression failure) For the rectangular column sections given in Table 11.3, determine the load capacity, P_n, for each assigned problem when the eccentricity is $e = 6$ in. (Answers are given in Table 11.3.)

11.3 (Rectangular sections: tension failure) For the rectangular column sections given in Table 11.3, determine the load capacity, P_n, for each assigned problem when the eccentricity is $e = 24$ in. (Answers are given in Table 11.3.)

11.4 (Rectangular sections with side bars) Determine the load capacity, ϕP_n, for the column section shown in Fig. 11.31 considering all side bars when the eccentricity is $e_y = 8$ in. Use $f'_c = 4$ ksi and $f_y = 60$ ksi. (Answer: 658 K.)

11.5 Repeat Problem 11.4 with Fig. 11.32. (Answer: 660 K.)

11.6 Repeat Problem 11.4 with Fig. 11.33. (Answer: 368 K.)

11.7 Repeat Problem 11.4 with Fig. 11.34. (Answer: 822 K.)

Table 11.3 Answers for Problems 11.1–11.3

					Answers to Problems			
					11.1		11.2	11.3
Number	f'_c(ksi)	b (in.)	h (in.)	$A_s = A'_s$	P_b	e_b	P_n (e = 6 in.)	P_n (e = 24 in.)
(a)	4	20	20	6 no. 10	572	17.4	1193	395
(b)	4	14	14	4 no. 8	249	10.9	407	93
(c)	4	24	24	8 no. 10	848	20.1	1860	696
(d)	4	18	26	6 no. 10	698	20.6	1528	591
(e)	4	12	18	4 no. 9	305	15.2	592	176
(f)	4	14	18	4 no. 10	354	16.2	715	221
(g)	5	16	16	5 no. 10	406	15.3	807	228
(h)	5	18	18	5 no. 9	540	12.5	930	230
(i)	5	14	20	4 no. 9	476	13.4	847	221
(j)	5	16	22	4 no. 10	606	14.8	1140	327
(k)	6	16	24	5 no. 10	746	16.8	1532	476
(l)	6	14	20	4 no. 9	534	12.8	944	226

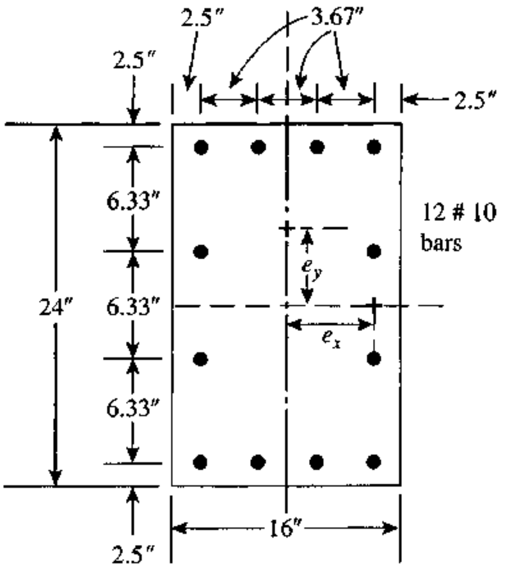


Figure 11.31 Problem 11.4.

11.8 (Design of rectangular column sections) For each assigned problem in Table 11.4, design a rectangular column section to support the factored load and moment shown. Determine A_s, A'_s, and h if not given; then choose adequate bars considering that $A_s = A'_s$. The final total steel ratio, ρ_g, should be close to the given values where applicable. Check the load capacity, ϕP_n, of the final section using statics and equilibrium equations. One solution for each problem is given in Table 11.4.

11.9 (ACI charts) Repeat Problems 11.2*b*, 11.2*d*, 11.2*f*, 11.8*a*, 11.8*c*, and 11.8*e* using the ACI charts.

11.10 (Circular columns: balanced condition) Determine the balanced load capacity, ϕP_b, the balanced moment, ϕM_b, and the balanced eccentricity, e_b, for the circular tied sections shown in Fig. 11.35. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

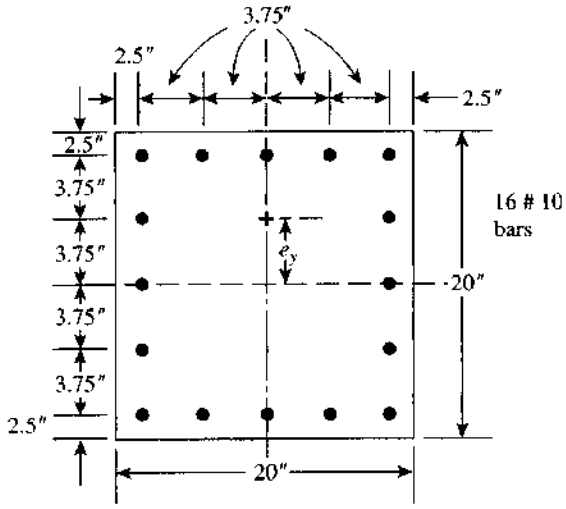


Figure 11.32 Problem 11.5.

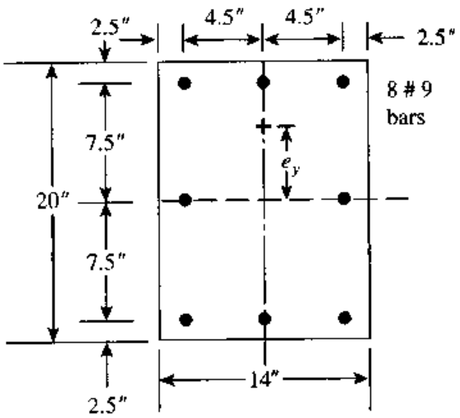


Figure 11.33 Problem 11.6.

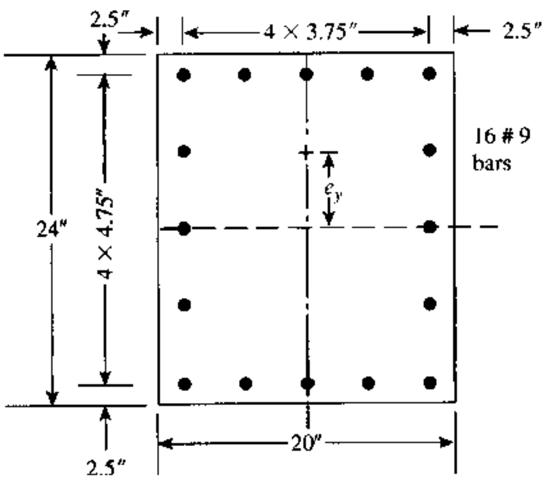


Figure 11.34 Problem 11.7.

Table 11.4 Problem 11.8

Number	f'_c(ksi)	P_u (K)	M_u (K·ft)	b (in.)	h (in.)	ρ_g %	One Solution h (in.)	$A_s = A'_s$
(a)	4	530	353	16	—	4.0	20	5 no. 10
(b)	4	410	205	14	18	—	18	5 no. 8
(c)	4	480	640	18	—	3.5	24	6 no. 10
(d)	4	440	440	20	20	—	20	6 no. 9
(e)	4	1125	375	20	24	—	24	6 no. 10
(f)	4	710	473	18	—	3.0	24	5 no. 10
(g)	5	300	300	14	—	2.0	20	3 no. 9
(h)	5	1000	665	20	26	—	26	6 no. 10
(i)	6	590	197	14	—	2.0	18	2 no. 10
(j)	6	664	332	16	20	—	20	4 no. 9

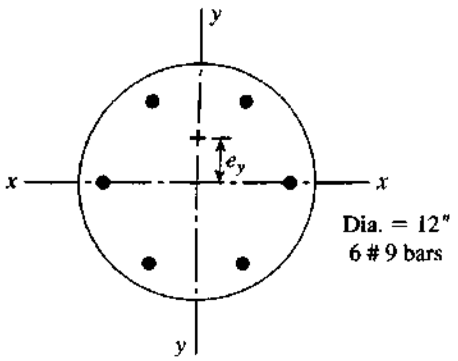


Figure 11.35 Problem 11.10.

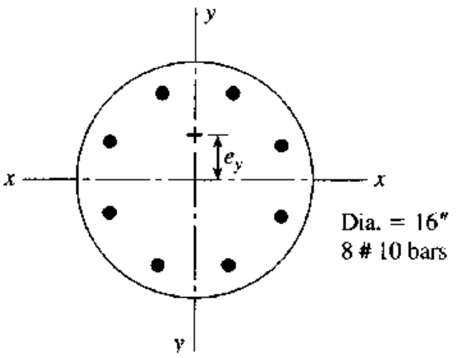


Figure 11.36 Problem 11.11.

11.11 Repeat Problem 11.10 for Fig. 11.36.

11.12 Repeat Problem 11.10 for Fig. 11.37.

11.13 Repeat Problem 11.11 for Fig. 11.38.

11.14 (Circular columns) Determine the load capacity, ϕP_n, for the circular tied column sections shown in Figs. 11.35 through 11.38 when the eccentricity is $e_y = 6$ in. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

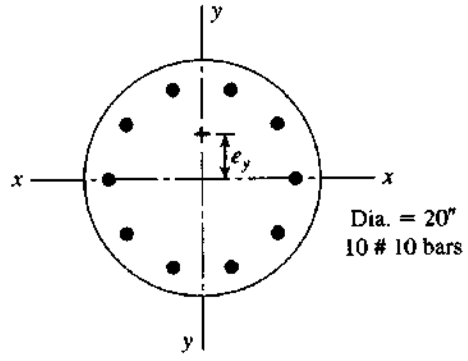


Figure 11.37 Problem 11.12.

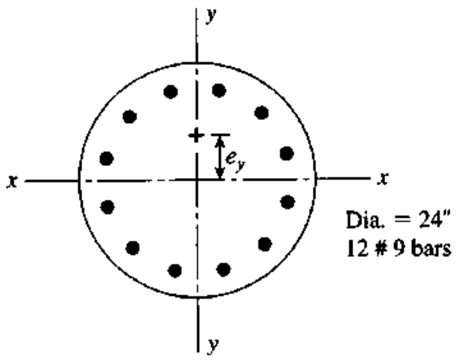


Figure 11.38 Problem 11.13.

11.15 (Biaxial bending) Determine the load capacity, P_n, for the column sections shown in Figs. 11.31 through 11.34 if $e_y = 8$ in. and $e_x = 6$ ini. using the Bresler reciprocal method. Use f_c'(4 ksi) and f_y = 60 ksi. For each problem the values of P_{nx}, P_{ny}, P_{n0} (P_{bx}, M_{bx}), and (P_{by}, M_{by}) are as follows:

- **a.** Figure 11.31: 952 K, 835 K, 2168 K (571 K, 792 K·ft), (536 K, 483 K·ft)
- **b.** Figure 11.32: 930 K, 1108 K, 2505 K (577 K, 742 K·ft), (577 K, 742 K·ft)
- **c.** Figure 11.33: 558 K, 495 K, 1408 K (408 K, 414 K·ft), (368 K, 260 K·ft)
- **d.** Figure 11.34: 1093 K, 1145 K, 2538 K (718 K, 865 K·ft), (701 K, 699 K·ft)

11.16 Repeat Problem 11.15 using the Parme method.

11.17 Repeat Problem 11.15 using the Hsu method.

11.18 For the column sections shown in Fig. 11.31, determine

- **a.** The uniaxial load capacities about the x- and y-axes, P_{nx} and P_{ny} using $e_y = 6$ in. and $e_x = 6$ in.
- **b.** The uniaxial balanced load and moment capacities about the x- and y-axes, P_{bx}, P_{by}, M_{bx}, and M_{by}.
- **c.** The axial load, P_{n_0}.
- **d.** The biaxial load capacity P_n when $e_y = e_x = 6$ in., using the Bresler reciprocal method, the Hsu method, or both.

11.19 Repeat Problem 11.18 for Fig. 11.32.

11.20 Repeat Problem 11.18 for Fig. 11.33.

11.21 Repeat Problem 11.18 for Fig. 11.34.

CHAPTER 12

SLENDER COLUMNS

Columns in a high-rise building, Toronto, Canada.

12.1 INTRODUCTION

In the analysis and design of short columns discussed in the previous two chapters, it was assumed that buckling, elastic shortening, and secondary moment due to lateral deflection had minimal effect on the ultimate strength of the column; thus, these factors were not included in the design procedure. However, when the column is long, these factors must be considered. The extra length will cause a reduction in the column strength that varies with the column effective height, width of the section, the slenderness ratio, and the column end conditions.

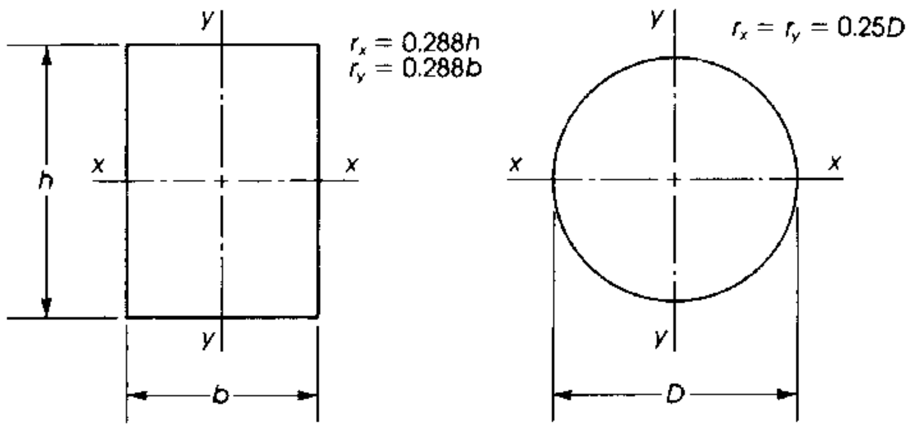


Figure 12.1 Rectangular and circular sections of columns, with radius of gyration r.

A column with a high slenderness ratio will have a considerable reduction in strength, whereas a low slenderness ratio means that the column is relatively short and the reduction in strength may not be significant. The slenderness ratio is the ratio of the column height, l, to the radius of gyration, r, where $r = I/A$, I being the moment of inertia of the section and A the sectional area.

For a rectangular section of width b and depth h (Fig. 12.1), $I_x = bh^3/12$ and $A = bh$. Therefore, $r_x = I/A = 0.288h$ (or, approximately, $r_x = 0.3h$). Similarly, $I_y = hb^3/12$ and $r_y = 0.288b$ (or, approximately, $0.3b$). For a circular column with diameter D, $I_x = I_y = \pi D^2/64$ and $A = \pi D^2/4$; therefore, $r_x = r_y = 0.25D$.

In general, columns may be considered as follows:

1. Long with a relatively high slenderness ratio, where lateral bracing or shear walls are required.
2. Long with a medium slenderness ratio that causes a reduction in the column strength. Lateral bracing may not be required, but strength reduction must be considered.
3. Short where the slenderness ratio is relatively small, causing a slight reduction in strength. This reduction may be neglected, as discussed in previous chapters.

12.2 EFFECTIVE COLUMN LENGTH (Kl_u)

The slenderness ratio l/r can be calculated accurately when the effective length of the column (Kl_u) is used. This effective length is a function of two main factors:

1. The unsupported length, l_u, represents the unsupported height of the column between two floors. It is measured as the clear distance between slabs, beams, or any structural member providing lateral support to the column. In a flat slab system with column capitals, the unsupported height of the column is measured from the top of the lower floor slab to the bottom of the column capital. If the column is supported with a deeper beam in one

direction than in the other direction, l_u should be calculated in both directions (about the x- and y-axes) of the column section. The critical (greater) value must be considered in the design.

2. The effective length factor, K, represents the ratio of the distance between points of zero moment in the column and the unsupported height of the column in one direction. For example, if the unsupported length of a column hinged at both ends, on which sidesway is prevented, is l_u, the points of zero moment will be at the top and bottom of the column—that is, at the two hinged ends. Therefore, the factor $K = l_u/l_u$ is 1.0. If a column is fixed at both ends and sidesway is prevented, the points of inflection (points of 0 moment) are at $l_u/4$ from each end. Therefore, $K = 0.5l_u/l_u = 0.5$ (Fig. 12.2). To evaluate the proper value of K, two main cases are considered.

   When structural frames are braced, the frame, which consists of beams and columns, is braced against sidesway by shear walls, rigid bracing, or lateral support from an adjoining structure. The ends of the columns will stay in position, and lateral translation of joints is prevented. The range of K in braced frames is always equal to or less than 1.0. The ACI Code, Section 10.10, recommends the use of $K = 1.0$ for braced frames.

   When the structural frames are unbraced, the frame is not supported against sidesway, and it depends on the stiffness of the beams and columns to prevent lateral deflection. Joint translations are not prevented, and the frame sways in the direction of lateral loads. The range of K for different columns and frames is given in Fig. 12.2, considering the two cases when sidesway is prevented or not prevented.

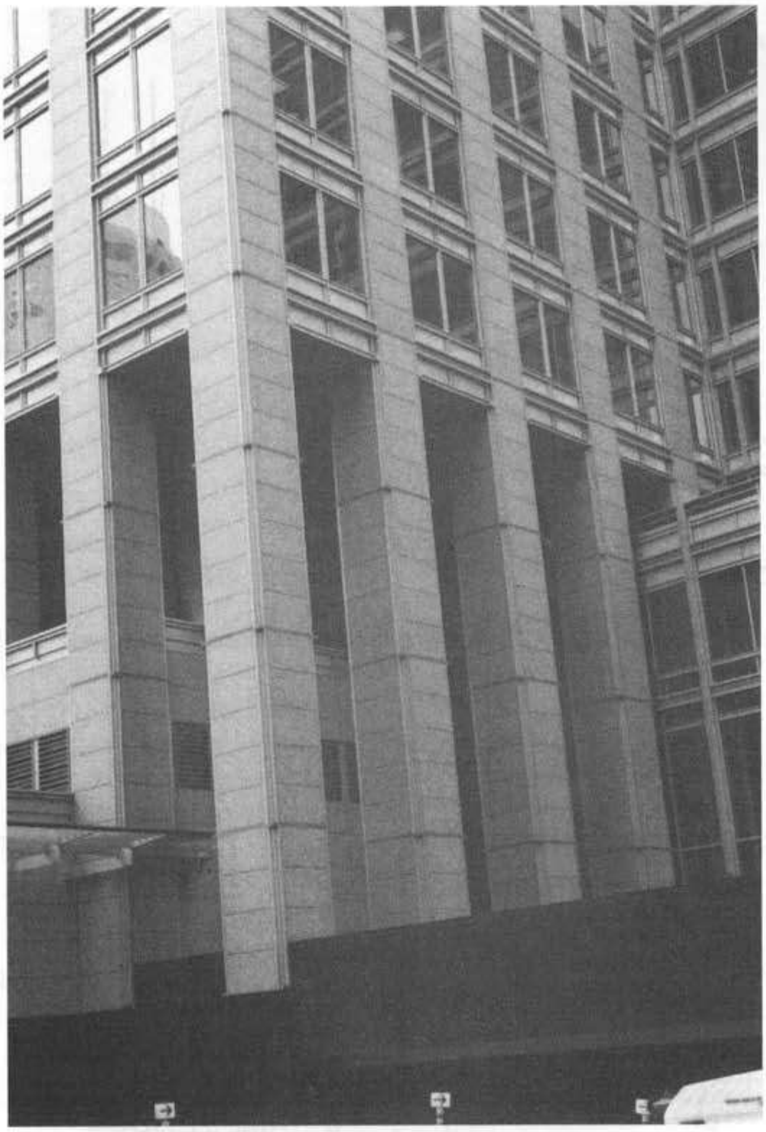
12.3 EFFECTIVE LENGTH FACTOR (K)

The effective length of columns can be estimated by using the alignment chart shown in Fig. 12.3 [10]. To find the effective length factor K, it is necessary first to calculate the end restraint factors Ψ_A and ψ_B at the top and bottom of the column, respectively, where

$$\psi = \frac{\Sigma EI/l_c \text{ of columns}}{\Sigma EI/l \text{ of beams}} \qquad (12.1)$$

(both in the plane of bending) where l_c = length center to center of joints in a frame and l = span length, center to center of joints. The ψ factor at one end shall include all columns and beams meeting at the joint. For a hinged end, ψ is infinite and may be assumed to be 10.0. For a fixed end, ψ is zero and may be assumed to be 1.0. Those assumed values may be used because neither a perfect frictionless hinge nor perfectly fixed ends can exist in reinforced concrete frames.

The procedure for estimating K is to calculate ψ_A for the top end of the column and ψ_B for the bottom end of the column. Plot ψ_A and ψ_B on the alignment chart of Fig. 12.3 and connect the two points to intersect the middle line, which indicates the K-value. Two nomograms are shown, one for braced frames where sidesway is prevented, and the second for unbraced frames, where sidesway is not prevented. The development of the charts is based on the assumptions that (1) the structure consists of symmetrical rectangular frames, (2) the girder moment at a joint is distributed to columns according to their relative stiffnesses, and (3) all columns reach their critical loads at the same time.



Long columns in an office building.

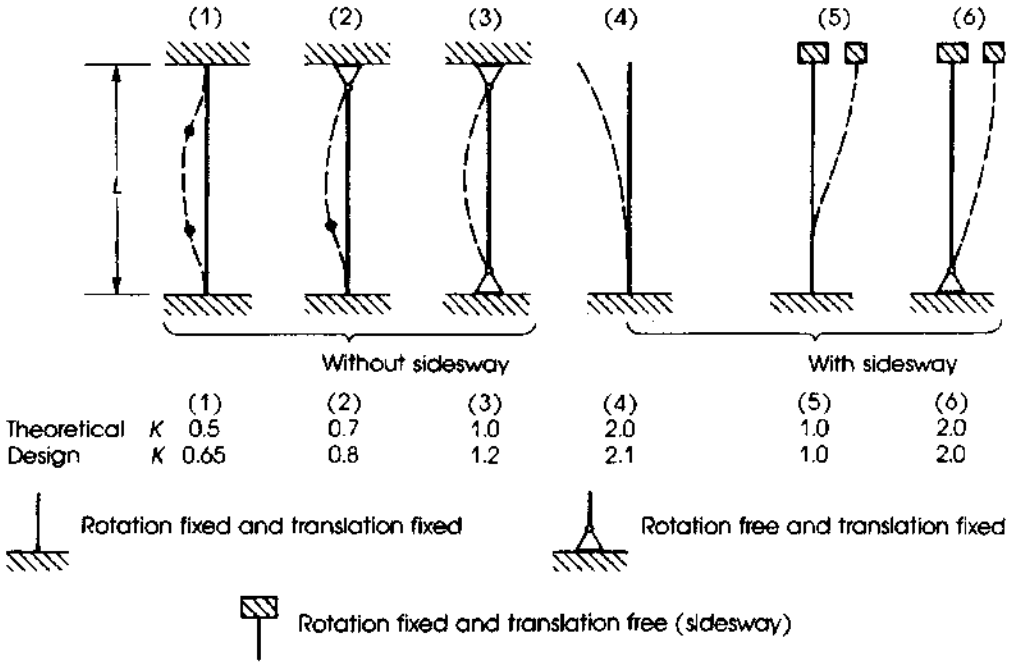
12.4 MEMBER STIFFNESS (*EI*)

The stiffness of a structural member is equal to the modulus of elasticity E times the moment of inertia I of the section. The values of E and I for reinforced concrete members can be estimated as follows:

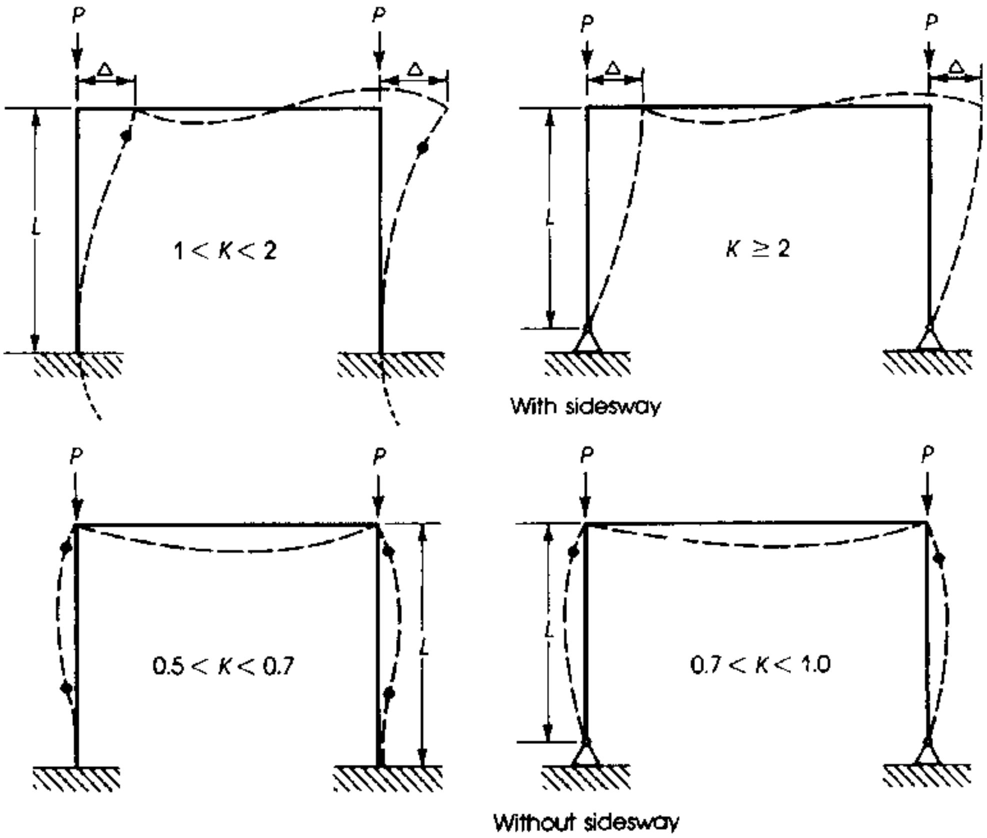
1. The modulus of elasticity of concrete was discussed in Chapter 2; the ACI Code gives the following expression:

$$E_c = 33w^{1.5}\sqrt{f_c'} \quad \text{or} \quad E_c = 57{,}000\sqrt{f_c'} \text{ (psi)}$$

for normal-weight concrete. The modulus of elasticity of steel is $E_s = 29 \times 10^6$ psi.



(a)



(b)

Figure 12.2 (*a*) Effective lengths of columns and length factor *K* and (*b*) effective lengths and *K* for portal columns.

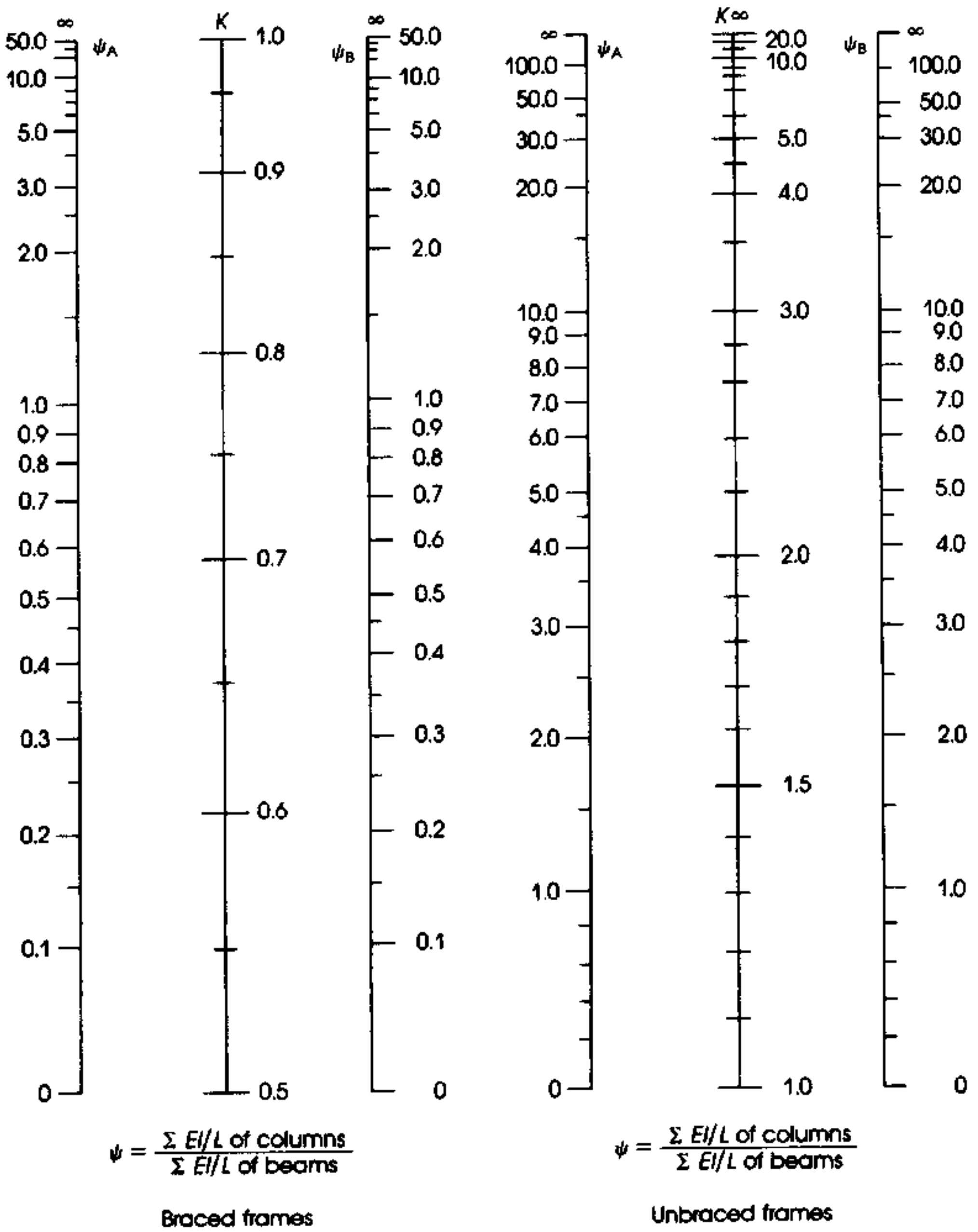


Figure 12.3 Alignment chart.

2. For reinforced concrete members, the moment of inertia I varies along the member, depending on the degree of cracking and the percentage of reinforcement in the section considered.

   To evaluate the factor ψ, EI must be calculated for beams and columns. For this purpose, I can be estimated as follows (ACI Code, Section 10.4.4.1):

   a. Compression members:

   Columns $I = 0.70I_g$

   Walls—Uncracked $I = 0.70I_g$

   —(Cracked) $I = 0.35I_g$

b. Flexural members:

Beams $I = 0.35I_g$

Flat plates and flat slabs $I = 0.25I_g$

Alternatively, the moments of inertia of compression and flexural members, I shall be permitted to be computed as follows:

c. Compression members:

$$I = \left(0.80 + 25\frac{A_{st}}{A_g}\right)\left(1 - \frac{M_u}{P_u h} - 0.5\frac{P_u}{P_o}\right) I_g \leq 0.875 I_g \qquad (12.2)$$

where P_u and M_u shall be determined from the particular load combination under consideration, or the combination of P_u and M_u determined in the smallest value of I, I need not be taken less than $0.35I_g$.

d. Flexural members:

$$I = (0.10 + 25\rho)\left(1.2 - 0.2\frac{b_w}{d}\right) I_g \leq 0.5 I_g \qquad (12.3)$$

where I_g = the moment of inertia of the gross concrete section about the centroidal axis, neglecting reinforcement.

$$\rho = \text{ratio of } A_s/bd \text{ in cross section}$$

The moment of inertia of T-beams should be based on the effective flange width defined in Section 3.15.2. It is generally sufficiently accurate to take I_g of a T-beam as two times the I_g of the web, $2(b_w h^3/12)$.

If the factored moments and shears from an analysis based on the moment of inertia of a wall, taken equal to $0.70I_g$, indicate that the wall will crack in flexure, based on the modulus of rupture, the analysis should be repeated with $I = 0.35I_g$ in those stories where cracking is predicted using factored loads.

The values of the moments of inertia were dervied for nonprestressed members. For prestressed members, the moments of inertia may differ depending on the amount, location, and type of the reinforcement and the degree of cracking prior to ultimate. The stifness value for prestressed concrete members should include an allowance for the variabilty of the stifnesses.

For continuous flexural members, I shall be permitted to be taken as the average of values obtained from Eq. (12.3) for the critical positive and negative moment sections. I need not be taken less than $0.25I_g$.

The cross-sectional dimensions and reinforcement ratio used in the above formulas shall be within 10 percent of the dimensions and reinforcement ratio shown on the design drawings or the stiffness evaluation shall be repeated.

3. Area, $A = 1.0A_g$ (gross-sectional area).

4. The moments of inertia shall be divided by $(1 + \beta_{dns})$ when, sustained lateral loads act on the structure or for stability check, where

$$\beta_{dns} = \frac{\text{maximum factored axial sustained load}}{\text{maximum factored axial load}} = \frac{1.2D \text{ (sustained)}}{1.2D + 1.6L} \leq 1.0 \qquad (12.4)$$

12.5 LIMITATION OF THE SLENDERNESS RATIO (Kl_u/r)

12.5.1 Nonsway Frames

The ACI Code, Section 10.10.1 recommends the following limitations between short and long columns in braced (nonsway) frames:

1. The effect of slenderness may be neglected and the column may be designed as a short column when

$$\frac{Kl_u}{r} \leq 34 - \frac{12M_1}{M_2} \tag{12.5}$$

where M_1 and M_2 are the factored end moments of the column and M_2 is greater than M_1.
2. The ratio M_1/M_2 is considered positive if the member is bent in single curvature and negative for double curvature (Fig. 12.4).
3. The term $(34 - 12M_1/M_2)$ shall not be taken greater than 40.
4. If the factored column moments are zero or $e = M_u/P_u < e_{min}$, the value of M_2 should be calculated using the minimum eccentricity given by ACI Code Section 10.10.6.5:

$$e_{min} = (0.6 + 0.03h) \quad \text{(inch)} \tag{12.6}$$

$$M_2 = P_u(0.6 + 0.03h) \tag{12.7}$$

where M_2 is the minimum moment. The moment M_2 shall be considered about each axis of the column separately. The value of K may be assumed to be equal to 1.0 for a braced frame unless it is calculated on the basis of EI analysis.

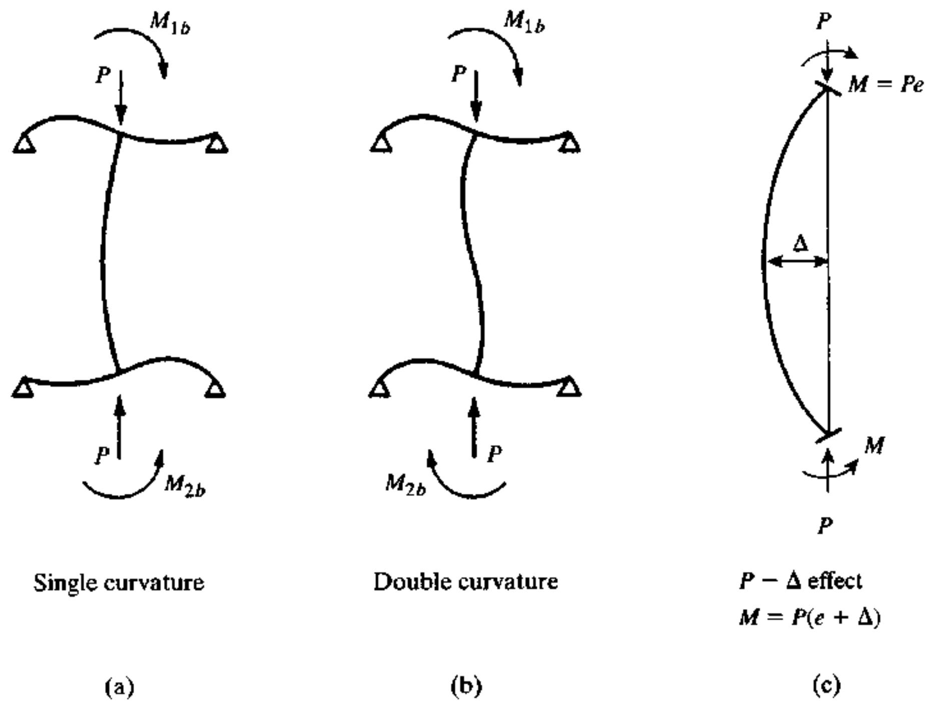


Figure 12.4 Single and double curvatures.

5. It shall be permitted to consider compression members braced against sidesway when bracing elements have a total stiffness, resisting lateral movement of that story, of at least 12 times the gross stiffness of the columns within the story.

12.5.2 Sway Frames

In compression members not braced (sway) against sidesway, the effect of the slenderness ratio may be neglected when

$$\frac{Kl_u}{r} < 22 \qquad \text{(ACI Code Section 10.10.1)} \tag{12.8}$$

12.6 MOMENT-MAGNIFIER DESIGN METHOD

12.6.1 Introduction

The first step in determining the design moments in a long column is to determine whether the frame is braced or unbraced against sidesway. If lateral bracing elements, such as shear walls and shear trusses, are provided or the columns have substantial lateral stiffness, then the lateral deflections produced are relatively small and their effect on the column strength is substantially low. It can be assumed that a story within a structure is nonsway if

$$Q = \frac{\Sigma P_u \Delta_0}{V_{us} l_c} \leq 0.05 \tag{12.9}$$

where ΣP_u and V_{us} are the story total vertical load and story shear, respectively, and Δ_0 is the first-order relative deflection between the top and bottom of the story due to V_{us}. The length l_c is that of the compression member in a frame, measured from center to center of the joints in the frame.

In general, compression members may be subjected to lateral deflections that cause secondary moments. If the secondary moment, M', is added to the applied moment on the column, M_a, the final moment is $M = M_a + M'$. An approximate method for estimating the final moment M is to multiply the applied moment M_a by a factor called the *magnifying moment factor* δ, which must be equal to or greater than 1.0, or $M_{max} = \delta M_a$ and $\delta \geq 1.0$. The moment M_a is obtained from the elastic structural analysis using factored loads, and it is the maximum moment that acts on the column at either end or within the column if transverse loadings are present.

If the P-Δ effect is taken into consideration, it becomes necessary to use a second-order analysis to account for the nonlinear relationship between the load, lateral displacement, and the moment. This is normally performed using computer programs. The ACI Code permits the use of first-order analysis of columns. The ACI Code *moment-magnifier design method* is a simplified approach for calculating the moment-magnifier factor in both braced and unbraced frames.

12.6.2 Magnified Moments in Nonsway Frames

The effect of slenderness ratio Kl_u/r in a compression member of a braced frame may be ignored if $Kl_u/r \leq 34 - 12M_1/M_2$, as given in Section 12.5.1. If Kl_u/r is greater than $(34 - 12M_1/M_2)$, then slenderness effect must be considered. The procedure for determining the magnification factor δ_{ns} in nonsway frames can be summarized as follows (ACI Code, Section 10.10.6):

1. Determine if the frame is braced against sidesway and find the unsupported length, l_u, and the effective length factor, K (K may be assumed to be 1.0).

2. Calculate the member stiffness, EI, using the reasonably approximate equation

$$EI = \frac{0.2E_cI_g + E_sI_{se}}{1 + \beta_{\text{dns}}} \tag{12.10}$$

or the more simplified approximate equation

$$EI = \frac{0.4E_cI_g}{1 + \beta_{\text{dns}}} \tag{12.11}$$

$$EI = 0.25E_cI_g \quad (\text{for } \beta_{\text{dns}} = 0.6) \tag{12.12}$$

where

$E_c = 57{,}000\sqrt{f'_c}$
$E_s = 29 \times 10^6$ psi
I_g = gross moment of inertia of the section about the axis considered, neglecting A_s
I_{se} = moment of inertia of the reinforcing steel

$$\beta_{\text{dns}} = \frac{\text{maximum factored axial sustained load}}{\text{maximum factored axial load}} = \frac{1.2D\ (\text{sustained})}{1.2D + 1.6L}$$

Note: The above β_{dns} is the ratio used to compute magnified moments in columns due to sustained loads.

Equations 12.11 and 12.12 are less accurate than Eq. 12.10. Moreover, Eq. 12.12 is obtained by assuming $\beta_d = 0.6$ in Eq. 12.11.
For improved accuracy EI can be approximated using suggested E and I values provided by:

$$I = \left(0.80 + 25\frac{A_{\text{st}}}{A_g}\right)\left(1 - \frac{M_u}{P_uh} - 0.5\frac{P_u}{p_o}\right) I_g \leq 0.875\, I_g$$

I need not be taken less than $0.35I_g$
where

A_{st} = Total area of longitudinal reinforcement (in.^{2})
P_o = nominal axial strenght at zero eccentricity (lb)
P_u = Factored axial force (+ve for compression) (lb)
M_u = Factored moment at section (lb.in.)
h = thikness of member (in.)

3. Determine the Euler buckling load, P_c:

$$P_c = \frac{\pi^2 EI}{(Kl_u)^2} \tag{12.13}$$

Use the values of EI, K, and l_u as calculated from steps 1 and 2.

4. Calculate the value of the factor C_m to be used in the equation of the moment-magnifier factor. For braced members without transverse loads,

$$C_m = 0.6 + \frac{0.4M_1}{M_2} \geq 0.4 \tag{12.14}$$

where M_1/M_2 is positive if the column is bent in single curvature. For members with transverse loads between supports, C_m shall be taken as 1.0.

5. Calculate the moment magnifier factor δ_{ns}:

$$\delta_{ns} = \frac{C_m}{1 - (P_u/0.75P_c)} \geq 1.0 \tag{12.15}$$

where P_u is the applied factored load and P_c and C_m are as calculated previously.

6. Design the compression member using the axial factored load, P_u, from the conventional frame analysis and a magnified moment, M_c, computed as follows:

$$M_c = \delta_{ns} M_2 \tag{12.16}$$

where M_2 is the larger factored end moment due to loads that result in no sidesway and should be $\geq P_u(0.6 + 0.03\,h)$. For frames braced against sidesway, the sway factor is $\delta_s = 0$. In nonsway frames, the lateral deflection is expected to be less than or equal to $H/1500$, where H is the total height of the frame.

12.6.3 Magnified Moments in Sway Frames

The effect of slenderness may be ignored in sway (unbraced) frames when $Kl_u/r < 22$. The procedure for determining the magnification factor, δ_s, in sway (unbraced) frames may be summarized as follows (ACI Code, Section 10.10.7):

1. Determine if the frame is unbraced against sidesway and find the unsupported length l_u and K, which can be obtained from the alignment charts (Fig. 12.3).

2–4. Calculate EI, P_c, and C_m as given by Eqs. 12.2, 12.10 through 12.14. Note that β_{dns} (to calculate I) is the ratio of maximum factored sustained shear within a story to the total factored shear in that story.

5. Calculate the moment-magnifier factor, δ_s using one of the following methods:

a. Magnifier method

$$\delta_s = \frac{1}{1 - (\Sigma P_u/0.75\Sigma P_c)} \geq 1.0 \tag{12.17}$$

where $\delta_s \leq 2.5$ and ΣP_u is the summation for all the factored vertical loads in a story and ΣP_c is the summation for all sway-resisting columns in a story. Also,

$$\delta_s M_s = \frac{M_s}{1 - (\Sigma P_u/0.75\Sigma P_c)} \geq M_s \tag{12.18}$$

where M_s is the factored end moment due to loads causing appreciable sway.

b. Approximate second order analysis

$$\delta_s = \frac{1}{1 - Q} \geq 1 \quad \text{or} \quad \delta_s M_s = \frac{M_s}{1 - Q} \geq M_s \tag{12.19}$$

where

$$Q = \frac{\Sigma P_u \Delta_o}{V_{us} l_c} \tag{12.20}$$

where

P_u = Factored axial load (lb)

Δ_o = Revative lateral deflection between the top and bottom of a story due to lateral forces using a first order elastic frame analysis

V_{us} = Factored horizontal shear in a story (lb)

l_c = Length of compression member in a frame (in.)

If δ_s exceeds 1.5, δ_s shall be calculated using second order elastic analysis or the magnifier method described in a.

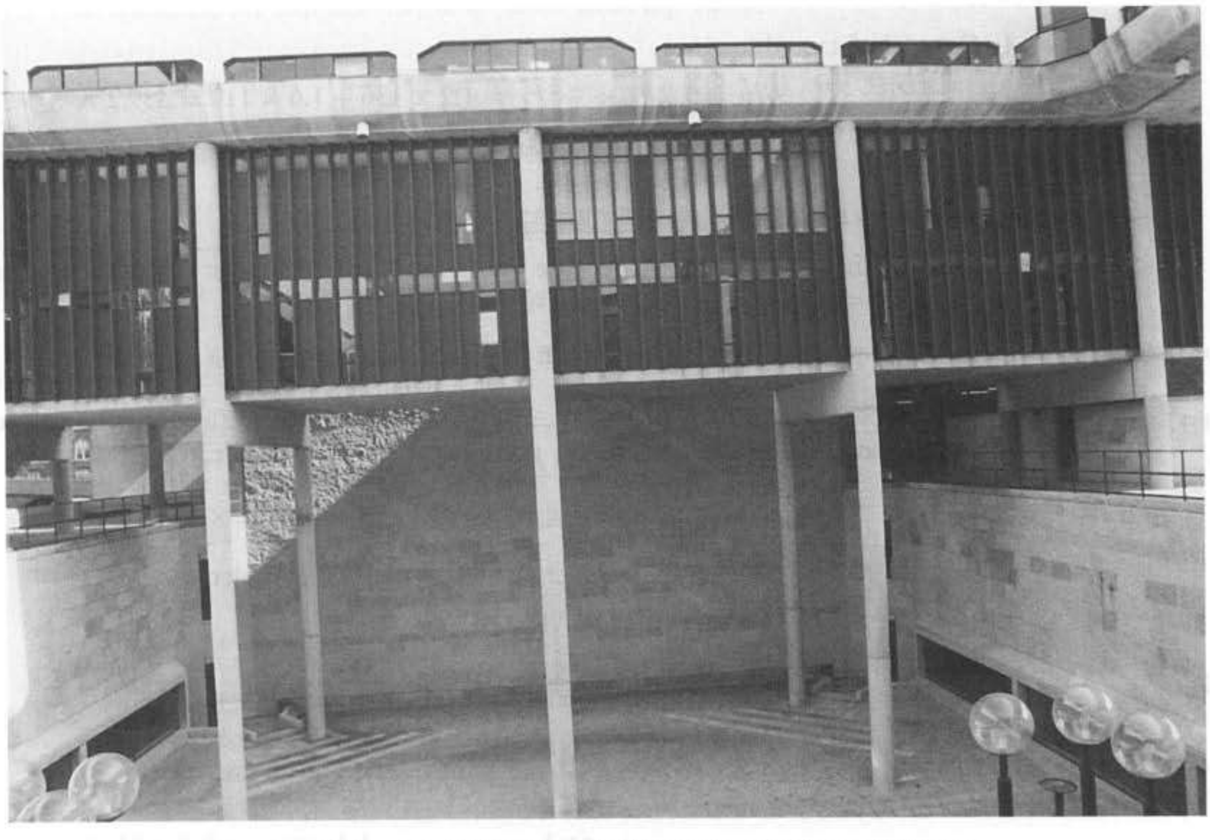
6. Calculate the magnified end moments M_1 and M_2 at the ends of an individual compression member, as follows:

$$M_1 = M_{1\,ns} + \delta_s M_{1s} \tag{12.21}$$

$$M_2 = M_{2\,ns} + \delta_s M_{2s} \tag{12.22}$$

where $M_{1\,ns}$ and $M_{2\,ns}$ are the moments obtained from the no-sway condition, whereas M_{1s} and M_{2s} are the moments obtained from the sway condition. If M_2 is greater than M_1 from structural analysis, then the design magnified moment is

$$M_c = M_{2\,ns} + \delta_s M_{2s} \tag{12.23}$$



Columns, University of Wisconsin, Madison, Wisconsin.

Example 12.1

The column section shown in Fig. 12.5 carries an axial load $P_D = 136\,\text{K}$ and a moment $M_D = 116\,\text{K·ft}$ due to dead load and an axial load $P_L = 110\,\text{K}$ and a moment $M_L = 93\,\text{K·ft}$ due to live load. The column is part of a frame that is braced against sidesway and bent in single curvature about its major axis. The unsupported length of the column is $l_c = 19\,\text{ft}$, and the moments at both ends of the column are equal. Check the adequacy of the column using $f_c' = 4$ ksi and $f_y = 60\,\text{ksi}$.

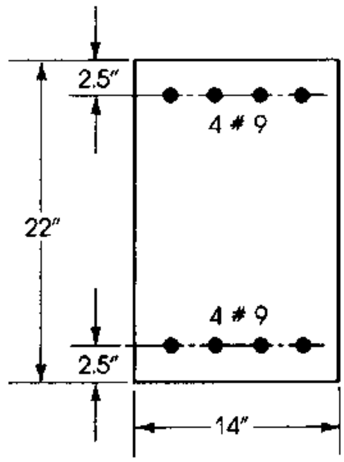


Figure 12.5 Example 12.1.

Solution

1. Calculate factored loads:

$$P_u = 1.2P_D + 1.6P_L = 1.2 \times 136 + 1.6 \times 110 = 339.2 \text{ K}$$

$$M_u = 1.2M_D + 1.6M_L = 1.2 \times 116 + 1.6 \times 93 = 288 \text{ K·ft}$$

$$e = \frac{M_u}{P_u} = \frac{288 \times 12}{339.2} = 10.2 \text{ in.}$$

2. Check if the column is long. Because the frame is braced against sidesway, assume $K = 1.0$, $r = 0.3h = 0.3 \times 22 = 6.6$ in., and $l_u = 19$ ft.

$$\frac{Kl_u}{r} = \frac{1 \times 19 \times 12}{6.6} = 34.5$$

For braced columns, if $Kl_u/r \le 34 - 12M_1/M_2$, slenderness effect may be neglected. Given end moments $M_1 = M_2$ and M_1/M_2 positive for single curvature,

$$\text{Right-hand side} = 34 - 12\frac{M_1}{M_2} = 34 - 12 \times 1 = 22$$

Because $Kl_u/r = 34.5 > 22$, slenderness effect must be considered.

3. Calculate EI from Eq. 12.10:

a. Calculate E_c:

$$E_c = 57{,}000\sqrt{f'_c} = 57{,}000\sqrt{4000} = 3605 \text{ ksi}$$

$$E_s = 29{,}000 \text{ ksi}$$

b. The moment of inertia is

$$I_g = \frac{14(22)^3}{12} = 12{,}422 \text{ in.}^4 \quad A_s = A'_s = 4.0 \text{ in.}^2$$

$$I_{se} = 2 \times 4.0\left(\frac{22-5}{2}\right)^2 = 578 \text{ in.}^4$$

The dead-load moment ratio is

$$\beta_{dns} = \frac{1.2 \times 136}{339.2} = 0.48$$

c. The stiffness is

$$EI = \frac{0.2E_cI_g + E_sI_{se}}{1+\beta_{dns}}$$

$$= \frac{(0.2 \times 3605 \times 12{,}422) + (29{,}000 \times 578)}{1+0.48}$$

$$= 17.40 \times 10^6 \text{ K·in.}^2$$

4. Calculate P_c:

$$P_c = \frac{\pi^2 EI}{(Kl_u)^2} = \frac{\pi^2(17.40 \times 10^6)}{(12 \times 19)^2} = 3303 \text{ K}$$

5. Calculate C_m from Eq. 12.14:

$$C_m = 0.6 + \frac{0.4M_1}{M_2} \geq 0.4$$

$$= 0.6 + 0.4(1) = 1.0$$

6. Calculate the moment-magnifier factor from Eq. 12.15:

$$\delta_{ns} = \frac{C_m}{1-(P_u/0.75P_c)} = \frac{1}{1-339.2/(0.75 \times 3303)} = 1.16$$

7. Calculate the design moment and load: Assume ($\phi = 0.65$),

$$P_n = \frac{339.2}{0.65} = 522 \text{ K}$$

$$M_n = \frac{288}{0.65} = 443.1 \text{ K·ft}$$

Design $M_c = 443.1(1.16) = 514$ K·ft. Design eccentricity = 514/522 = 0.98 ft = 11.82 in., or 12 in.

8. Determine the nominal load strength of the section using $e = 12$ in. according to Example 11.4:

$$P_n = 47.6a + 226.4 - 4f_s \qquad \text{(I)}$$

$$e' = e + d - \frac{h}{2} = 12 + 19.5 - \frac{22}{2} = 20.50 \text{ in.}$$

$$P_n = \frac{1}{20.50}\left[47.6a\left(19.5 - \frac{a}{2}\right) + 226.4(19.5 - 2.5)\right]$$

$$= 45a - 1.15a^2 + 186.6 \qquad \text{(II)}$$

Solving for a from Eqs. I and II, $a = 10.6$ in. and $P_n = 535$ K. The load strength, P_n, is greater than the required load of 522 K; therefore, the section is adequate. If the section is not adequate, increase steel reinforcement.

9. Check the assumed ϕ:

$$a = 10.6 \text{ in.} \quad c = 12.47 \text{ in.} \quad d_t = 19.5 \text{ in.}$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003$$

$$= 0.00169 < 0.002$$

$$\phi = 0.65$$

Example 12.2

Check the adequacy of the column in Example 12.1 if the unsupported length is $l_u = 10$ ft. Determine the maximum nominal load on the column.

Solution

1. Applied loads are $P_n = 522$ K and $M_n = 443.1$ K.
2. Check if the column is long: $l_u = 10$ ft, $r = 0.3h = 0.3 \times 22 = 6.6$ in., and $K = 1.0$ (frame is braced against sidesway).

$$\frac{Kl_u}{r} = \frac{1 \times (10 \times 12)}{6.6} = 18.2$$

Check if $Kl_u/r = 34 - 12M_{1b}/M_{2b}$:

$$\text{Right-hand side} = 34 - 12 \times 1 = 22$$

$$\frac{Kl_u}{r} = 18.2 < 22$$

Therefore, the slenderness effect can be neglected.

3. Determine the nominal load capacity of the short column, as explained in Example 11.4. From Example 11.4, the nominal compressive strength is $P_n = 612.1$ K (for $e = 10$ in.), which is greater than the required load of 522 K, because the column is short with $e = 10.2$ in. (Example 12.1).

Example 12.3

Check the adequacy of the column in Example 12.1 if the frame is unbraced (sway) against sidesway, the end-restraint factors are $\psi_A = 0.8$ and $\psi_B = 2.0$, and the unsupported length is $l_u = 16$ ft.

Solution

1. Determine the value of K from the alignment chart (Fig. 12.3) for unbraced frames. Connect the values of $\psi_A = 0.8$ and $\psi_B = 2.0$, to intersect the K-line at $K = 1.4$.

$$\frac{Kl_u}{r} = \frac{1.4 \times (16 \times 12)}{6.6} = 40.7$$

2. For unbraced frames, if $Kl_u/r \le 22$, the column can be designed as a short column. Because actual $Kl_u r = 40.7 > 22$, the slenderness effect must be considered.
3. Calculate the moment magnifier δ_{ns}, given $C_m = 1.0$, $K = 1.4$, $EI = 17.40 \times 10^6$ K·in.^{2} (from Example 12.1), and

$$P_c = \frac{\pi^2 EI}{(Kl_u)^2} = \frac{\pi^2 \times 17.40 \times 10^6}{(1.4 \times 16 \times 12)^2} = 2377 \text{ K}$$

$$\delta_{ns} = \frac{C_m}{1 - \left(\dfrac{P_u}{0.75 \times P_c}\right)} = \frac{1.0}{1 - \left(\dfrac{339.2}{0.75 \times 2377}\right)} = 1.24$$

4. From Example 12.1, the applied loads are $P_u = 339.2$ K and $M_u = 288$ K·ft, or

$$P_n = 522 \text{ K} \quad \text{and} \quad M_n = 443.1 \text{ K·ft}$$

The design moment $M_c = 1.24(443.1) = 549.4$ K·ft; hence,

$$e = \frac{\delta_{ns} M_n}{P_n} = 549.4 \times \frac{12}{522} = 12.63 \text{ in.} \qquad \text{say, 13 in.}$$

5. The requirement now is to check the adequacy of a short column for $P_n = 522\,\text{K}$, $M_c = 549.4\,\text{K·ft}$, and $e = 13\,\text{in}$. The procedure is explained in Example 11.4.
6. From Example 11.4,

$$P_n = 47.6a + 226.4 - 4f_s$$

$$e' = e + d - \frac{h}{2} = 13 + 19.5 - \frac{22}{2} = 21.5 \text{ in.}$$

$$P_n = \frac{1}{21.5}\left[47.6a\left(19.5 - \frac{a}{2}\right) + 226.4(19.5 - 2.5)\right]$$

$$= 43.16a - 1.1a^2 + 179 \quad a = 10.4 \text{ in.}$$

Thus, $c = 12.24\,\text{in.}$ and $P_n = 508\,\text{K}$. This load capacity of the column is less than the required P_n of 522 K. Therefore, the section is not adequate.

7. Increase steel reinforcement to four no. 10 bars on each side and repeat the calculations to get $P_n = 568\,\text{K}$, $\varepsilon_t < 0.002$, and $\phi = 0.65$.

Example 12.4

Design an interior square column for the first story of an eight-story office building. The clear height of the first floor is 16 ft, and the height of all other floors is 11 ft. The building layout is in 24 bays (Fig. 12.6), and the columns are not braced against sidesway. The loads acting on a first-floor interior column due to gravity and wind are as follows:

$$\text{Axial dead load} = 380 \text{ K}$$

$$\text{Axial live load} = 140 \text{ K}$$

$$\text{Axial wind load} = 0 \text{ K}$$

$$\text{Dead-load moments} = 32 \text{ K·ft (top) and } 54 \text{ K·ft (bottom)}$$

$$\text{Live-load moments} = 20 \text{ K·ft (top) and } 36 \text{ K·ft (bottom)}$$

$$\text{Wind-load moments} = 50 \text{ K·ft (top) and } 50 \text{ K·ft (bottom)}$$

$$EI/l \text{ for beams} = 360 \times 10^3 \text{ K·in.}$$

Use $f'_c = 5$ ksi, $f_y = 60$ ksi, and the ACI Code requirements. Assume an exterior column load of two-thirds the interior column load, a corner column load of one-third the interior column load, and $\beta_{dns} = 0.55$.

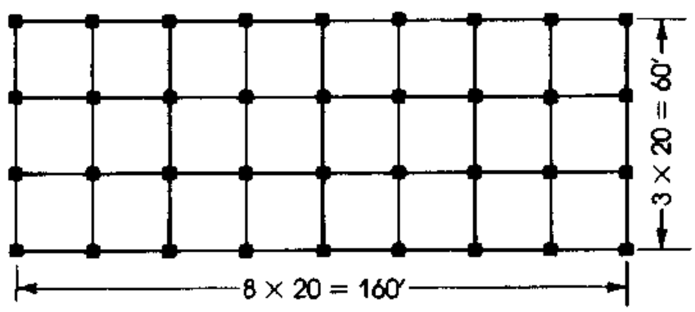


Figure 12.6 Example 12.4.

Solution

1. Calculate the factored forces using load combinations. For gravity loads,

$$P_u = 1.2D + 1.6L = 1.2(380) + 1.6(140) = 680\ \text{K}$$

$$M_u = M_{2\text{ns}} = 1.2M_D + 1.6M_L = 1.2(54) + 1.6(36) = 122.4\ \text{K·ft}$$

For gravity plus wind load,

$$\begin{aligned} P_u &= (1.2D + 0.5L + 1.6W) \\ &= [1.2(380) + 0.5(140 + 0)] = 526\ \text{K} \\ M_{\text{uns}} &= M_{2\text{ns}} = (1.2 \times 54 + 1.6 \times 36) = 122.4\ \text{K·ft} \\ M_{\text{us}} &= M_{2s} = (1.6M_w) = (1.6 \times 50) = 80\ \text{K·ft} \end{aligned}$$

Other combinations are not critical:

$$\begin{aligned} P_u &= 0.9D + 1.6W = 0.9(380) + 1.6(0) = 342\ \text{K} \\ M_2 &= M_{\text{uns}} = 0.9M_D = 0.9(54) = 48.6\ \text{K·ft} \\ M_{2s} &= 1.6M_w = 1.6(50) = 80\ \text{K·ft} \\ e &= \frac{M_u}{P_u} = \frac{M_{2\text{ns}}}{P_u} = 122.4 \times \frac{12}{680} = 2.16\ \text{in.} \\ \min e &= 0.6 + 0.03(18) = 1.14\ \text{in.} < 2.16 \end{aligned}$$

2. Select a preliminary section of column based on gravity load combination using tables or charts. Select a section 18 by 18 in. reinforced by four no. 10 bars (Fig. 12.7).
3. Check Kl_u/r:

$$I_g = \frac{(18)^4}{12} = 8748\ \text{in.}^4 \quad E_c = 4.03 \times 10^6\ \text{psi}$$

for columns, $I = 0.7\ I_g$.
For a 16-ft column,

$$\frac{EI}{l_c} = \frac{(0.7)(8748)(4.03 \times 10^6)}{16 \times 12} = 128.5 \times 10^6$$

For an 11-ft column,

$$\frac{EI}{l_c} = \frac{(0.7)(8748)(4.03 \times 10^6)}{11 \times 12} = 187 \times 10^6$$

For beams, and $EI_g/l_b = 360 \times 10^6$, $I = 0.35\ I_g$, and $EI/l_b = 0.35EI_g/l_b = 126 \times 10^6$

$$\psi(\text{top}) = \psi(\text{bottom}) = \frac{\Sigma(EI/l_c)}{\Sigma(EI/l_b)} = \frac{(128.5 + 187)}{2(126)} = 1.25$$

From the chart (Fig. 12.3), K is 1.37 for an unbraced frame and 0.8 for a braced frame.

$$\frac{Kl_u}{r} = \frac{1.37(16 \times 12)}{0.3 \times 18} = 48.7$$

which is more than 22. Therefore, the slenderness ratio must be considered.

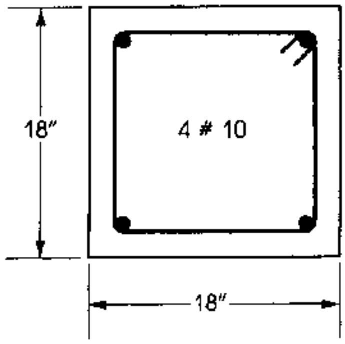


Figure 12.7 Column cross section, Example 12.4.

4. Compute P_c:

$$E_c = 4.03 \times 10^3 \text{ ksi} \quad E_s = 29 \times 10^3 \text{ ksi}$$

$$I_g = 8748 \text{ in.}^4 \qquad I_{se} = 5.06\left(\frac{13}{2}\right)^2 = 214 \text{ in.}^4$$

$$\beta_{dns} = 0.55$$

$$EI = \frac{0.2E_cI_g + E_sI_{se}}{1+\beta_{dns}}$$

$$EI = \frac{0.2(4.03 \times 10^3 \times 8748) + 29 \times 10^3(214)}{1+0.55} = 8.55 \times 10^6 \text{ K·in.}^2$$

For calculation of δ_s, $\beta_{dns} = 0$ and $E = 8.55 \times 10^6(1.55) = 13.25 \times 10^6$ K·in.^{2}

$$P_c = \frac{\pi^2\ EI}{(Kl_u)^2} = \frac{\pi^2(8.55 \times 10^6)}{(0.8 \times 16 \times 12)^2} = 3577 \text{ K} \quad \text{(braced)}$$

$$P_c = \frac{\pi^2(13.25 \times 10^6)}{(1.37 \times 16 \times 12)^2} = 1890 \text{ K} \quad \text{(unbraced)}$$

For one floor in the building, there are 14 interior columns, 18 exterior columns, and four corner columns.

$$\Sigma P_u = 14(526) + 18\left(\frac{2}{3} \times 526\right) + 4\left(\frac{1}{3} \times 526\right) = 14{,}377 \text{ K}$$

$$\Sigma P_c = 14(1890) + 22\left(\frac{2}{3} \times 1890\right) = 54{,}180 \text{ K}$$

$$\delta_s = \frac{1.0}{1 - \left(\dfrac{14{,}377}{0.75 \times 54{,}180}\right)} = 1.54$$

which is greater than 1.0 (Eq. 12.17).

$$M_c = M_{2\text{ns}} + \delta_s M_{2s} = (122.4) + 1.54(80) = 245.6 \text{ K·ft}$$

5. Design loads are $P_u = 526$ K and $M_c = 245.6$ K·ft.

$$e = \frac{245.6(12)}{526} = 5.6 \text{ in.}$$

$$e_{min} = 0.6 + 0.03(18) = 1.14 \text{ in.} < e$$

By analysis, for $e = 5.6$ in. and $A_s = A_s' = 2.53$ in.^{2}, ($\phi = 0.65$ in.) the load capacity of the 18 × 18-in. column is $\phi P_n = 556$ K and $\phi M_n = 259$ K·ft, so the section is adequate. (Solution steps are similar to Example 11.4. Values are $a = 10.37$ in., $c = 13$ in., $f_s = 17$ ksi, $f_s' = 60$ ksi, $\phi P_b = 385$ K, and $e_b = 8.9$ in.).

$$\varepsilon_t = 0.003\frac{(15.5 - 13)}{13} = 0.00058 < 0.002, \quad \phi = 0.65.$$

SUMMARY

Sections 12.1–12.3

1. The radius of gyration is $r = \sqrt{I/A}$, where $r = 0.3h$ for rectangular sections and $0.25D$ for circular sections.
2. The effective column length is Kl_u. For braced frames, $K = 1.0$; for unbraced frames, K varies as shown in Fig. 12.2.
3. K can be determined from the alignment chart (Fig. 12.3) or Eqs. 12.2 through 12.6.

Section 12.4

Member stiffness is EI:

$$E_c = 33w^{1.5}\sqrt{f_c'}$$

The moment of inertia, I, may be taken as $I = 0.35I_g$ for beams, $0.70I_g$ for columns, $0.70I_g$ for uncracked walls, $0.35I_g$ for cracked walls, and $0.25I_g$ for plates and flat slabs.

Alternatively, the moments of inertia of compression and flexural members, I, shall be permitted to be computed as follows:

1. Compression members:

$$I = \left(0.80 + 25\frac{A_{st}}{A_g}\right)\left(1 - \frac{M_u}{P_u h} - 0.5\frac{P_u}{P_o}\right) I_g \leq 0.875 I_g \qquad (12.2)$$

2. Flexural members:

$$I = (0.10 + 25\rho)\left(1.2 - 0.2\frac{b_w}{d}\right) I_g \leq 0.5 I_g \qquad (12.3)$$

Section 12.5

The effect of slenderness may be neglected when

$$\frac{Kl_u}{r} \leq 22 \quad \text{(for unbraced frames)} \qquad (12.8)$$

$$\frac{Kl_u}{r} \leq 34 - \frac{12M_1}{M_2} \quad \text{(for braced columns)} \qquad (12.5)$$

where M_1 and M_2 are the end moments and $M_2 > M_1$.

Section 12.6

1. For nonsway frames,

$$EI = \frac{0.2E_c I_g + E_s I_{se}}{1 + \beta_{dns}} \tag{12.10}$$

or the more simplified equation

$$EI = \frac{0.4E_c I_g}{1 + \beta_{dns}} \tag{12.11}$$

$$\beta_{dns} = \frac{1.2D}{1.2D + 1.6L} \tag{12.4}$$

More simply,

$$EI = 0.25E_c I_g \ (\beta_{dns} = 0.6) \tag{12.12}$$

The Euler buckling load is

$$P_c = \frac{\pi^2 EI}{(Kl_u)^2} \tag{12.13}$$

$$C_m = 0.6 + \frac{0.4M_1}{M_2} \geq 0.4 \tag{12.14}$$

The moment-magnifier factor (nonsway frames) is

$$\delta_{ns} = \frac{C_m}{1 - \frac{P_u}{0.75P_c}} \tag{12.15}$$

The design moment is

$$M_c = \delta_{ns} M_2 \tag{12.16}$$

2. For sway (unbraced) frames, the moment-magnifier factor is calculated either from
   a. Magnifier method

$$\delta_s = \frac{1.0}{1 - \dfrac{\Sigma P_u}{0.75\Sigma P_c}} \geq 1.0 \tag{12.17}$$

   b. Approximate second order analysis

$$\delta_s = \frac{1}{1 - Q} \tag{12.19}$$

$$Q = \frac{\Sigma P_u \Delta_0}{V_{us} l_c} \tag{12.20}$$

the design moment is

$$M_1 = M_{1ns} + \delta_s \ M_{1s} \tag{12.21}$$

$$M_2 = M_{2ns} + \delta_s \ M_{2s} \tag{12.22}$$

If $M_2 > M_1$ then:

$$M_c = M_{2ns} + \delta_s \ M_{2s} \tag{12.23}$$

where M_{2ns} is the unmagnified moment due to gravity loads (nonsway moment) and $\delta_s M_{2s}$ is the magnified moment due to sway frame loads.

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PROBLEMS

12.1 The column section in Fig. 12.8 carries an axial load $P_D = 128\,\text{K}$ and a moment $M_D = 117\,\text{K·ft}$ due to dead load and an axial load $P_L = 95\,\text{K}$ and a moment $M_1 = 100\,\text{K·ft}$ due to live load. The column is part of a frame, braced against sidesway, and bent in single curvature about its major axis. The unsupported length of the column is $l_u = 18\,\text{ft}$, and the moments at both ends are equal. Check the adequacy of the section using $f'_c = 4$ ksi and $f_y = 60\,\text{ksi}$.

12.2 Repeat Problem 12.1 if $l_u = 12\,\text{ft}$.

12.3 Repeat Problem 12.1 if the frame is unbraced against sidesway and the end-restraint factors are ψ (top) = 0.7 and ψ (bottom) = 1.8 and the unsupported height is $l_u = 14\,\text{ft}$.

12.4 The column section shown in Fig. 12.9 is part of a frame unbraced against sidesway and supports an axial load $P_D = 166\,\text{K}$ and a moment $M_D = 107\,\text{K·ft}$ due to dead load and $P_L = 115\,\text{K}$ and $M_L = 80\,\text{K·ft}$ due to live load. The column is bent in single curvature and has an unsupported length $l_u = 16\,\text{ft}$. The moment at the top of the column is $M_2 = 1.5M_1$, the moment at the bottom of the column. Check if the section is adequate using $f'_c = 5$ ksi, $f_y = 60\,\text{ksi}$, ψ (top) = 2.0, and ψ (bottom) = 1.0.

12.5 Repeat Problem 12.4 if the column length is $l_u = 14\,\text{ft}$.

12.6 Repeat Problem 12.4 if the frame is braced against sidesway and $M_1 = M_2$.

12.7 Repeat Problem 12.4 using $f'_c = 4$ ksi and $f_y = 60\,\text{ksi}$.

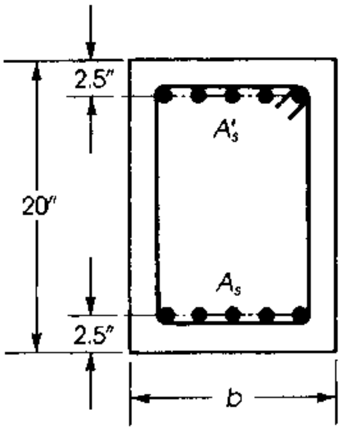


Figure 12.8 Problem 12.1 ($A_s = A'_s$) = 5 no. 9 bars and b = 14 in.

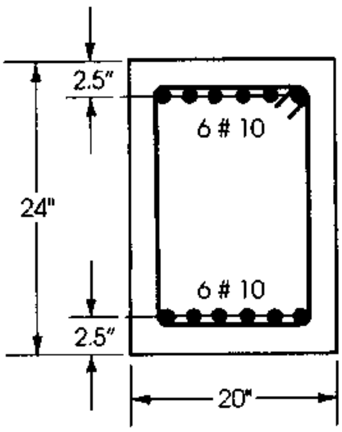


Figure 12.9 Problem 12.4.

12.8 Design a 20-ft-long rectangular tied column for an axial load $P_D = 214.5$ K and a moment $M_D =$ 64 K·ft due to dead load and an axial load $P_L = 120$ K and a moment $M_L = 40$ K·ft due to live load. The column is bent in single curvature about its major axis, braced against sidesway, and the end moments are equal. The end-restraint factors are ψ (top) = 2.5 and ψ (bottom) = 1.4. Use $f'_c = 5$ ksi, $f_y = 60$ ksi, and $b = 15$ in.

12.9 Design the column in Problem 12.8 if the column length is 10 ft.

12.10 Repeat Problem 12.8 if the column is unbraced against sidesway.

CHAPTER 13

FOOTINGS

Office building under construction, New Orleans, Louisiana.

13.1 INTRODUCTION

Reinforced concrete footings are structural members used to support columns and walls and to transmit and distribute their loads to the soil. The design is based on the assumption that the footing is rigid, so that the variation of the soil pressure under the footing is linear. Uniform soil pressure is achieved when the column load coincides with the centroid of the footing. Although this assumption is acceptable for rigid footings, such an assumption becomes less accurate as the footing becomes relatively more flexible. The proper design of footings requires that

1. The load capacity of the soil is not exceeded.
2. Excessive settlement, differential settlement, or rotations are avoided.
3. Adequate safety against sliding and/or overturning is maintained.

The most common types of footings used in buildings are the single footings and wall footings (Figs. 13.1 and 13.2). When a column load is transmitted to the soil by the footing, the soil becomes compressed. The amount of settlement depends on many factors, such as the type of soil, the load intensity, the depth below ground level, and the type of footing. If different footings of the same structure have different settlements, new stresses develop in the structure. Excessive differential settlement may lead to the damage of nonstructural members in the buildings or even failure of the affected parts.

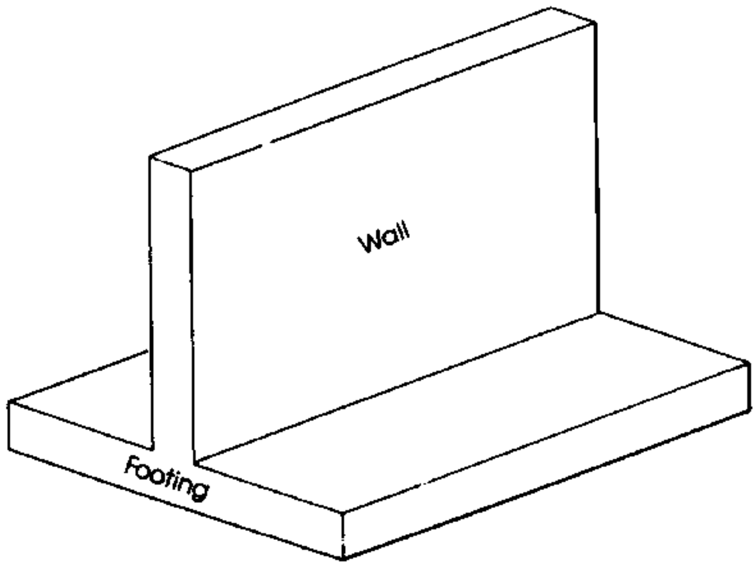


Figure 13.1 Wall footing.

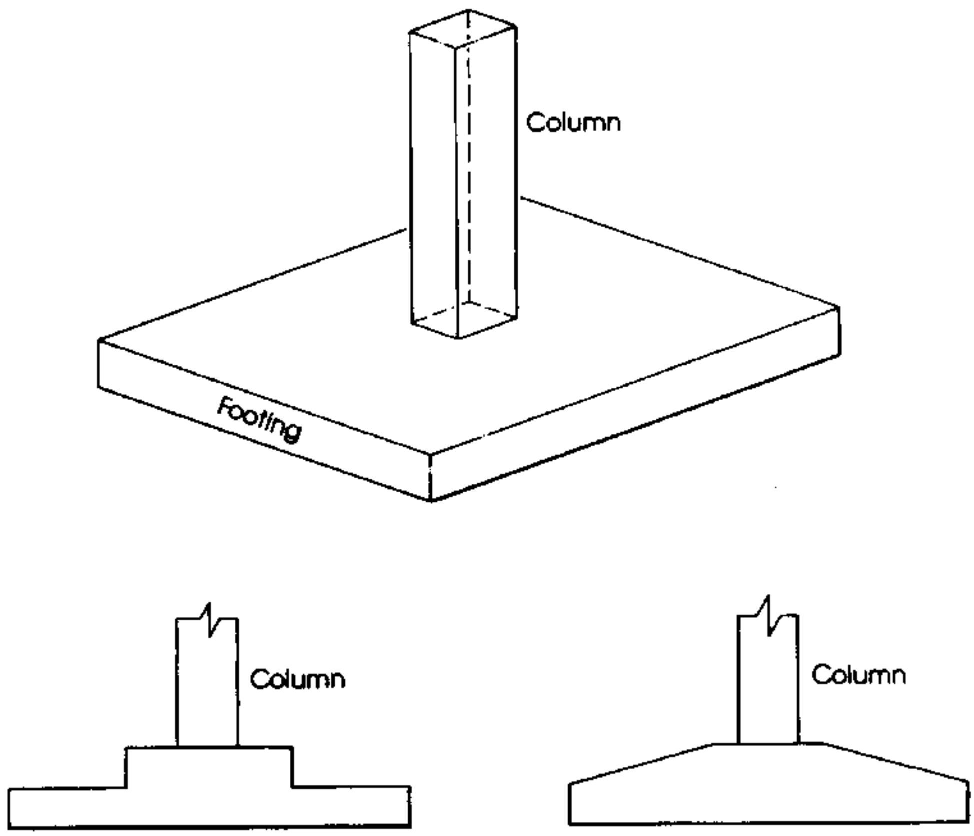


Figure 13.2 Single footing.

Vertical loads are usually applied at the centroid of the footing. If the resultant of the applied loads does not coincide with the centroid of the bearing area, a bending moment develops. In this case, the pressure on one side of the footing will be greater than the pressure on the other side.

If the bearing soil capacity is different under different footings—for example, if the footings of a building are partly on soil and partly on rock—a differential settlement will occur. It is usual in such cases to provide a joint between the two parts to separate them, allowing for independent settlement.

The depth of the footing below the ground level is an important factor in the design of footings. This depth should be determined from soil tests, which should provide reliable information on safe bearing capacity at different layers below ground level. Soil test reports specify the allowable bearing capacity to be used in the design. In cold areas where freezing occurs, frost action may cause heaving or subsidence. It is necessary to place footings below freezing depth to avoid movements.

13.2 TYPES OF FOOTINGS

Different types of footings may be used to support building columns or walls. The most common types are as follows:

1. *Wall footings* are used to support structural walls that carry loads from other floors or to support nonstructural walls. They have a limited width and a continuous length under the wall (Fig. 13.1). Wall footings may have one thickness, be stepped, or have a sloped top.
2. *Isolated*, or *single, footings* are used to support single columns (Fig. 13.2). They may be square, rectangular, or circular. Again, the footing may be of uniform thickness, stepped, or have a sloped top. This is one of the most economical types of footings, and it is used when columns are spaced at relatively long distances. The most commonly used are square or rectangular footings with uniform thickness.
3. *Combined footings* (Fig. 13.3) usually support two columns or three columns not in a row. The shape of the footing in plan may be rectangular or trapezoidal, depending on column loads. Combined footings are used when two columns are so close that single footings cannot be used or when one column is located at or near a property line.
4. *Cantilever*, or *strap, footings* (Fig. 13.4) consist of two single footings connected with a beam or a strap and support two single columns. They are used when one footing supports an eccentric column and the nearest adjacent footing lies at quite a distance from it. This type replaces a combined footing and is sometimes more economical.
5. *Continuous footings* (Fig. 13.5) support a row of three or more columns. They have limited width and continue under all columns.
6. *Raft*, or *mat, foundations* (Fig. 13.6) consist of one footing, usually placed under the entire building area, and support the columns of the building. They are used when
   a. The soil-bearing capacity is low.
   b. Column loads are heavy.
   c. Single footings cannot be used.
   d. Piles are not used.
   e. Differential settlement must be reduced through the entire footing system.
7. *Pile caps* (Fig. 13.7) are thick slabs used to tie a group of piles together and to support and transmit column loads to the piles.

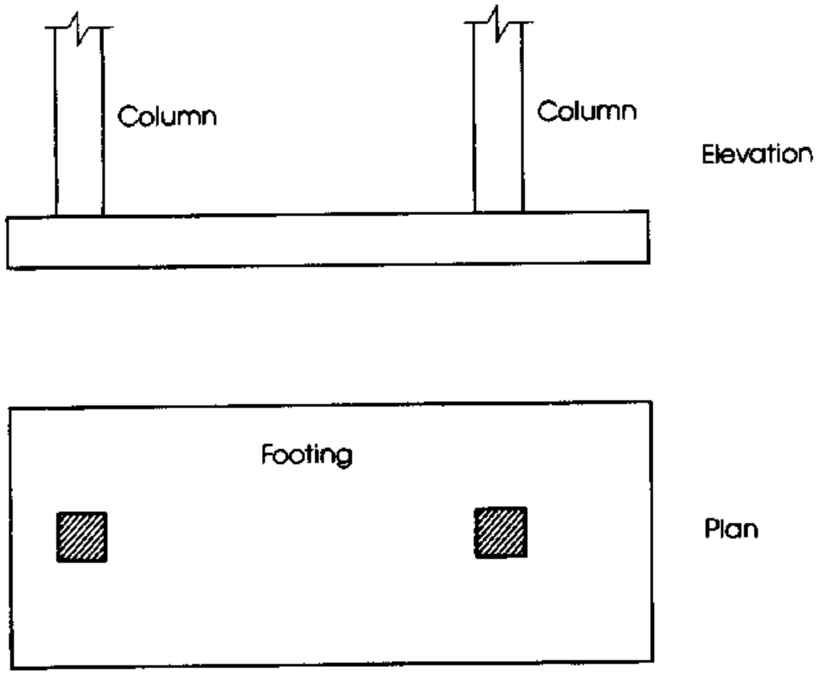


Figure 13.3 Combined footing.

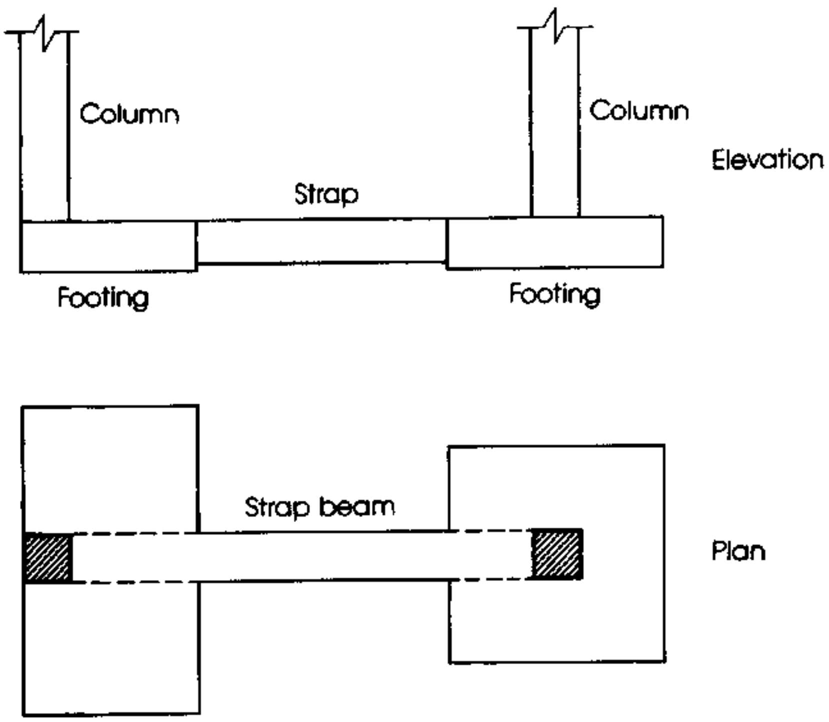


Figure 13.4 Strap footing.

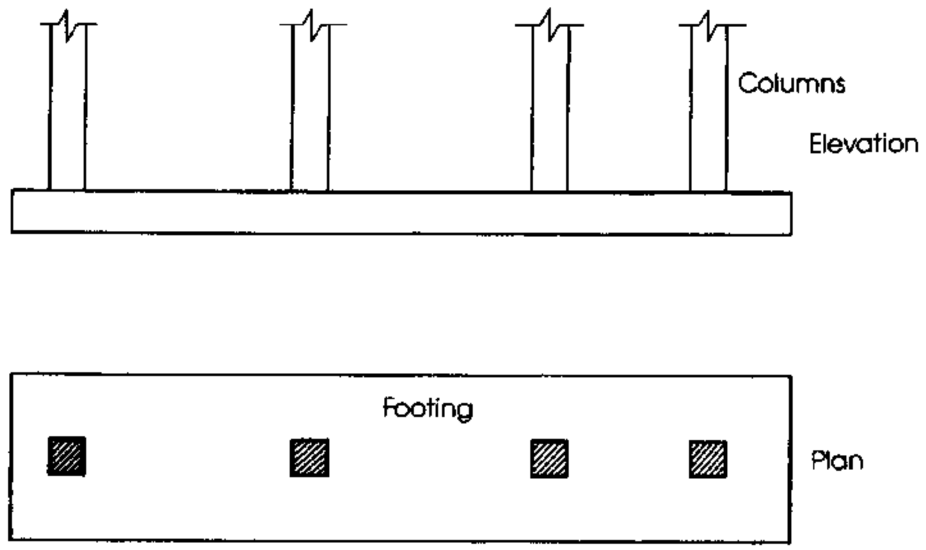


Figure 13.5 Continuous footing.

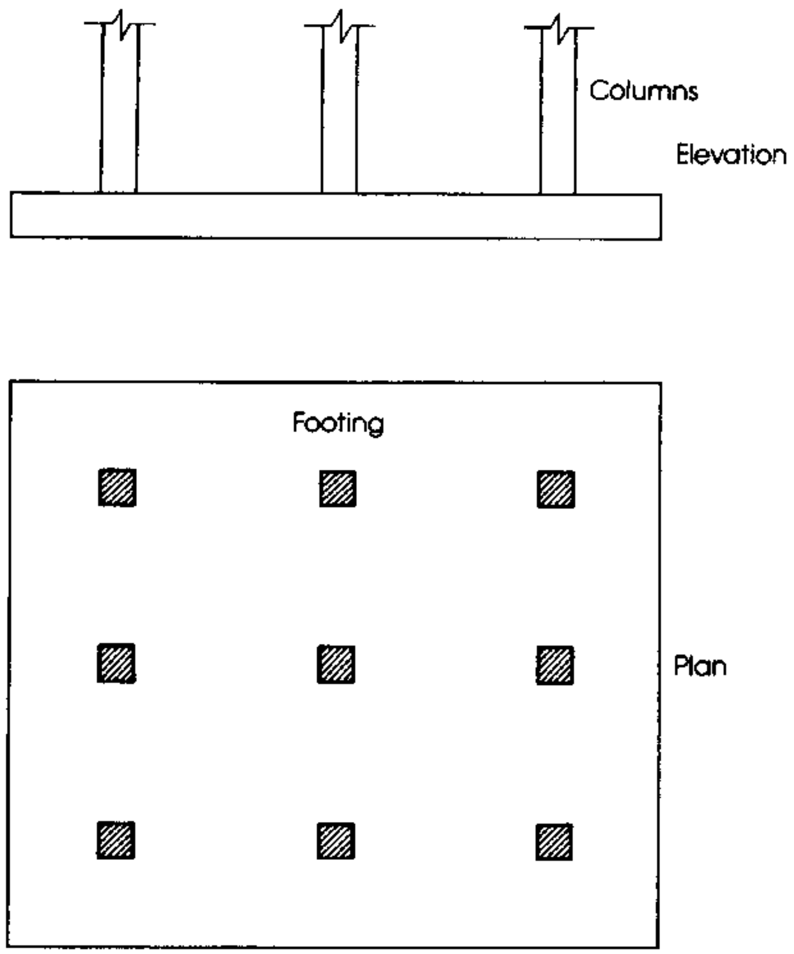


Figure 13.6 Raft, or mat, foundation.

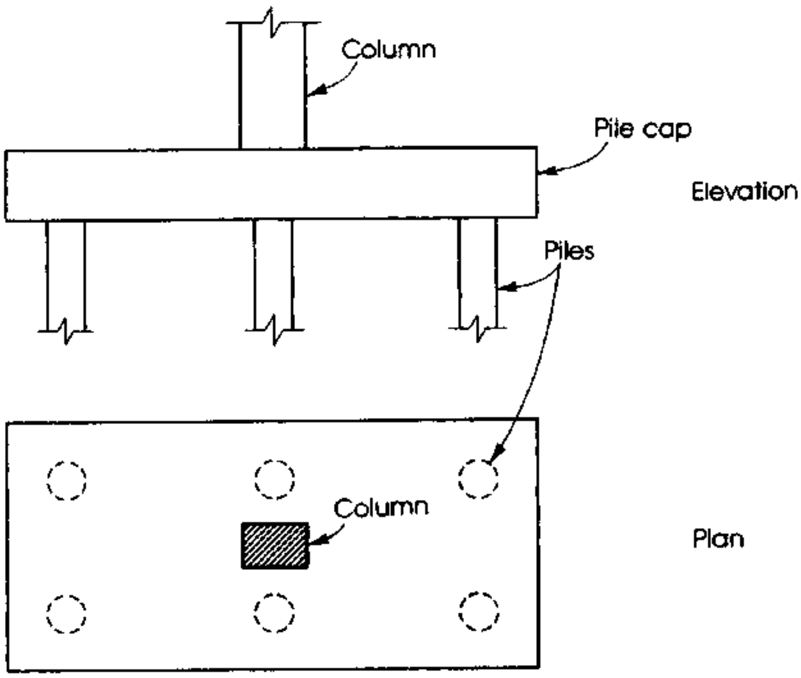


Figure 13.7 Pile cap footing.

13.3 DISTRIBUTION OF SOIL PRESSURE

Fig. 13.8 shows a footing supporting a single column. When the column load, P, is applied on the centroid of the footing, a uniform pressure is assumed to develop on the soil surface below the footing area. However, the actual distribution of soil pressure is not uniform but depends on many factors, especially the composition of the soil and the degree of flexibility of the footing.

For example, the distribution of pressure on cohesionless soil (sand) under a rigid footing is shown in Fig.13.9. The pressure is maximum under the center of the footing and decreases toward the ends of the footing. The cohesionless soil tends to move from the edges of the footing, causing a reduction in pressure, whereas the pressure increases around the center to satisfy equilibrium conditions. If the footing is resting on a cohesive soil such as clay, the pressure under the edges is greater than at the center of the footing (Fig. 13.10). The clay near the edges has a strong cohesion with the adjacent clay surrounding the footing, causing the nonuniform pressure distribution.

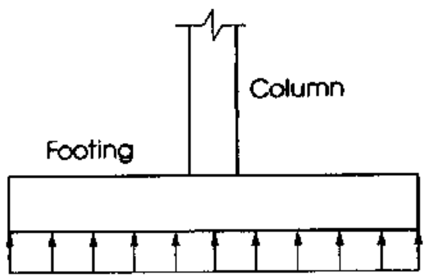


Figure 13.8 Distribution of soil pressure assuming uniform pressure.

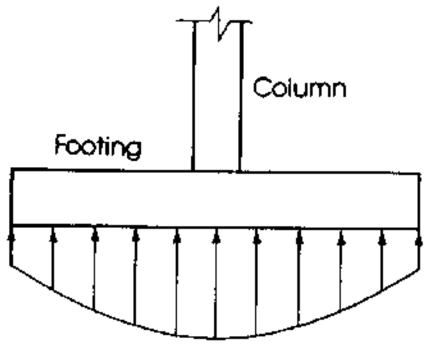


Figure 13.9 Soil pressure distribution in cohesionless soil (sand).

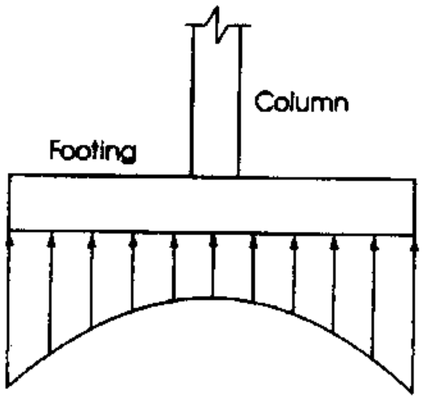


Figure 13.10 Soil pressure distribution in cohesive soil (clay).

The allowable bearing soil pressure, q_a, is usually determined from soil tests. The allowable values vary with the type of soil, from extremely high in rocky beds to low in silty soils. For example, q_a, for sedimentary rock is 30 ksf, for compacted gravel is 8 ksf, for well-graded compacted sand is 6 ksf, and for silty-gravel soils is 3 ksf.

Referring to Fig. 13.8, when the load P is applied, the part of the footing below the column tends to settle downward. The footing will tend to take a uniform curved shape, causing an upward pressure on the projected parts of the footing. Each part acts as a cantilever and must be designed for both bending moments and shearing forces. The design of footings is explained in detail later.

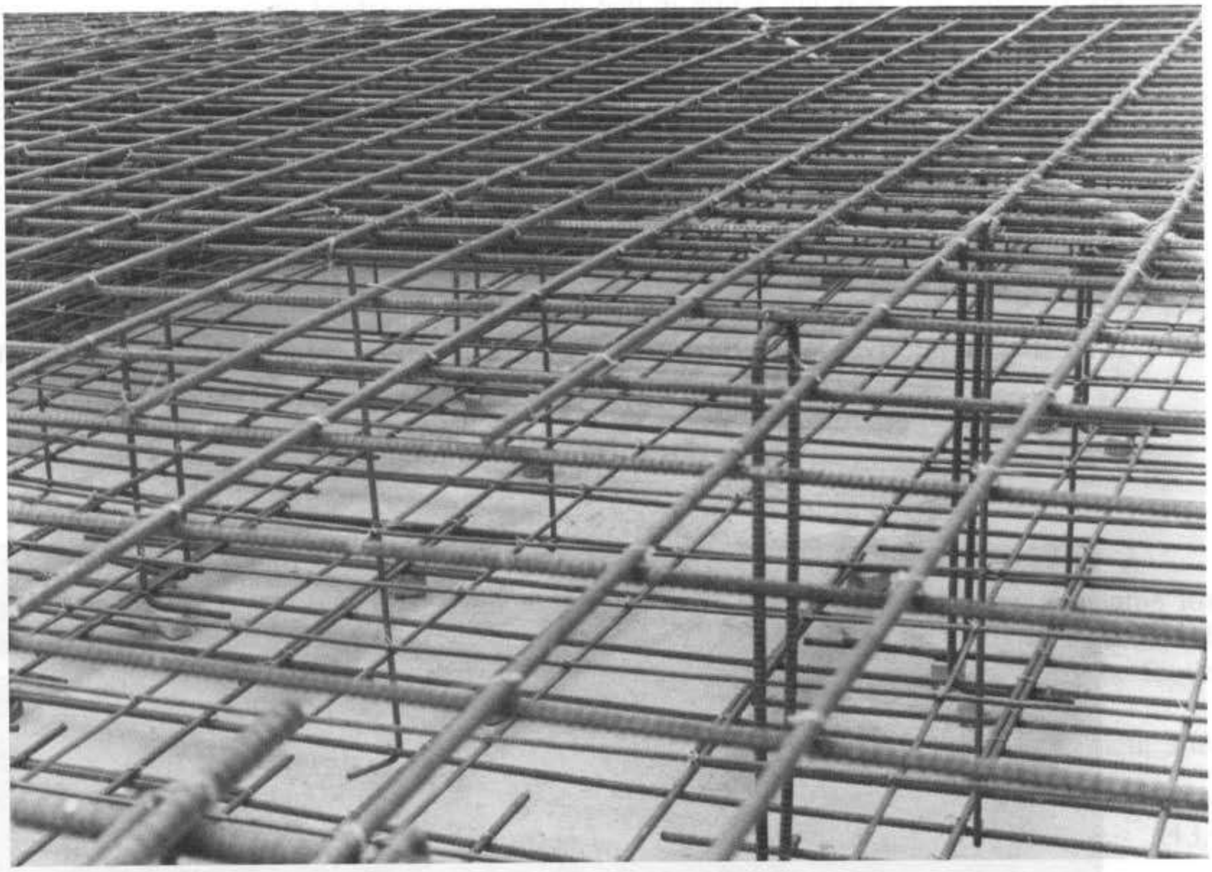
13.4 DESIGN CONSIDERATIONS

Footings must be designed to carry the column loads and transmit them to the soil safely. The design procedure must take the following strength requirements into consideration:

1. The area of the footing based on the allowable bearing soil capacity
2. On-way shear
3. Two-way shear, or punching shear
4. Bending moment and steel reinforcement required

5. Bearing capacity of columns at their base and dowel requirements
6. Development length of bars
7. Differential settlement

These strength requirements are explained in the following sections.



Reinforcing rebars placed in two layers in a raft foundation.

13.4.1 Size of Footings

The area of the footings can be determined from the actual external loads (unfactored forces and moments) such that the allowable soil pressure is not exceeded. In general, for vertical loads

$$\text{Area of footing} = \frac{\text{total service load (including self-weight)}}{\text{allowable soil pressure, } q_a} \tag{13.1}$$

or

$$\text{Area} = \frac{P(\text{total})}{q_a}$$

where the total service load is the unfactored design load. Once the area is determined, a factored soil pressure is obtained by dividing the factored load, $P_u = 1.2D + 1.6L$, by the area of the footing. This is required to design the footing by the strength design method.

$$q_u = \frac{P_u}{\text{area of footing}} \tag{13.2}$$

The allowable soil pressure, q_a, is obtained from soil test and is based on service load conditions.

13.4.2 One-Way Shear (Beam Shear) (V_{u_1})

For footings with bending action in one direction, the critical section is located at a distance d from the face of the column. The diagonal tension at section m-m in Fig. 13.11 can be checked as was done before in beams. The allowable shear in this case is equal to

$$\phi V_c = 2\phi\lambda\sqrt{f_c'}bd \quad (\phi = 0.75) \tag{13.3}$$

where b = width of section m-m. The factored shearing force at section m-m can be calculated as follows:

$$V_{u_1} = q_u b\left(\frac{L}{2} - \frac{C}{2} - d\right) \tag{13.4}$$

If no shear reinforcement is to be used, then d can be determined, assuming $V_u = \phi V_c$:

$$d = \frac{V_{u_1}}{2\phi\lambda\sqrt{f_c'}b} \tag{13.5}$$

Wall and column footings, partly covered.

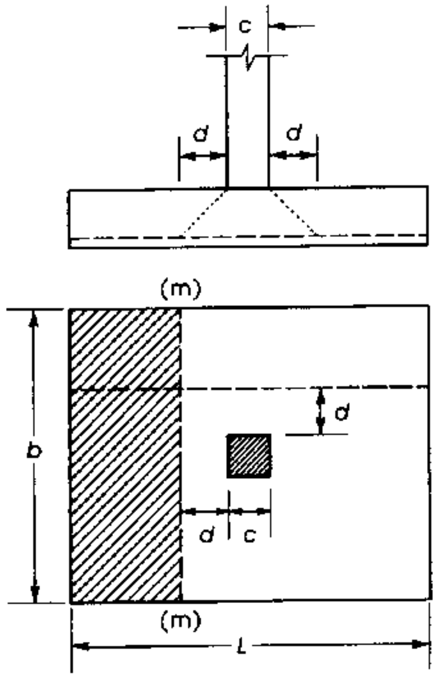


Figure 13.11 One-way shear.

13.4.3 Two-Way Shear (Punching Shear) (V_{u_2})

Two-way shear is a measure of the diagonal tension caused by the effect of the column load on the footing. Inclined cracks may occur in the footing at a distance $d/2$ from the face of the column on all sides. The footing will fail as the column tries to punch out part of the footing (Fig. 13.12).

The ACI Code, Section 11.11.2 allows a shear strength, V_c, in footings without shear reinforcement for two-way shear action, the smallest of

$$V_{c_1} = 4\lambda\sqrt{f'_c}b_0 d \tag{13.6}$$

$$V_{c_2} = \left(2 + \frac{4}{\beta}\right)\lambda\sqrt{f'_c}b_0 d \tag{13.7}$$

$$V_{c_3} = \left(\frac{\alpha_s d}{b_0} + 2\right)\lambda\sqrt{f'_c}b_0 d \tag{13.8}$$

where

β = Ratio of long side to short side of the rectangular column

b_0 = perimeter of the critical section taken at $d/2$ from the loaded area (column section) (see Fig. 13.12)

d = effective depth of footing

λ = is a modification factor for type of concrete (ACI 8.6.1)

λ = 1.0 Normal-weight concrete

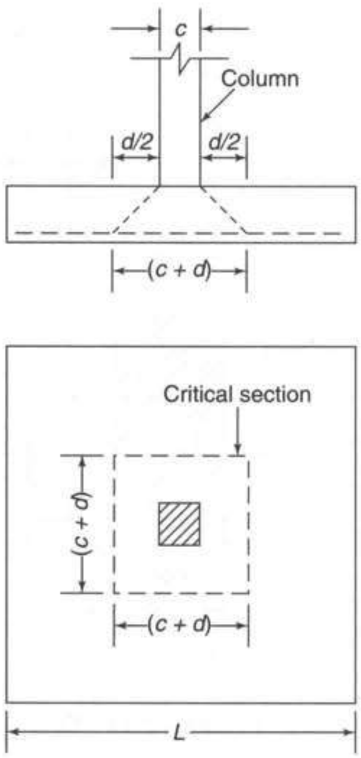
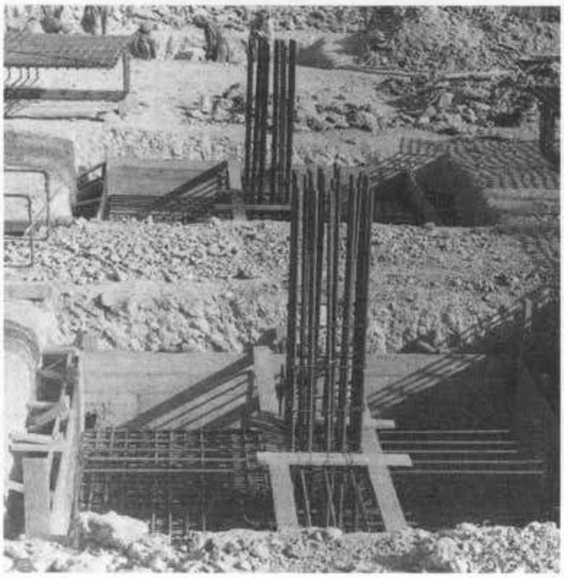


Figure 13.12 Punching shear (two-way).

$\lambda = 0.85$ sand-lightweight concrete
$\lambda = 0.75$ for all-lightweight concrete

Linear interpolation shall be permitted between 0.85 and 1.0 on the basis of volumetric fractions, for concrete containing normal-weight fine aggregate and a blend of lightweight and normal-weight coarse aggregate.



Reinforced concrete single footings.

For the values of V_{c_1} and V_{c_2} it can be observed that V_{c_1} controls (less than V_{c_2}) whenever $\beta_c \leq 2$, whereas V_{c_2} controls (less than V_{c_1}) whenever $\beta_c > 2$. This indicates that the allowable shear V_c is reduced for relatively long footings. The actual soil pressure variation along the long side increases with an increase in β. For shapes other than rectangular, β is taken to be the ratio of the longest dimension of the effective loaded area in the long direction to the largest width in the short direction (perpendicular to the long direction).

For Eq. 13.8, α_s is assumed to be 40 for interior columns, 30 for edge columns, and 20 for corner columns. The concrete shear strength, V_{c_3} represents the effect of an increase in b_0 relative to d. For a high ratio of b_0/d, V_{c_3} may control.

Based on the preceding three values of V_c, the effective depth, d, required for two-way shear is the largest obtained from the following formulas ($\phi = 0.75$):

$$d_1 = \frac{V_{u2}}{\phi 4\lambda\sqrt{f_c'}b_0} \quad \text{(where } \beta \leq 2\text{)} \tag{13.9}$$

or

$$d_1 = \frac{V_{u2}}{\phi\left(2+\frac{4}{\beta}\right)\lambda\sqrt{f_c'}b_0} \quad \text{(where } \beta > 2\text{)} \tag{13.10}$$

$$d_2 = \frac{V_{u2}}{\phi\left(\frac{\alpha_s d}{b_0}+2\right)\lambda\sqrt{f_c'}b_0} \tag{13.11}$$

The two-way shearing force, V_{u_2}, and the effective depth, d, required (if shear reinforcement is not provided) can be calculated as follows (refer to Fig. 13.12):

1. Assume d.
2. Determine b_0: $b_0 = 4(c + d)$ for square columns, where one side $= c$. $b_0 = 2\ (c_1 + d) + 2(c_2 + \text{d})$ for rectangular columns of sides c_1 and c_2.
3. The shearing force V_{u_2} acts at a section that has a length $b_0 = 4\ (c + d)$ or $[2(c_1 + d) + 2(c_2 + d)]$ and a depth d; the section is subjected to a vertical downward load, P_u, and a vertical upward pressure, q_u (Eq. 13.2). Therefore,

$$V_{u_2} = P_u - q_u(c+d)^2 \text{ for square columns} \tag{13.12a}$$

$$V_{u_2} = P_u - q_u(c_1+d)(c_2+d) \text{ for rectangular columns} \tag{13.12b}$$

4. Determine the largest d (of d_1 and d_2). If d is not close to the assumed d, revise your assumption and repeat.

13.4.4 Flexural Strength and Footing Reinforcement

The critical sections for moment occur at the face of the column (section n-n, Fig. 13.13). The bending moment in each direction of the footing must be checked and the appropriate reinforcement must be provided. In square footings and square columns, the bending moments in both directions are equal. To determine the reinforcement required, the depth of the footing in each direction may be used. Because the bars in one direction rest on top of the bars in the other direction, the effective depth, d, varies with the diameter of the bars used. An average value of d may be adopted. A practical value of d may be assumed to be $(h - 4.5)$ in.

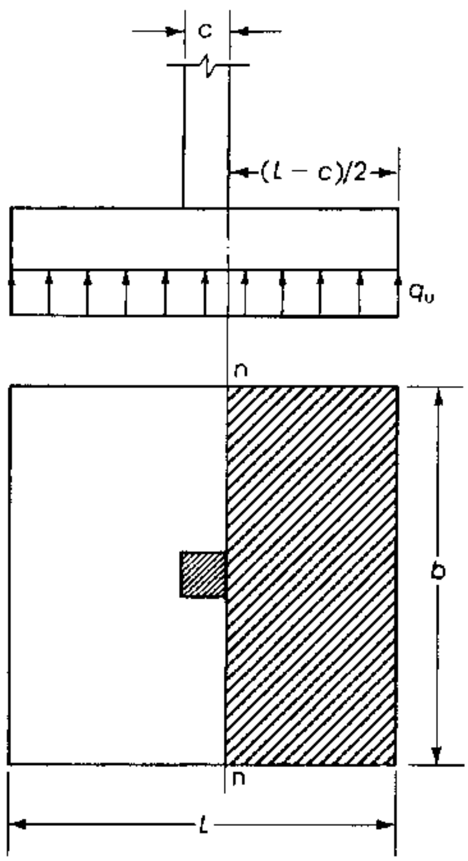


Figure 13.13 Critical section of bending moment.

The depth of the footing is often controlled by shear, which requires a depth greater than that required by the bending moment. The steel reinforcement in each direction can be calculated in the case of flexural members as follows:

$$M_u = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b}\right) \tag{13.13}$$

Also, the steel ratio, ρ, can be determined as follows (Eq. 4.2):

$$\rho = \frac{0.85 f'_c}{f_y}\left[1 - \sqrt{1 - \frac{2R_u}{\phi(0.85 f'_c)}}\right] \tag{13.14}$$

where $R_u = M_u/bd^2$. When R_u is determined, ρ can also be obtained from Eq. 13.15.

The minimum steel ratio requirement in flexural members is equal to $200/f_y$ when $f'_c <$ 4500 psi and equal to $3\sqrt{f'_c}/f_y$ when $f'_c \geq 4500$ psi. However, the ACI Code, Section 10.5, indicates that for structural slabs of uniform thickness, the minimum area and maximum spacing of steel bars in the direction of bending shall be as required for shrinkage and temperature reinforcement. This last minimum steel requirement is very small, and a higher minimum reinforcement ratio is recommended, but it should not be greater than $200/f_y$.

The reinforcement in one-way footings and two-way footings must be distributed across the entire width of the footing. In the case of two-way rectangular footings, the ACI Code, Section 15.4.4, specifies that in the long direction, a portion of the total reinforcement $\gamma_s A_s$

distributed uniformly along the width of the footing. In the short direction, a certain ratio of the total reinforcement in this direction must be placed uniformly within a bandwidth equal to the length of the short side of the footing according to

$$\gamma_s = \frac{2}{\beta + 1} \tag{13.15}$$

where

$$\beta = \frac{\text{long side of footing}}{\text{short side of footing}} \tag{13.16}$$

The bandwidth must be centered on the centerline of the column (Fig. 13.14). The remaining reinforcement in the short direction must be uniformly distributed outside the bandwidth. This remaining reinforcement percentage shall not be less than that required for shrinkage and temperature.

When structural steel columns or masonry walls are used, then the critical sections for moments in footings are taken at halfway between the middle and the edge of masonry walls and halfway between the face of the column and the edge of the steel base place (ACI Code, Section 15.4.2).

13.4.5 Bearing Capacity of Column at Base

The loads from the column act on the footing at the base of the column, on an area equal to the area of the column cross-section. Compressive forces are transferred to the footing directly by bearing on the concrete.

Forces acting on the concrete at the base of the column must not exceed the bearing strength of concrete as specified by the ACI Code, Section 10.14:

$$\text{Bearing strength } N_1 = \phi(0.85 f_c' A_1) \tag{13.17}$$

where $\phi = 0.65$ and $A_1 =$ the bearing area of the column. The value of the bearing strength given in Eq. 13.17 may be multiplied by a factor $\sqrt{A_2/A_1} \leq 2.0$ for bearing on footings when the supporting surface is wider on all sides than the loaded area. Here A_2 is the area of the part of the supporting footing that is geometrically similar to and concentric with the loaded area (Fig. 13.15). Because $A_2 > A_1$, the factor $\sqrt{A_2/A_1}$ is greater than unity, indicating that the allowable

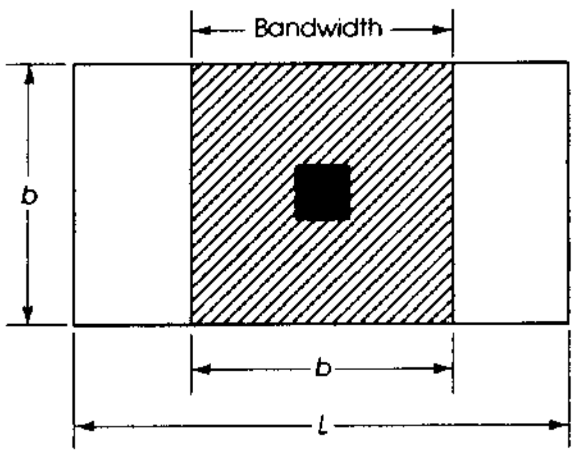


Figure 13.14 Bandwidth for reinforcement distribution.

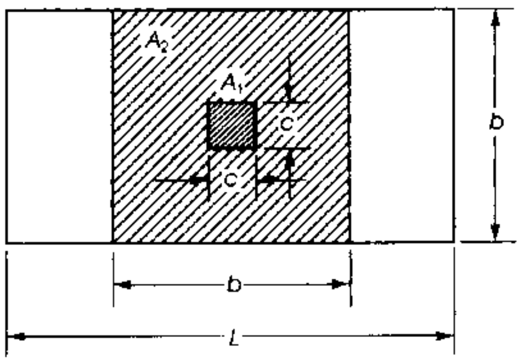


Figure 13.15 Bearing areas on footings. $A_1 = c^2, A_2 = b^2$.

bearing strength is increased because of the lateral support from the footing area surrounding the column base. The modified bearing strength is

$$N_2 = \phi(0.85 f'_c A_1)\sqrt{\frac{A_2}{A_1}} \le 2\phi(0.85 f'_c A_1) \tag{13.18}$$

If the factored force, P_u, is greater than either N_1 or N_2 reinforcement must be provided to transfer the excess force. This is achieved by providing dowels or extending the column bars into the footing. The excess force is $P_{ex} = P_u - N_1$ and the area of the dowel bars is $A_{sd} = (P_{ex}/f_y) \ge 0.005\ A_1$, where A_1 is the area of the column section. At least four bars should be used at the four corners of the column. If the factored force is less than either N_1 or N_2, then minimum reinforcement must be provided. The ACI Code, Section 15.8.2, indicates that the minimum area of the dowel reinforcement is at least $0.005 A_g$ (and not less than four bars), where A_g is the gross area of the column section. The minimum reinforcement requirements apply also to the case when the factored forces are greater than N_1 and N_2. The dowel bars may be placed at the four corners of the column and extended in both the column and footing. The dowel diameter shall not exceed the diameter of the longitudinal bars in the columns by more than 0.15 in. This requirement is necessary to ensure proper action between the column and footing. The development length of the dowels must be checked to determine proper transfer of the compression force into the footing.

13.4.6 Development Length of the Reinforcing Bars

The critical sections for checking the development length of the reinforcing bars are the same as those for bending moments. The development length for compression bars was given in Chapter 7:

$$l_{dc} = \frac{0.02 f_y d_b}{\lambda\sqrt{f'_c}} \tag{7.15}$$

but this value cannot be less than $0.0003\ f_y d_b \ge 8$ in. For other values, refer to Chapter 7.

13.4.7 Differential Settlement (Balanced Footing Design)

Footings usually support the following loads:

- Dead loads from the substructure and superstructure
- Live load resulting from occupancy

- Weight of materials used in backfilling
- Wind loads

Each footing in a building is designed to support the maximum load that may occur on any column due to the critical combination of loadings, using the allowable soil pressure.

The dead load, and maybe a small portion of the live load (called the *usual* live load), may act continuously on the structure. The rest of the live load may occur at intervals and on some parts of the structure only, causing different loadings on columns. Consequently, the pressure on the soil under different footings will vary according to the loads on the different columns, and differential settlement will occur under the various footings of one structure. Because partial settlement is inevitable, the problem turns out to be the amount of differential settlement that the structure can tolerate. The amount of differential settlement depends on the variation in the compressibility of the soils, the thickness of the compressible material below foundation level, and the stiffness of the combined footing and superstructure. Excessive differential settlement results in cracking of concrete and damage to claddings, partitions, ceilings, and finishes.

Differential settlement may be expressed in terms of angular distortion of the structure. Bjerrum [5] indicated that the danger limits of distortion for some conditions vary between $\frac{1}{600}$ to $\frac{1}{150}$ depending on the damage that will develop in the building.

For practical purposes it can be assumed that the soil pressure under the effect of sustained loadings is the same for all footings, thus causing equal settlements. The sustained load (or the usual load) can be assumed to be equal to the dead load plus a percentage of the live load, which occurs very frequently on the structure. Footings then are proportioned for these sustained loads to produce the same soil pressure under all footings. In no case is the allowable soil bearing capacity to be exceeded under the dead load plus the maximum live load for each footing. Example 13.4 explains the procedure for calculating the areas of footings, taking into consideration the effect of differential settlement.

13.5 PLAIN CONCRETE FOOTINGS

Plain concrete footings may be used to support masonry walls or other light loads and transfer them to the supporting soil. The ACI Code Section 22.7 allows the use of plain concrete pedestals and footings on soil, provided that the design stresses shall not exceed the following:

1. Maximum flexural stress in tension is less than or equal to $5\phi\lambda\sqrt{f'_c}$ (where $\phi = 0.60$).
2. Maximum stress in one-way shear (beam action) is less than or equal to $\frac{4}{3}\,\phi\lambda\sqrt{f'_c}$ (where $\phi = 0.60$).
3. Maximum shear stress in two-way action according to ACI Code Section 22.5.4 is

$$\left(\frac{4}{3}+\frac{8}{3\beta}\right)\phi\lambda\sqrt{f'_c} \leq 2.66\,\phi\sqrt{f'_c} \quad \text{(where } \phi = 0.60\text{)} \tag{13.19}$$

where

β = Ratio of long side to short side of the rectangular column

λ = modification factor described in 13.4.3.

4. Maximum compressive strength shall not exceed the concrete bearing strengths specified; f'_c of plain concrete should not be less than 2500 psi.

5. The minimum thickness of plain concrete footings shall not be less than 8 in.
6. The critical sections for bending moments are at the face of the column or wall.
7. The critical sections for one-way shear and two-way shear action are at distances d and $d/2$ from the face of the column or wall, respectively. Although plain concrete footings do not require steel reinforcement, it will be advantageous to provide shrinkage reinforcement in the two directions of the footing.
8. Stresses due to factored loads are computed assuming a linear distribution in concrete.
9. The effective depth, d, must be taken equal to the overall thickness minus 3 in.
10. For flexure and one-way shear, use a gross section bh, whereas for two-way shear, use b_0h to calculate ϕV_c.

Example 13.1

Design a reinforced concrete footing to support a 20-in.-wide concrete wall carrying a dead load of 26 K/ft, including the weight of the wall, and a live load of 20 K/ft. The bottom of the footing is 6 ft below final grade. Use normal-weight concrete with $f'_c = 4$ ksi, $f_y = 60$ ksi, and an allowable soil pressure of 5 ksf.

Solution

1. Calculate the effective soil pressure. Assume a total depth of footing of 20 in. Weight of footing is $(\frac{20}{12})(150) = 250$ psf. Weight of the soil fill on top of the footing, assuming that soil weighs 100 lb/ft^3, is $(6 - \frac{20}{12}) \times 100 = 433$ psf. Effective soil pressure at the bottom of the footing is $5000 - 250 - 433 = 4317\,\text{psf} = 4.32\,\text{ksf}$.
2. Calculate the width of the footing for a 1-ft length of the wall:

$$\text{Width of footing} = \frac{\text{total load}}{\text{effective soil pressure}}$$

$$= \frac{26 + 20}{4.32} = 10.7\text{ft}$$

Use 11 ft.

3. Net upward pressure = (factored load)/(footing width) (per 1 ft):

$$P_u = 1.2D + 1.6L = 1.2 \times 26 + 1.6 \times 20 = 63.2\ \text{K}$$

$$\text{Net pressure} = q_u = \frac{63.2}{11} = 5.745\ \text{ksf}$$

4. Check the assumed depth for shear requirements. The concrete cover in footings is 3 in., and assume no. 8 bars; then $d = 20 - 3.5 = 16.5$. The critical section for one-way shear is at a distance d from the face of the wall:

$$V_u = q_u\left(\frac{B}{2} - d - \frac{c}{2}\right) = 5.745\left(\frac{11}{2} - \frac{16.5}{12} - \frac{20}{2 \times 12}\right) = 18.91\ \text{K}$$

$$\text{Allowable one-way shear} = 2\lambda\sqrt{f'_c} = (2)(1)\sqrt{4000} = 126.5\ \text{psi}$$

$$\text{Required } d = \frac{V_u}{\phi(2\sqrt{f'_c})b} = \frac{18.91 \times 1000}{0.75(126.5)(12)} = 16.6\ \text{in.}$$

$$b = \text{1-ft length of footing} = 12\ \text{in.}$$

Total depth is $16.6 + 3.5 = 20.1$ in., or 20 in. Actual d is $20 - 3.5 = 16.5$ in. (as assumed). Note that few trials are needed to get the assumed and calculated d quite close.

5. Calculate the bending moment and steel reinforcement. The critical section is at the face of the wall:

$$M_u = \frac{1}{2}q_u\left(\frac{B}{2} - \frac{c}{2}\right)^2 = \frac{5.745}{2}\left(\frac{11}{2} - \frac{20}{24}\right)^2 = 62.6 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{62.6 \times 12{,}000}{12(16.5)^2} = 230 \text{ psi}$$

From Table A.1 in Appendix A, for $R_u = 230$ psi, $f'_c = 4$ ksi, and $f_y = 60$ ksi, the steel percentage is $\rho = 0.0045$ (or from Eq. 13.14). Minimum steel percentage for flexural members is

$$\rho_{\min} = \frac{200}{f_y} = \frac{200}{60{,}000} = 0.0033$$

Percentage of shrinkage reinforcement is 0.18% (for $f_y = 60$ ksi). Therefore, use $\rho = 0.0045$ as calculated.

$$A_s = 0.0045 \times 12 \times 16.5 = 0.89 \text{ in.}^2$$

Use no. 8 bars spaced at 9 in. ($A_s = 1.05$ in.^{2}) (Table A.14).

6. Check the development length for no. 8 bars:

$$l_d = 48d_b = 48(1) = 48 \text{ in. (Refer to Chapter 7).}$$

Provided

$$l_d = \frac{B}{2} - \frac{c}{2} - 3 \text{ in.} = \frac{11(12)}{2} - \frac{20}{2} - 3 = 53 \text{ in.}$$

7. Calculate secondary reinforcement in the longitudinal direction: $A_s = 0.0018(12)(20) = 0.43$ in.^{2}/ft. Choose no. 5 bars spaced at 8 in. ($A_s = 0.46$ in.^{2}). Details are shown in Fig. 13.16.

Example 13.2

Design a square single footing to support an 18-in.-square tied interior column reinforced with eight no. 9 bars. The column carries an unfactored axial dead load of 245 K and an axial live load of 200 K. The base of the footing is 4 ft below final grade and the allowable soil pressure is 5 ksf. Use normal-weight concrete, with $f'_c = 4$ ksi and $f_y = 60$ ksi.

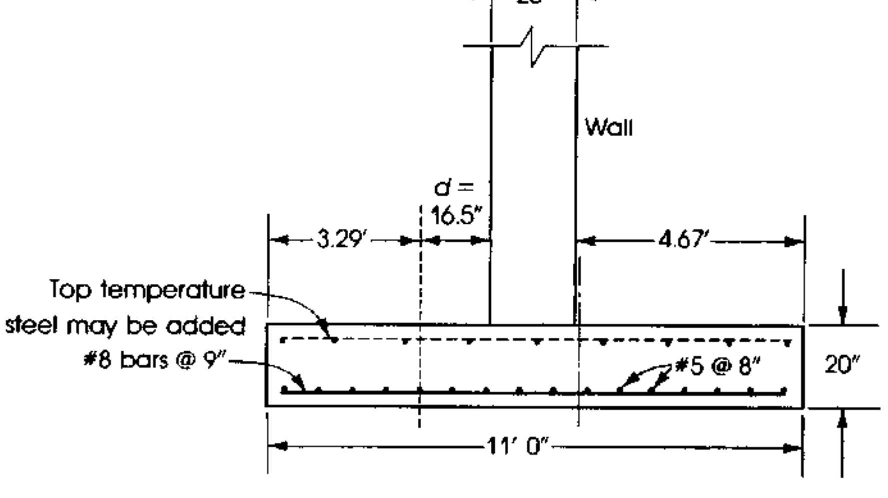


Figure 13.16 Example 13.1: Wall footing.

Solution

1. Calculate the effective soil pressure. Assume a total depth of footing of 2 ft. The weight of the footing is $2 \times 150 = 300$ psf. The weight of the soil on top of the footing (assuming the weight of soil = 100 pcf) is $2 \times 100 = 200$ psf.

$$\text{Effective soil pressure} = 5000 - 300 - 200 = 4500 \text{ psf}$$

2. Calculate the area of the footing:

$$\text{Actual loads} = D + L = 245 + 200 = 445 \text{ K}$$

$$\text{Area of footing} = \frac{445}{4.5} = 98.9 \text{ ft}^2$$

$$\text{Side of footing} = 9.94 \text{ ft.}$$

Thus, use 10 ft (Fig.13.17).

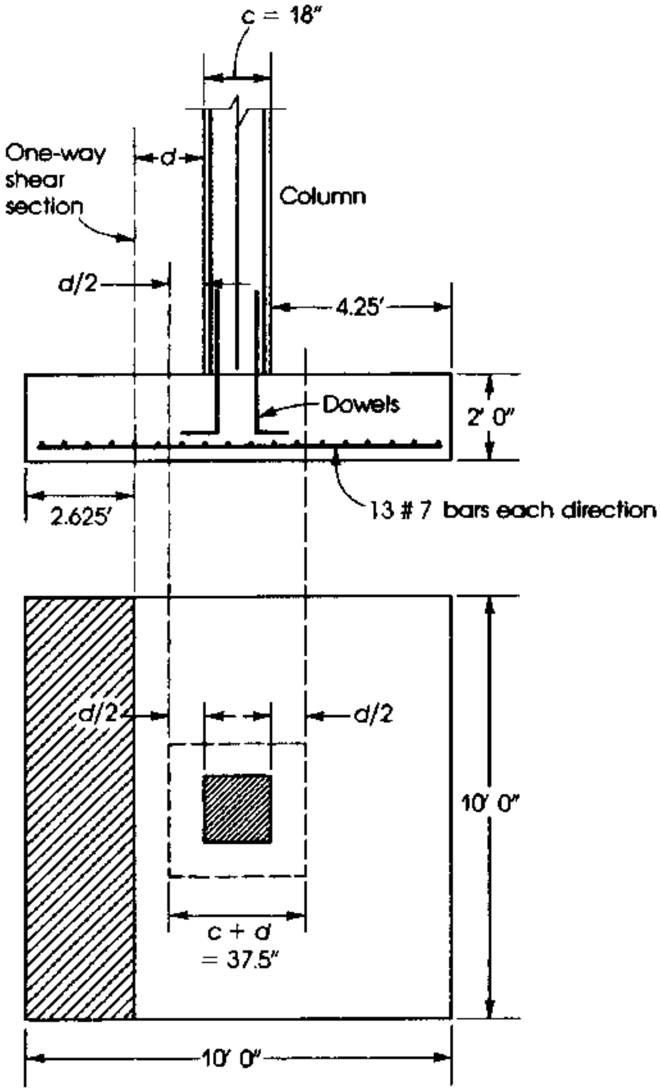


Figure 13.17 Example 13.2: Square footing.

3. Net upward pressure equals (factored load)/(area of footing).

$$P_u = 1.2D + 1.6L$$

$$= 1.2 \times 245 + 1.6 \times 200 = 614 \text{ K}$$

$$\text{Net upward pressure, } q_u = \frac{614}{10 \times 10} = 6.14 \text{ ksf}$$

4. Check depth due to two-way shear. If no shear reinforcement is used, two-way shear determines the critical footing depth required. For an assumed total depth of 24 in., calculate d to the centroid of the top layer of the steel bars to be placed in the two directions within the footing. Let the bars to be used be no. 8 bars for calculating d.

$$d = 24 - 3 \text{ (cover)} - 1.5 \text{ (bar diameters)} = 19.5 \text{ in.}$$

It is quite practical to assume $d = h - 4.5$ in.

$$b_0 = 4(c + d) = 4(18 + 19.5) = 150 \text{ in.}$$

$$c + d = 18 + 19.5 = 37.5 \text{ in.} = 3.125 \text{ ft}$$

$$V_{u_2} = P_u - q_u(c + d)^2 = 614 - 6.14(3.125)^2 = 554 \text{ K}$$

$$\text{Required } d_1 = \frac{V_{u_2}}{4\phi\lambda(\sqrt{f'_c}b_0)}$$

$$= \frac{554(1000)}{(4)(0.75)(1)\sqrt{4000}(150)} = 19.5 \text{ in.} \quad (\beta = 1; \text{Eq. } 13.9)$$

$$\text{Required } d_2 = \frac{554(1000)}{0.75\left(\dfrac{40 \times 19.5}{150} + 2\right)(\sqrt{4000})(150)}$$

$$= 10.8 \text{ in.} \quad \text{(not critical)}$$

($\alpha_s = 40$ for interior columns.) Thus, the assumed depth is adequate. Two or more trials may be needed to reach an acceptable d that is close to the assumed one.

5. Check depth due to one-way shear action: The critical section is at a distance d from the face of the column.

$$\text{Distance from edge of footing} = \left(\frac{L}{2} - \frac{c}{2} - d\right) = 2.625 \text{ ft}$$

$$V_{u_1} = 6.14 \times (2.625)(10) = 161.2 \text{ K}$$

The depth required for one-way shear is

$$d = \frac{V_{u1}}{(0.75)(2)\lambda\sqrt{f'_c}b}$$

$$= \frac{161.2(1000)}{(0.75)(2)(1)(\sqrt{4000})(10 \times 12)} = 14.2 \text{ in.} < 19.5 \text{ in.}$$

6. Calculate the bending moment and steel reinforcement. The critical section is at the face of the column. The distance from edge of footing is

$$\left(\frac{L}{2} - \frac{c}{2}\right) = 5 - \frac{1.5}{2} = 4.25 \text{ ft}$$

$$M_u = \frac{1}{2}q_u\left(\frac{L}{2} - \frac{c}{2}\right)^2 b = \frac{1}{2}(6.14)(4.25)^2(10) = 554.5 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{554.5(12{,}000)}{(10 \times 12)(19.5)^2} = 145.8 \text{ psi}$$

Applying Eq. 13.14, $\rho = 0.0028$.

$$A_s = \rho bd = 0.0028(10 \times 12)(19.5) = 6.55 \text{ in.}^2$$

$$\text{Minimum } A_s \text{ (shrinkage steel)} = 0.0018(10 \times 12)(24)$$

$$= 5.18 \text{ in.}^2 < 6.55 \text{ in.}^2$$

$$\text{Minimum } A_s \text{ (flexure)} = 0.0033(10 \times 12)(19.5) = 7.72 \text{ in.}^2$$

Therefore, $A_s = 7.72\,\text{in.}^2$ can be adopted. Use 13 no. 7 bars ($A_s = 7.82\,\text{in.}^2$), spaced at $s = (120 - 6)/12 = 9.5\,\text{in.}$ in both directions.

7. Check bearing stress:
   a. Bearing strength, N_1 at the base of the column ($A_1 = 18 \times 18\,\text{in.}$) is

   $$N_1 = \phi(0.085 f'_c A_1) = 0.65(0.85 \times 4)(18 \times 18) = 716 \text{ K}$$

   b. Bearing strength, N_2, at the top of footing ($A_2 = 10 \times 10\,\text{ft}$) is

   $$N_2 = N_1\sqrt{\frac{A_2}{A_1}} \le 2N_1$$

   $$A_2 = 10 \times 10 = 100 \text{ ft}^2 \quad A_1 = \frac{18 \times 18}{144} = 2.25 \text{ ft}^2$$

   $$\sqrt{\frac{A_2}{A_1}} = 6.67 > 2$$

   Therefore, $N_2 = 2N_1 = 1432\,\text{K}$. Because $P_u = 614\,\text{K} < N_1$, bearing stress is adequate. The minimum area of dowels required is $0.005\,A_1 = 0.005\,(18 \times 18) = 1.62\,\text{in.}^2$. The minimum number of bars is four, so use four no. 8 bars placed at the four corners of the column.

   c. Development length of dowels in compression:

   $$l_{dc} = \frac{0.02 d_b f_y}{\lambda\sqrt{f'_c}} = \frac{0.02(1)(60{,}000)}{(1)\sqrt{4000}} = 19 \text{ in.}$$

   (controls). Minimum l_{dc} is $0.0003 d_b f_y = 0.0003(1)\,(60{,}000) = 18\,\text{in.} \ge 8\,\text{in.}$ Therefore, use four no. 8 dowels extending 19 in. into column and footing. Note that l_d is less than d of 19.5 in., which is adequate.
8. The development length of main bars in footing for no. 7 bars is $l_d = 48d_b = 42\,\text{in.}$ (refer to Chapter 7), provided $l_d = L/2 - c/2 - 3\,\text{in.} = 48\,\text{in.}$ Details of the footing are shown in Fig. 13.17 on page 430.

Example 13.3

Design a rectangular footing for the column of Example 13.2 if one side of the footing is limited to 8.5 ft.

Solution

1. The design procedure for rectangular footings is similar to that of square footings, taking into consideration the forces acting on the footing in each direction separately.
2. From the previous example, the area of the footing required is $98.9\,\text{ft}^2$:

$$\text{Length of footing} = \frac{98.9}{8.5} = 11.63 \quad \text{ft}$$

so use 12 ft (Fig. 13.18). Footing dimensions are 8.5×12 ft.

3. $P_u = 614$ K. Thus, net upward pressure is

$$q_u = \frac{614}{8.5 \times 12} = 6.02 \text{ ksf}$$

4. Check the depth due to one-way shear. The critical section is at a distance d from the face of the column. In the longitudinal direction,

$$V_{u_1} = \left(\frac{L}{2} - \frac{c}{2} - d\right) \times q_u b$$

$$= \left(\frac{12}{2} - \frac{1.5}{2} - \frac{19.5}{12}\right) \times 6.02 \times 8.5 = 185.5 \text{ K}$$

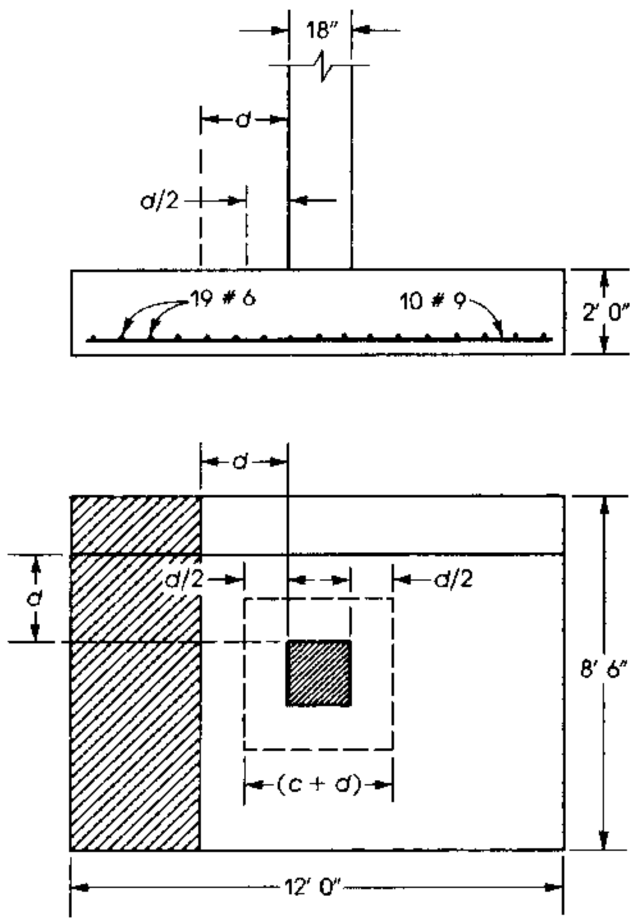


Figure 13.18 Example 13.3: Rectangular footing.

This shear controls. In the short direction, $V_u = 135.4\,\text{K}$ (not critical).

$$\text{Required } d = \frac{V_{u1}}{2\phi\lambda\sqrt{f'_c}b} = \frac{185.5 \times 1000}{(2)(0.75)(1)\sqrt{4000} \times (8.5 \times 12)} = 19.2 \text{ in.}$$

$$d \text{ provided} = 19.5 \text{ in.} > 19.2 \text{ in.}$$

5. Check the depth for two-way shear action (punching shear). The critical section is at a distance $d/2$ from the face of the column on four sides.

$$b_0 = 4(18 + 19.5) = 150 \text{ in.}$$

$$(c + d) = 18 + 19.5 = 37.5 \text{ in.} = 3.125 \text{ ft}$$

$$\beta = \frac{12}{8.5} = 1.41 < 2$$

(Use $V_c = 4\phi\lambda\sqrt{f'_c}b_0 d$.)

$$V_{u_2} = P_u - q_u(c + d)^2 = 614 - 6.02(3.125)^2 = 555.2 \text{ K}$$

$$d_1 = \frac{V_{u_2}}{4\phi\lambda\sqrt{f'_c}b_0} = \frac{555.2 \times 1000}{4(0.75)(1)\sqrt{4000} \times 150} = 19.5 \text{ in.}$$

$$d_2 = 10.6 \text{ in.} \quad \text{(Does not control.)}$$

6. Design steel reinforcement in the longitudinal direction. The critical section is at the face of the support. The distance from the edge of the footing is

$$\frac{L}{2} - \frac{c}{2} = \frac{12}{2} - \frac{1.5}{2} = 5.25 \text{ ft}$$

$$M_u = \frac{1}{2}(6.02)(5.25)^2(8.5) = 705.2 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{705.2(12{,}000)}{(8.5 \times 12)(19.5)^2} = 218 \text{ psi}$$

Applying Eq. 13.14, $\rho = 0.0042$:

$$A_s = 0.0042(8.5 \times 12)(19.5) = 8.35 \text{ in.}^2$$

$$\text{Min } A_s \text{ (shrinkage)} = 0.0018(8.5 \times 12)(24) = 4.4 \text{ in.}^2$$

$$\text{Min } A_s \text{ (flexure)} = 0.0033(8.5 \times 12)(19.5) = 6.56 \text{ in.}^2$$

Use $A_s = 8.35$ in. and 10 no. 9 bars ($A_s = 10\,\text{in.}^2$) spaced at $S = (102 - 6)/9 = 10.7$ in.

7. Design steel reinforcement in the short direction. The distance from the face of the column to the edge of the footing is

$$\frac{8.5}{2} - \frac{1.5}{2} = 3.5 \text{ ft}$$

$$M_u = \frac{1}{2}(6.02)(3.5)^2(12) = 422.5 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{442.5(12{,}000)}{(12 \times 12)(19.5)^2} = 97 \text{ psi}$$

Applying Eq. 13.4, $\rho = 0.0019$:

$$A_s = 0.0019(12 \times 12)(19.5) = 5.34 \text{ in.}^2$$

$$\text{Min } A_s \text{ (shrinkage)} = 0.0018(12 \times 12)(24) = 6.22 \text{ in.}^2$$

$$\text{Min } A_s \text{ (flexure)} = 0.0033(12 \times 12)(19.5) = 9.26 \text{ in.}^2$$

The value of A_s to be used must be greater than or equal to 6.22 in.^{2} Use 18 no. 6 bars (A_s = 7.92 in.^{2}).

$$\gamma_s = \frac{2}{\beta + 1} = \frac{2}{\left(\frac{12}{8.5}\right) + 1} = 0.83$$

The number of bars in an 8.5-ft band is 18(0.83) = 15 bars. The number of bars left on each side is (18 − 15)/2 = 2 bars. Therefore, place 15 no. 6 bars within the 8.5-ft band; then place two no. 6 bars (A_s = 0.88 in.^{2}) within (12 − 8.5)/2 = 1.625 ft on each side of the band. The total number of bars is 19 no. 6 bars (A_s = 8.36 in.^{2}). In this example, the bars may be distributed at equal spacings all over the 12-ft length; S = (144 − 6)/18 = 7.6 in. Details of reinforcement are shown in Fig.13.18.

8. Check the bearing stress at the base of the column, as explained in the previous example. Use four no. 8 dowel bars.
9. Development length of the main reinforcement: l_d = 29 in. for no. 6 bars and 54 in. for no. 9 bars.

$$\text{Provided } l_d \text{ (long direction)} = \left(\frac{l}{2} - \frac{c}{2} - 3 \text{ in.}\right) = 60 \text{ in.}$$

$$\text{Provided } l_d \text{ (short direction)} = 39 \text{ in.} > 29 \text{ in.}$$

Example 13.4

Determine the footing areas required for equal settlement (balanced footing design) if the usual live load is 20% for all footings. The footings are subjected to dead loads and live loads as indicated in the following table. The allowable net soil pressure is 6 ksf.

	Footing Number				
	1	2	3	4	5
Dead load	120 K	180 K	140 K	190 K	210 K
Live load	150 K	220 K	200 K	170 K	240 K

Solution

1. Determine the footing that has the largest ratio of live load to dead load. In this example, the footing 3 ratio of 1.43 is higher than the other ratios.
2. Calculate the usual load for all footings. The usual load is the dead load and the portion of live load that most commonly occurs on the structure. In this example,

$$\text{Usual load} = \text{D.L.} + 0.2(\text{L.L.})$$

The values of the usual loads are shown in the following table.

3. Determine the area of the footing that has the highest ratio of L.L./D.L.

$$\text{Area of footing 3} = \frac{\text{D.L.} + \text{L.L}}{\text{allowable soil pressure}} = \frac{140 + 200}{6} = 56.7 \text{ ft}^2$$

The usual soil pressure under footing 3 is

$$\frac{\text{Usual load}}{\text{Area of footing}} = \frac{180}{56.7} = 3.18 \text{ ksf}$$

4. Calculate the area required for each footing by dividing its usual load by the soil pressure of footing 3. The areas are tabulated in the following table. For footing 1, for example, the required area is $150/3.18 = 47.2\,\text{ft}^2$.
5. Calculate the maximum soil pressure under each footing:

$$q_{\max} = \frac{D+L}{\text{area}} \leq 6 \text{ ksf} \quad \text{(allowable soil pressure)}$$

	Footing Number				
Description	**1**	**2**	**3**	**4**	**5**
$\dfrac{\text{Live load}}{\text{Dead load}}$	1.25	1.22	1.43	0.90	1.14
Usual load = D.L. + 0.2 (L.L.) (kips)	150	224	180	224	258
Area required $= \dfrac{\text{usual load}}{3.18 \text{ ksf}} (\text{ft}^2)$	47.2	70.4	56.7	70.4	81.1
Max. soil pressure $= \dfrac{\text{D}+\text{L}}{\text{area}} (\text{ksf})$	5.72	5.68	6.00	5.11	5.55

Example 13.5

Design a plain concrete footing to support a 16-in.-thick concrete wall. The loads on the wall consist of a 16-K/ft dead load (including the self-weight of wall) and a 10-K/ft live load. The base of the footing is 4 ft below final grade. Use $f_c' = 3$ ksi and an allowable soil pressure of 5 ksf.

Solution

1. Calculate the effective soil pressure. Assume a total depth of footing of 28 in.

$$\text{Weight of footing} = \frac{28}{12} \times 145 = 338 \text{ psf}$$

The weight of the soil, assuming that soil weighs 100 pcf, is $(4 - 2.33) \times 100 = 167$ psf. Effective soil pressure is $5000 - 338 - 167 = 4495$ psf.

2. Calculate the width of the footing for a 1-ft length of the wall ($b = 1$ ft):

$$\text{Width of footing} = \frac{\text{total load}}{\text{effective soil pressure}}$$

$$= \frac{16+10}{4.495} = 5.79 \text{ ft}$$

Use 6.0 ft (Fig. 13.19).

3. $U = 1.2D + 1.6L = 1.2 \times 16 + 1.6 \times 10 = 35.2$ K/ft. The net upward pressure is $q_u = 35.2/6 = 5.87$ ksf.
4. Check bending stresses. The critical section is at the face of the wall. For a 1-ft length of wall and footing,

$$M_u = \frac{1}{2} q_u \left(\frac{L}{2} - \frac{c}{2}\right)^2 = \frac{1}{2}(5.87)\left(\frac{6}{2} - \frac{16}{2 \times 12}\right)^2 = 16 \text{ K·ft}$$

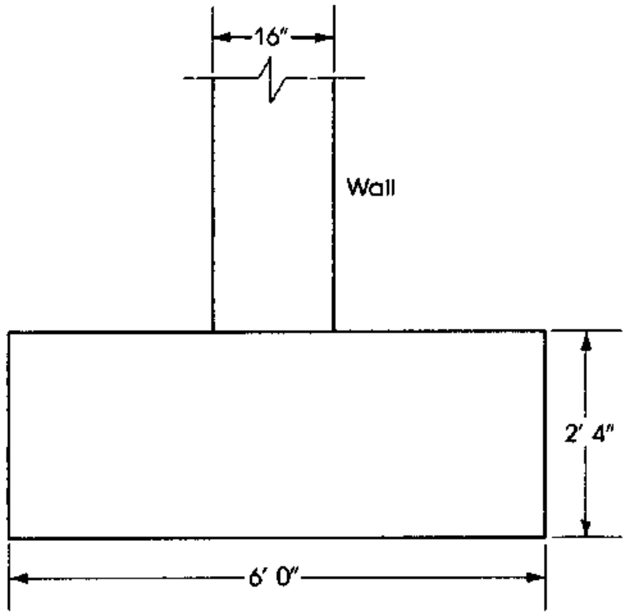


Figure 13.19 Example 13.5: Plain concrete wall footing.

Let the effective depth, d, be 28 − 3 = 25 in., assuming that the bottom 3 in. is not effective.

$$I_g = \frac{bd^3}{12} = \frac{12}{12}(25)^3 = 15{,}625 \text{ in.}^4$$

The flexural tensile stress is

$$f_t = \frac{M_u c}{I} = \frac{(16 \times 12{,}000)}{15{,}625}\left(\frac{25}{2}\right) = 153 \text{ psi}$$

The allowable flexural tensile stress is $5\phi\sqrt{f'_c} = 5 \times 0.55\sqrt{3000} = 151$ psi (close).

5. Check shear stress: The critical section is at a distance $d = 25$ in. from the face of the wall.

$$V_u = q_u\left(\frac{L}{2} - \frac{c}{2} - d\right) = 5.87\left(\frac{6}{2} - \frac{16}{2 \times 12} - \frac{25}{12}\right) = 1.47 \text{ K}$$

$$\phi V_c = \phi\left(\frac{4}{3}\right)\lambda\sqrt{f'_c}bd = \frac{(0.55)\left(\frac{4}{3}\right)(1)\sqrt{3000}(12)(25)}{1000} = 12.05 \text{ K}$$

Therefore, the section is adequate. It is advisable to use minimum reinforcement in both directions.

13.6 COMBINED FOOTINGS

When a column is located near a property line, part of the single footing might extend into the neighboring property. To avoid this situation, the column may be placed on one side or edge of the footing, causing eccentric loading. This may not be possible under certain conditions, and sometimes it is not an economical solution. A better design can be achieved by combining the footing with the nearest internal column footing, forming a combined footing. The center of gravity of the combined footing coincides with the resultant of the loads on the two columns.

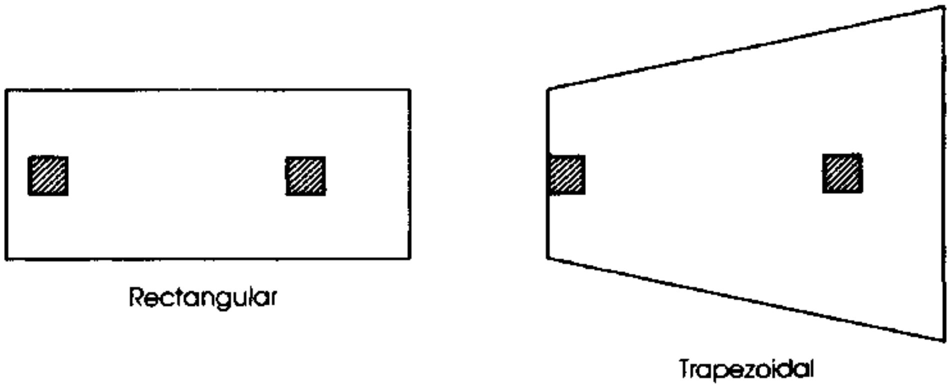


Figure 13.20 Combined footings.

Another case where combined footings become necessary is when the soil is poor and the footing of one column overlaps the adjacent footing. The shape of the combined footing may be rectangular or trapezoidal (Fig. 13.20). When the load of the external column near the property line is greater than the load of the interior column, a trapezoidal footing may be used to keep the centroid of footing in line with the resultant of the two column loads. In most other cases, a rectangular footing is preferable.

The length and width of the combined footing are chosen to the nearest 3 in., which may cause a small variation in the uniform pressure under the footing, but it can be tolerated. For a uniform upward pressure, the footing will deflect, as shown in Fig. 13.21. The ACI Code, Section 15.10, does not provide a detailed approach for the design of combined footings. The design, in general, is based on structural analysis.

A simple method of analysis is to treat the footing as a beam in the longitudinal direction, loaded with uniform upward pressure, q_u. For the transverse direction, it is assumed that the

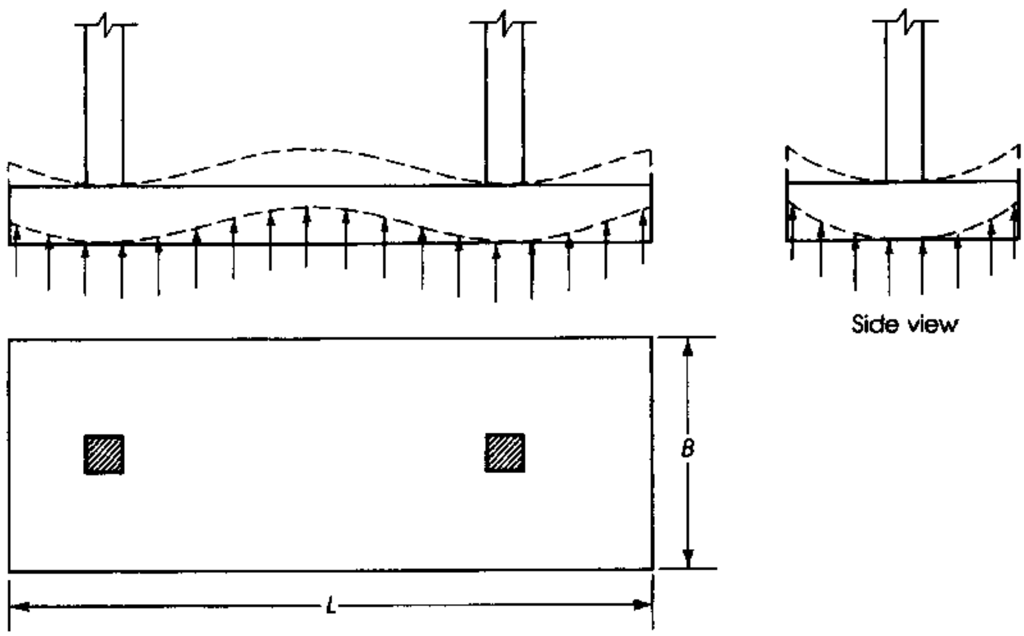


Figure 13.21 Upward deflection of a combined footing in two directions.

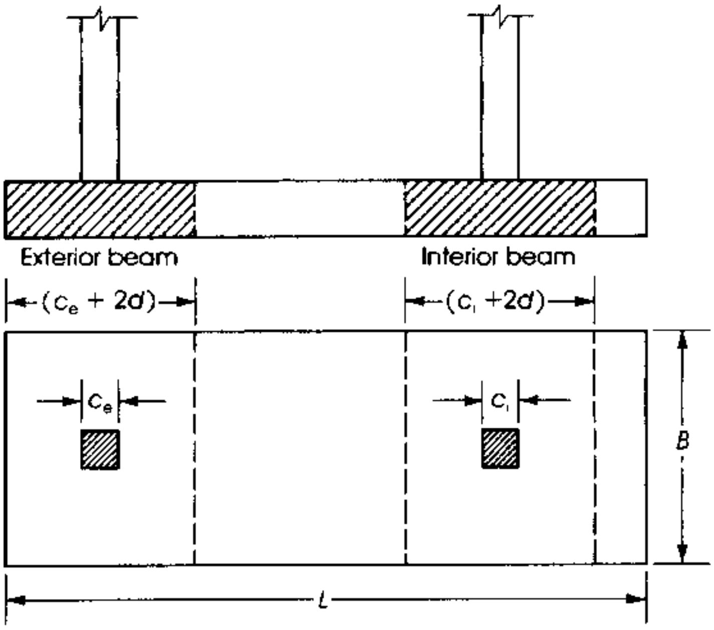


Figure 13.22 Analysis of combined footing in the transverse direction.

column load is spread over a width under the column equal to the column width plus d on each side, whenever that is available. In other words, the column load acts on a beam under the column within the footing, which has a maximum width of $(c + 2d)$ and a length equal to the short side of the footing (Fig. 13.22). A smaller width, down to $(c + d)$, may be used. The next example explains the design method in detail.

Example 13.6

Design a rectangular combined footing to support two columns, as shown in Fig. 13.23. The edge column, I, has a section 16 by 16 in. and carries a D.L. of 180 K and an L.L. of 120 K. The interior column, II, has a section 20 by 20 in. and carries a D.L. of 250 K and an L.L. of 140 K. The allowable soil pressure is 5 ksf and the bottom of the footing is 5 ft below final grade. Design the footing using $f_c' = 4$ ksi, $f_y = 60$ ksi, and the ACI strength design method.

Solution

1. Determine the location of the resultant of the column loads. Take moments about the center of the exterior column I:

$$x = \frac{(250 + 140) \times 16}{(250 + 140) + (180 + 120)} = 9 \text{ ft from column I}$$

The distance of the resultant from the property line is $9 + 2 = 11.0$ ft. The length of the footing is $2 \times 11 = 22.0$ ft. In this case the resultant of column loads will coincide with the resultant of the upward pressure on the footing.

2. Determine the area of the footing. Assume the footing total depth is 36 in. ($d = 36 - 4.5 = 31.5$ in.)

$$\text{Total actual (working) loads} = 300 + 390 = 690 \text{ K}$$

$$\text{New upward pressure} = 5000 - \left(\frac{36}{12} \times 150\right) - (2 \times 100) = 4500 \text{ psf}$$

(Assumed weight of soil is 100 psf.)

$$\text{Required area} = \frac{690}{4.5} = 153.3 \text{ ft}^2$$

$$\text{Width of footing} = \frac{153.3}{22} = 6.97 \text{ ft}$$

Use 7 ft. Choose a footing 22 by 7 ft (area = 154 ft^2).

3. Determine the factored upward pressure using factored loads:

$$P_{u1} \text{ (column I)} = 1.2 \times 180 + 1.6 \times 120 = 408 \text{ K}$$

$$P_{u2} \text{ (column II)} = 1.2 \times 250 + 1.6 \times 140 = 524 \text{ K}$$

The net factored soil pressure is $q_u = (408 + 524)/154 = 6.05$ ksf.

4. Draw the factored shearing force diagram as for a beam of $L = 22$ ft supported on two columns and subjected to an upward pressure of 6.05 ksf × 7 (width of footing) = 42.35 K/ft (per foot length of footing).

$$V_u \text{ (at outer face column I)} = 42.35\left(2 - \frac{8}{12}\right) = 56.5 \text{ K}$$

$$V_u \text{ (at interior face column I)} = 408 - 42.35\left(2 + \frac{8}{12}\right) = 295 \text{ K}$$

$$V_u \text{ (at outer face column II)} = 42.35\left(4 - \frac{10}{12}\right) = 134.1 \text{ K}$$

$$V_u \text{ (at interior face column I)} = 524 - \left(4 + \frac{10}{12}\right) \times 42.35 = 319.3 \text{ K}$$

Find the point of zero shear, x; distance between interior faces of columns I and II is

$$16 - \frac{8}{12} - \frac{10}{12} = 14.5 \text{ ft}$$

$$x = \frac{295}{(295 + 319.3)}(14.5) = 6.9 \text{ ft}$$

5. Draw the factored moment diagram considering the footing as a beam of $L = 22$ ft supported by the two columns. The uniform upward pressure is 47.5 K/ft.

$$M_{u1} \text{ (at outer face column I)} = 423.5\frac{(1.33)^2}{2} = 37.6 \text{ K·ft}$$

$$M_{u2} \text{ (at outer face column II)} = 42.35\frac{(3.17)^2}{2} = 212.8 \text{ K·ft}$$

The maximum moment occurs at zero shear:

$$\text{Maximum } M_u \text{ (calculated from column I side)} = 408\left(6.9 + \frac{8}{12}\right) - \frac{42.35}{2}\left(6.9 + \frac{8}{12} + 2\right)^2$$
$$= 1149 \text{ K·ft}$$

$$\text{Maximum } M_u \text{ (from column II side)} = 524\left(7.6 + \frac{10}{12}\right) - \frac{42.35}{2}\left(7.6 + \frac{10}{12} + 4\right)^2$$
$$= 1146 \text{ K·ft}$$

The moments calculated from both sides of the footings are close enough, and $M_{u\,\max} =$ 1149 K·ft may be adopted. This variation occurred mainly because of the adjustment of the length and width of the footing.

6. Check the depth for one-way shear. Maximum shear occurs at a distance $d = 31.5$ in. from the interior face of column II (Fig. 13.23).

$$V_{u_1} = 319.3 - \frac{31.5}{12}(42.35) = 208.7 \text{ K}$$

$$d = \frac{V_{u1}}{\phi(2\lambda\sqrt{f'_c})b} = \frac{202.7 \times 100}{0.75(2(1)\sqrt{4000})(7 \times 12)} = 26.1 \text{ in.}$$

The effective depth provided is 31.5 in. > 26.1 in.; thus, the footing is adequate.

7. Check depth for two-way shear (punching shear). For the interior column,

$$b_0 = 4(c + d) = \left(\frac{4}{12}\right)(20 + 31.5) = 17.17 \text{ ft}$$

$$(c + d) = \frac{20 + 31.5}{12} = 4.29 \text{ ft}$$

The shear V_{u_2} at a section $d/2$ from all sides of the column is equal to

$$V_{u_2} = P_{u2} - q_u(c + d)^2 = 524 - 6.05(4.29)^2 = 413 \text{ K}$$

$$d = \frac{V_{u2}}{\phi(4\lambda\sqrt{f'_c})b_0} = \frac{413(1000)}{0.75(4(1)\sqrt{4000})(17.7 \times 12)} = 10.3 \text{ in.} < 31.5 \text{ in.}$$

The exterior column is checked and proved not to be critical.

8. Check the depth for moment and determine the required reinforcement in the long direction.

$$\text{Maximum bending moment} = 1149 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{1149(12{,}000)}{(7 \times 12)(31.5)^2} = 166 \text{ psi}$$

Applying Eq. 13.14, the steel percentage is $\rho = 0.0033 = 0.0033$ ($\rho_{\min}$).

$$A_s = 0.0033(84 \times 31.5) = 8.73 \text{ in.}^2$$

$$\text{Min } A_s \text{ (shrinkage)} = 0.0018(84)(36) = 5.44 \text{ in.}^2$$

$A_s = 8.73$ in.^{2} controls. Use 10 no. 9 bars ($A_s = 10$ in.^{2}).

$$\text{Spacing of bars} = \frac{84 - 6 \text{ (concrete cover)}}{9 \text{ (no. of spacings)}} = 8.67 \text{ in.}$$

The bars are extended between the columns at the top of the footing with a concrete cover of 3 in. Place minimum reinforcement at the bottom of the projecting ends of the footing beyond the columns to take care of the positive moments. Extend the bars a development length l_d beyond the side of the column.

The minimum shrinkage reinforcement is 5.44 in.^{2} Use seven no. 8 bars ($A_s = 5.5$ in.^{2})

The development length required for the main top bars is $1.3l_d = 1.3(54) = 70$ in. beyond the point of maximum moment. Development lengths provided to both columns are adequate.

9. For reinforcement in the short direction, calculate the bending moment in the short (transverse) direction, as in the case of single footings. The reinforcement under each column is to be placed within a maximum bandwidth equal to the column width twice the effective depth d of the footing (Fig. 13.24).

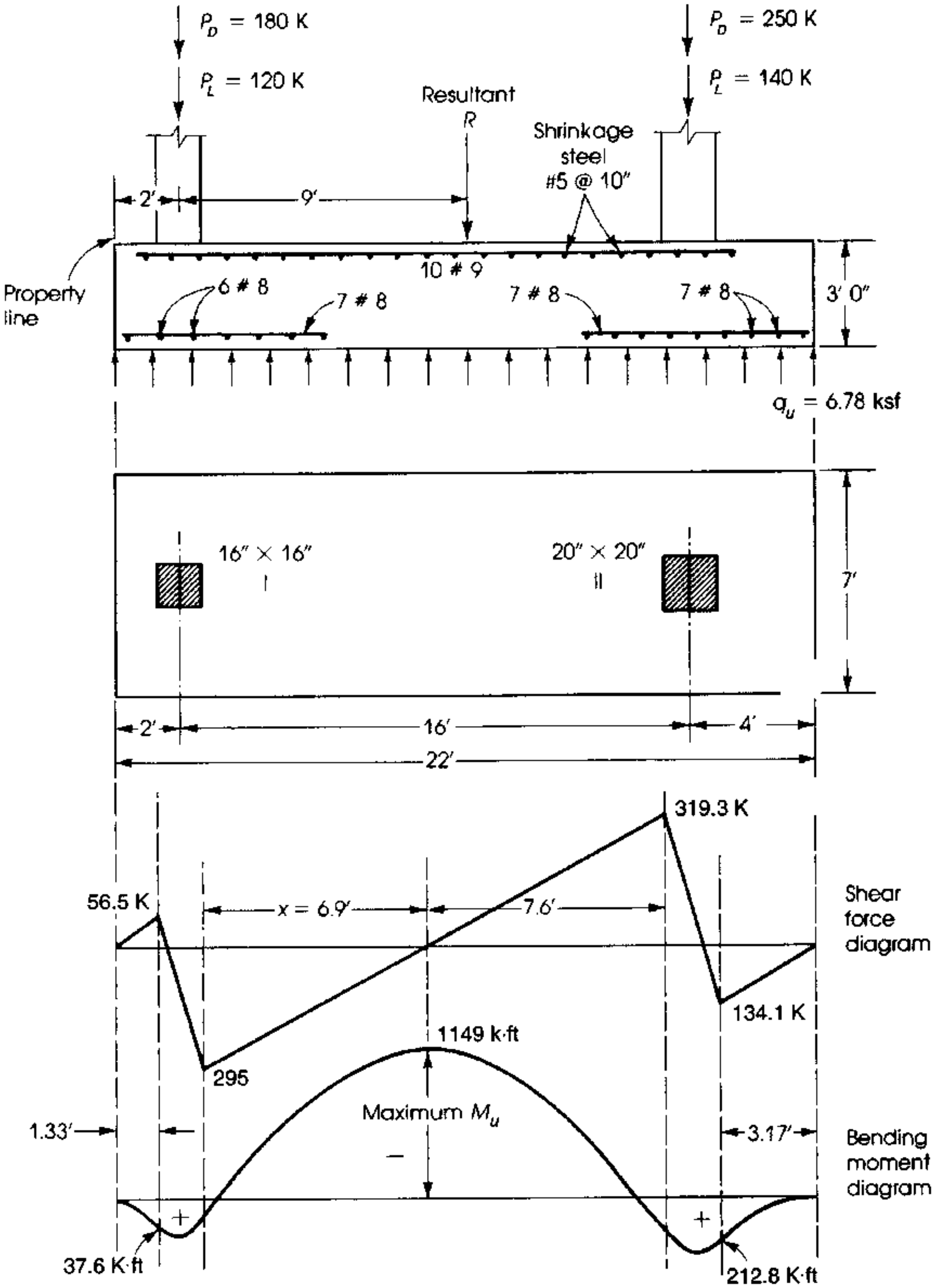


Figure 13.23 Example 13.6: Design of a combined footing.

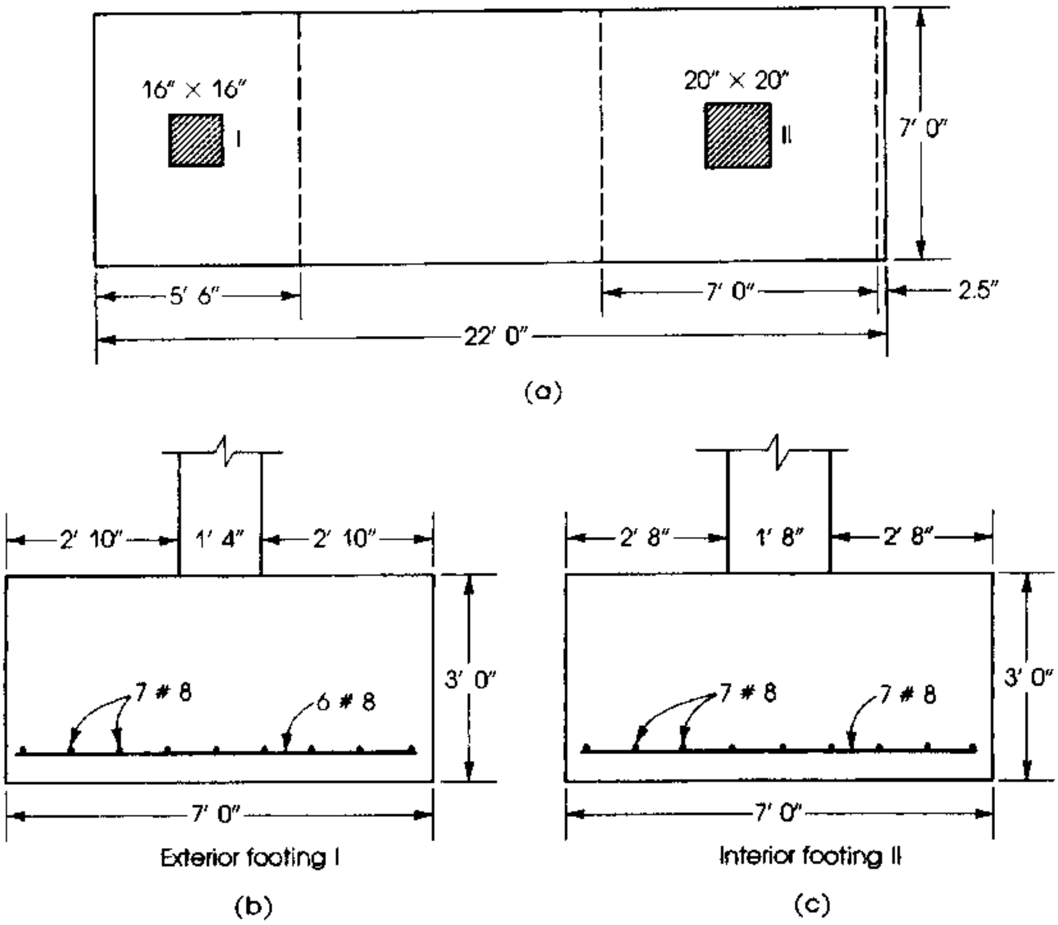


Figure 13.24 Design of combined footing, transverse direction: (*a*) plan, (*b*) exterior footing, and (*c*) interior footing.

a. Reinforcement under exterior column I:

$$\begin{aligned}\text{Bandwidth} &= 16 \text{ in. (column width)}\\ &\quad + 16 \text{ in. (on exterior side of column)}\\ &\quad + 31.5 \text{ in. } (d)\\ &= 63.5 \text{ in.} = 5.3 \text{ ft}\end{aligned}$$

Use 5.5 ft. Net upward pressure in the short direction under column I is

$$\frac{P_{u1}}{\text{width of footing}} = \frac{408}{7} = 58.3 \text{ K/ft}$$

Distance from the free end to the face of the column is $\frac{7}{2} - \frac{8}{12} = 2.83$ ft.

$$M_u \text{ (at face of column I)} = \frac{58.3}{2}(2.83)^2 = 233.5 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{233.5 \times 12{,}000}{(5.5 \times 12)(31.5)^2} = 43 \text{ psi}$$

The steel percentage, ρ, is less than minimum ρ for shrinkage reinforcement ratio of 0.0018.

$$A_{s\min} = (0.0018)(5.5 \times 12)(36) = 4.3 \text{ in.}^2$$

Use six no. 8 bars ($A_s = 4.71 \text{ in.}^2$) placed within the bandwidth of 66 in.

b. Reinforcement under the interior column II:

$$\text{Bandwidth} = 20 + 31.5 + 31.5 = 83 \text{ in.} = 6.91 \text{ ft}$$

Use 7 ft (84 in.).

$$\text{Net upward pressure} = \frac{P_{u2}}{\text{width of footing}} = \frac{524}{7} = 75 \text{ K/ ft}$$

$$\text{Distance to face of column} = \frac{7}{2} - \frac{10}{12} = 2.67 \text{ ft}$$

$$M_u \text{ (at face of column II in short direction)} = \tfrac{1}{2}(75)(2.67)^2 = 267 \text{ K}\cdot\text{ft}$$

$$R_u = \frac{(267)(12{,}000)}{(84)(31.5)^2} = 38 \text{ psi}$$

which is very small. Use a minimum shrinkage reinforcement ratio of 0.0018.

$$A_s = (0.0018)(84)(36) = 5.44 \text{ in.}^2$$

Use seven no. 8 bars placed within the bandwidth of 84 in. under column II, as shown in Figs. 13.23 and 13.24. The development length l_d of no. 8 bars in the short direction is 48 in.

13.7 FOOTINGS UNDER ECCENTRIC COLUMN LOADS

When a column transmits axial loads only, the footing can be designed such that the load acts at the centroid of the footing, producing uniform pressure under the footing. However, in some cases, the column transmits an axial load and a bending moment, as in the case of the footings of fixed-end frames. The pressure q that develops on the soil will not be uniform and can be evaluated from the following equation:

$$q = \frac{P}{A} \pm \frac{Mc}{I} \geq 0 \tag{13.19}$$

where A and I are the area and moment of inertia of the footing, respectively. Different soil conditions exist, depending on the magnitudes of P and M, and allowable soil pressure. The different design conditions are shown in Fig. 13.25 and are summarized as follows:

1. When $e = M/P < L/6$, the soil pressure is trapezoidal.

$$q_{\max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{LB} + \frac{6M}{BL^2} \tag{13.20}$$

$$q_{\min} = \frac{P}{A} - \frac{Mc}{I} = \frac{P}{LB} - \frac{6M}{BL^2} \tag{13.21}$$

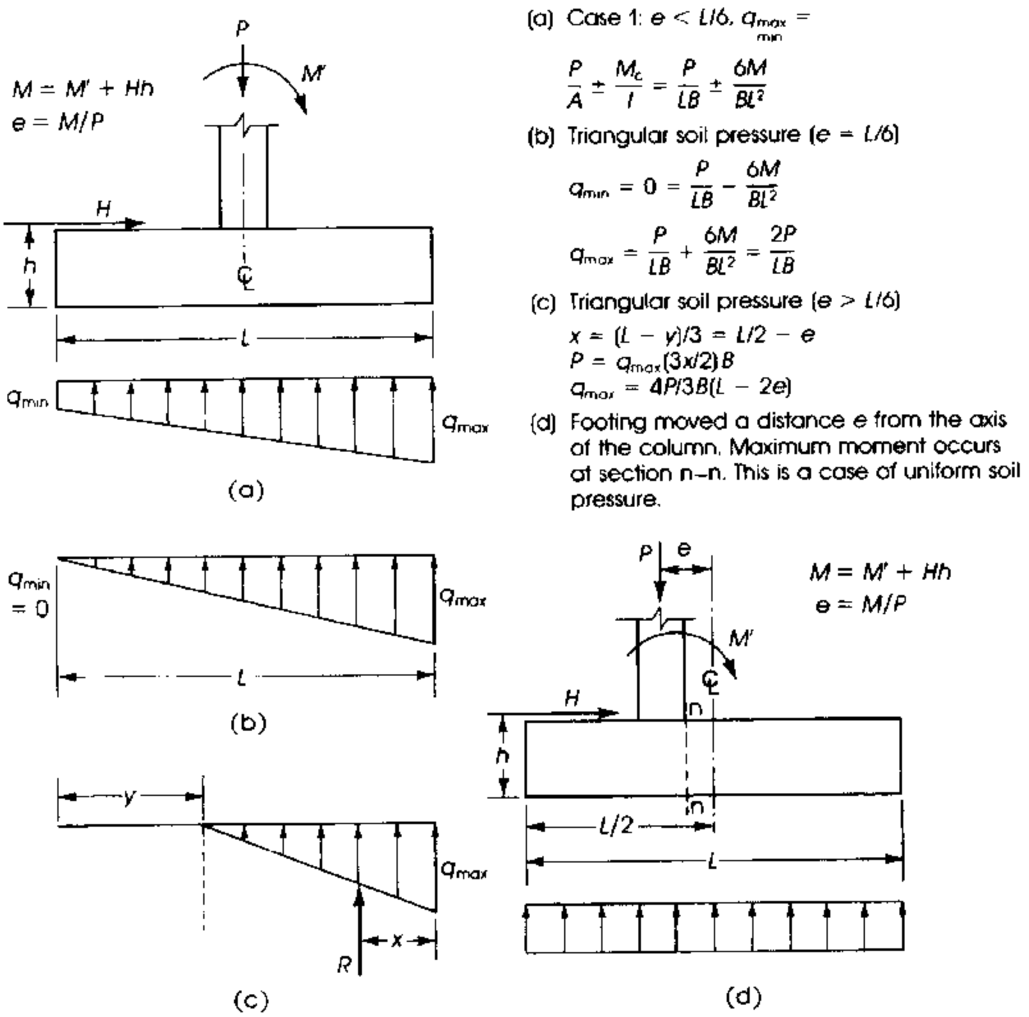


Figure 13.25 Single footing subjected to eccentric loading: L = length of footing, B = width, and h = height.

2. When $e = M/P = L/6$, the soil pressure is triangular.

$$q_{\max} = \frac{P}{LB} + \frac{6M}{BL^2} = \frac{2P}{LB} \tag{13.22}$$

$$q_{\min} = 0 = \frac{P}{LB} - \frac{6M}{BL^2} \text{ or } \frac{P}{LB} = \frac{6M}{BL^2} \tag{13.23}$$

3. When $e > L/6$, the soil pressure is triangular.

$$x = \frac{L-y}{3} = \frac{L}{2} - e$$

$$P = q_{\max}\left(\frac{3x}{2}\right)B \tag{13.24}$$

$$q_{\max} = \frac{2P}{3xB} = \frac{4P}{3B(L-2e)}$$

4. When the footing is moved a distance e from the axis of the column to produce uniform soil pressure under the footing. Maximum moment occurs at section n-n.

$$M = M' - Hh \quad \text{and} \quad e = \frac{M}{P}$$

13.8 FOOTINGS UNDER BIAXIAL MOMENT

In some cases, a footing may be subjected to an axial force and biaxial moments about its x- and y-axes; such a footing may be needed for a factory crane that rotates 360°. The footing then must be designed for the critical loading.

Referring to Fig. 13.26, if the axial load P acts at a distance e_x from the y-axis and e_y from the x-axis, then

$$M_x = Pe_y \quad \text{and} \quad M_y = Pe_x$$

The soil pressure at corner 1 is

$$q_{\max} = \frac{P}{A} + \frac{M_x c_y}{I_x} + \frac{M_y c_x}{I_y}$$

At corner 2,

$$q_2 = \frac{P}{A} - \frac{M_x c_y}{I_x} + \frac{M_y c_x}{I_y}$$

At corner 3,

$$q_3 = \frac{P}{A} - \frac{M_x c_y}{I_x} - \frac{M_y c_x}{I_y}$$

At corner 4,

$$q_4 = \frac{P}{A} + \frac{M_x c_y}{I_x} - \frac{M_y c_x}{I_y}$$

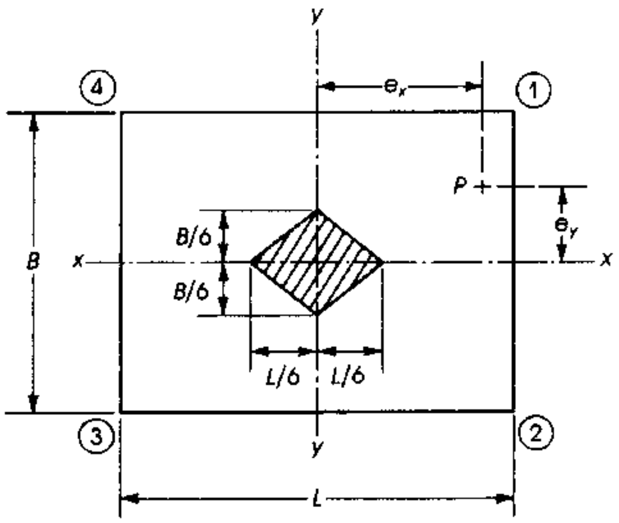


Figure 13.26 Footing subjected to P and biaxial moment. If $e_x < L/6$ and $e_y < B/6$, footing will be subjected to upward soil pressure on all bottom surface (nonuniform pressure).

Again, note that the allowable soil pressure must not be exceeded and the soil cannot take any tension; that is, $q \geq 0$.

Example 13.7

A 12-in. by 24-in. column of an unsymmetrical shed shown in Fig. 13.27*a* is subjected to an axial load $P_D = 220\,\text{K}$ and a moment $M_d = 180\,\text{K·ft}$ due to dead load and an axial load $P_L = 165\,\text{K}$ and a moment $M_L = 140\,\text{K·ft}$ due to live load. The base of the footing is 5 ft below final grade, and the allowable soil bearing pressure is 5 ksf. Design the footing using $f'_c = 4$ ksi and $f_y = 60\,\text{ksi}$.

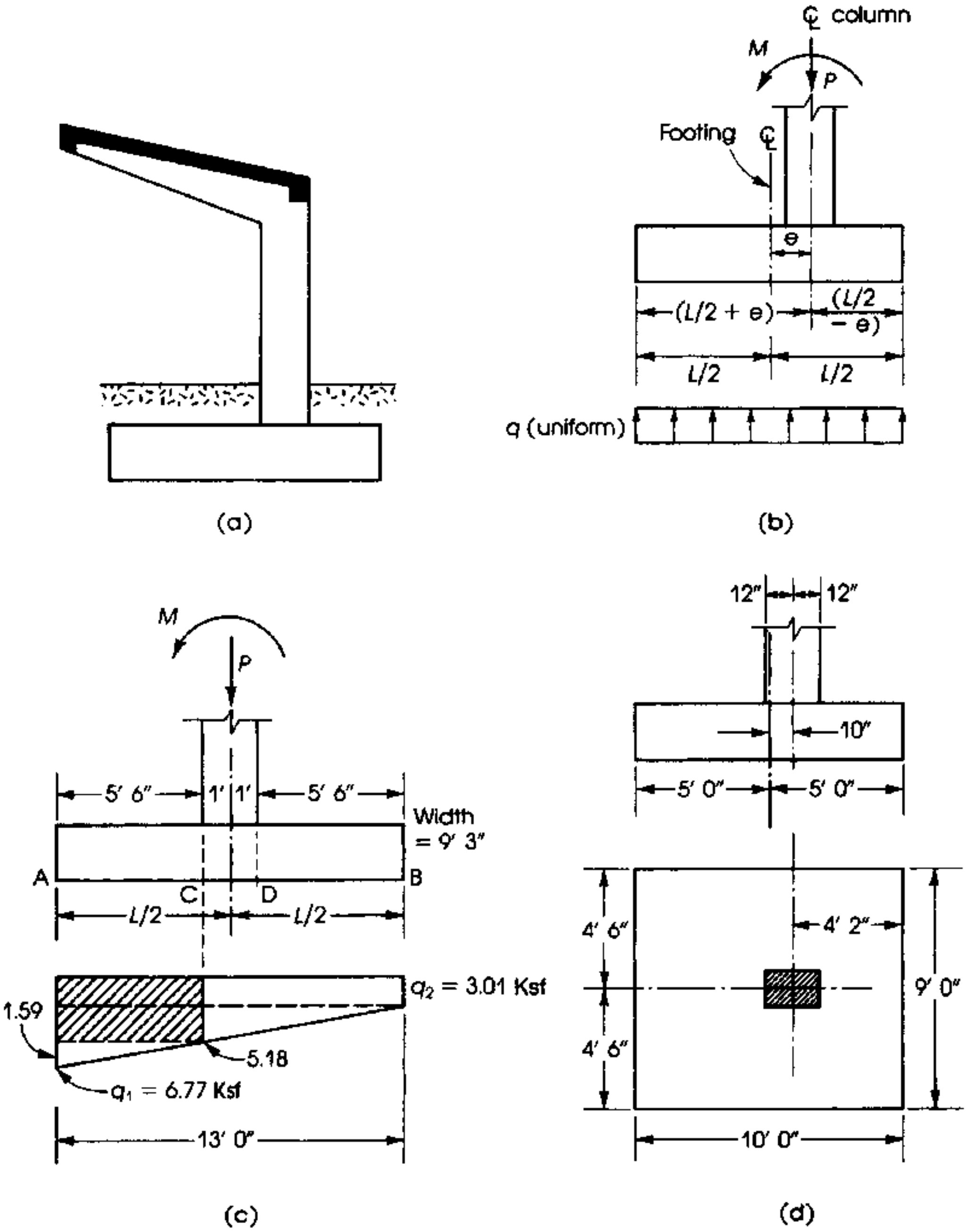


Figure 13.27 Example 13.7.

Solution

The footing is subjected to an axial load and a moment

$$P = 220 + 165 = 385 \text{ K}$$

$$M = 180 + 140 = 320 \text{ K·ft}$$

The eccentricity is

$$e = \frac{M}{P} = \frac{320 \times 12}{385} = 9.97 \text{ in.} \qquad \text{say, } 10 \text{ in.}$$

The footing may be designed by two methods.

Method 1: Move the center of the footing a distance $e = 10$ in. from the center of the column. In this case, the soil pressure will be considered uniformly distributed under the footing (Fig. 13.27*b*).

Method 2: The footing is placed concentric with the center of the column. In this case, the soil pressure will be trapezoidal or triangular (Fig. 13.27c), and the maximum and minimum values can be calculated as shown in Fig. 13.27.

The application of the two methods to Example 13.7 can be explained briefly as follows:

1. For the first method, assume a footing depth of 20 in. ($d = 16.5$ in.) and assume the weight of soil is 100 pcf. Net upward pressure is $5000 - \frac{20}{12} \times 150$ (footing) $- \left(5 - \frac{20}{12}\right) \times 100 = 4417$ psf.

$$\text{Area of footing} = \frac{385}{4.42} = 87.1 \text{ ft}^2$$

   Assume a footing width of 9 ft; then the footing length is 87.1/9 = 9.7 ft, say, 10 ft. Choose a footing 9 by 10 ft and place the column eccentrically, as shown in Fig. 13.27*d*. The center of the footing is 10 in. away from the center of the column.

2. The design procedure now is similar to that for a single footing. Check the depth for two-way and one-way shear action. Determine the bending moment at the face of the column for the longitudinal and transverse directions. Due to the eccentricity of the footing, the critical section will be on the left face of the column in Fig. 13.27*d*. The distance to the end of footing is $(5 \times 12) - 2 = 58$ in. $= 4.833$ ft.

$$P_u = 1.2D + 1.6L = 1.2 \times 200 + 1.6 \times 165 = 504 \text{ K}$$

$$q_u = \frac{504}{9 \times 10} = 5.6 \text{ ksf}$$

$$\text{Maximum } M_u = (5.6 \times 9) \times \frac{(4.833)^2}{2} = 588.6 \text{ K·ft} \quad \text{(in 9-ft width)}$$

   In the transverse direction,

$$M_u = (5.6 \times 10) \times \frac{(4)^2}{2} = 448 \text{ K·ft}$$

   Revise the assumed depth if needed and choose the required reinforcement in both directions of the footing, as was explained in the single-footing example.

3. For the second method, calculate the area of the footing in the same way as explained in the first method; then calculate the maximum soil pressure and compare it with that allowable using actual loads.

$$\text{Total load } P = 385 \text{ K}$$

$$\text{Size of footing} = 10 \times 9 \text{ ft}$$

   Because the eccentricity is $e = 10$ in. $< L/6 = 10 \times \frac{12}{6} = 20$ in., the shape of the upward soil pressure is trapezoidal. Calculate the maximum and minimum soil pressure:

$$q_{\max} = \frac{P}{LB} + \frac{6M}{BL^2} = \frac{385}{10 \times 9} + \frac{6 \times 320}{9(10)^2} = 6.42 \text{ ksf} > 4.42 \text{ ksf}$$

The footing is not safe. Try a footing 9.25 × 13 ft (area = 120.25 ft^2).

$$q_{max} = \frac{385}{120.25} + \frac{6 \times 320}{9.25(13)^2} = 4.22 \text{ ksf} < 4.42 \text{ ksf}$$

$$q_{min} = 3.2 - 1.22 = 1.98 \text{ ksf}$$

4. Calculate the factored upward pressure using factored loads; then calculate moments and shears, as explained in previous examples.

13.9 SLABS ON GROUND

A concrete slab laid directly on ground may be subjected to

1. Uniform load over its surface, producing small internal forces.
2. Nonuniform or concentrated loads, producing some moments and shearing forces. Tensile stresses develop, and cracks will occur in some parts of the slab.

Tensile stresses are generally induced by a combination of

1. Contraction due to temperature and shrinkage, restricted by the friction between the slab and the subgrade, causing tensile stresses
2. Warping of the slab
3. Loading conditions
4. Settlement

Contraction joints may be formed to reduce the tensile stresses in the slab. Expansion joints may be provided in thin slabs up to a thickness of 10 in.

Basement floors in residential structures may be made of 4- to 6-in. concrete slabs reinforced in both directions with a wire fabric reinforcement. In warehouses, slabs may be 6 to 12 in. thick, depending on the loading on the slab. Reinforcement in both directions must be provided, usually in the form of wire fabric reinforcement. Basement floors are designed to resist upward earth pressure and any water pressure. If the slab rests on very stable or incompressible soils, then differential settlement is negligible. In this case the slab thickness will be a minimum if no water table exists. Columns in the basement will have independent footings. If there is any appreciable differential settlement, the floor slab must be designed as a stiff ra ft foundation.

13.10 FOOTINGS ON PILES

When the ground consists of so ft material for a great depth, and its bearing capacity is very low, it is not advisable to place the footings directly on the soil. It may be better to transmit the loads through piles to a deep stratum that is strong enough to bear the loads or to develop sufficient friction around the surface of the piles.

Many different kinds of piles are used for foundations. The choice depends on ground conditions, presence of ground water, function of the pile, and cost. Piles may be made of concrete, steel, or timber.

In general, a pile cap (or footing) is necessary to distribute the load from a column to the heads of a number of piles. The cap should be of sufficient size to accommodate deviation in the position of the pile heads. The caps are designed as beams spanning between the pile heads and carrying concentrated loads from columns. When the column is supported by two piles, the cap may be designed as a reinforced concrete truss of a triangular shape.

The ACI Code, Section 15.2, indicates that computations for moments and shears for footings on piles may be based on the assumption that the reaction from any pile is concentrated at the pile center. The base area of the footing or number of piles shall be determined from the unfactored forces and moments.

The minimum concrete thickness above the reinforcement in a pile footing is limited to 12 in. (ACI Code, Section 15.7). For more design details of piles and pile footings, refer to books on foundation engineering.

13.11 SI EQUATIONS

1. One-way shear:

$$\phi V_c = 0.17\lambda\phi\sqrt{f'_c}bd \tag{13.3}$$

2. Two-way shear:

$$V_{c_1} = 0.33\lambda\sqrt{f'_c}b_0d \tag{13.6}$$

$$V_{c_2} = 0.17\left(1 + \frac{2}{\beta}\right)\lambda\sqrt{f'_c}b_0d \tag{13.7}$$

$$V_{c_3} = 0.083\left(\frac{\alpha_s d}{b_o} + 2\right)\lambda\sqrt{f'_c}b_0d \tag{13.8}$$

Other equations remain the same.

SUMMARY

Sections 13.1–13.4

1. *General:*
   H = distance of the bottom of footing from final grade (ft)
   h = total depth of footing (in.)
   c = wall thickness (in.)
   q_a = allowable soil pressure (ksf)
   q_e = effective soil pressure
   W_s = weight of soil (pcf) (Assume 100 pcf if not given)
2. *Design of wall footings:* The design steps can be summarized as follows.
   a. Assume a total depth of footing h (in.). Consider 1-ft length of footing.
   b. Calculate $q_e = q_a - (h/12)(150) - W_s(H - h/12)$ (q_a in psf).
   c. Calculate width of footing: B = (total service load)/$q_e = (P_D + P_n)/q_e$. (Round to the nearest higher half foot.) The footing size is $(B \times 1)$ ft.
   d. Calculate the factored upward pressure, $q_u = P_u/B$

$$P_u = 1.2P_D + 1.6P_L$$

e. Check the assumed depth for one-way shear requirements considering $d_a = (h - 3.5)$ in. (Two-way shear does not apply.)

$$V_u = q_u\left(\frac{B}{2} - d - \frac{c}{2}\right) \qquad \text{(Use kips.)}$$

$$\text{Required } d = \frac{V_u(1000)}{\phi(2\lambda\sqrt{f_c'})(12)} \geq d_a$$

f. Calculate the bending moment and main steel. The critical section is at the face of the wall.

- $M_u = 0.5q_u\,(L/2 - c/2)^2$; get $R_u = M_u/bd^2$.
- Determine ρ from tables in Appendix A or from Eq. 13.14.
- $A_s = \rho bd = 12\,\rho d$ in.^{2}/ft; $A_s \geq A_{s\,\text{min}}$.
- Minimum steel for shrinkage is

$$A_{\text{sh}} = 0.0018\,(bh) \text{ for } f_y = 60 \text{ ksi}$$

$$A_{\text{sh}} = 0.0020\,(bh) \text{ for } f_y = 40, \text{ or } 50 \text{ ksi}$$

Minimum steel for flexure is

$$A_{\text{sf}} = \left(\frac{200}{f_y}\right)bd = \left(\frac{200}{f_y}\right)(12d) \qquad \text{when } f_c' < 4500 \text{ psi}$$

$$A_{\text{sf}} = \frac{(3\sqrt{f_c'})(12d)}{f_y} \qquad \text{when } f_c' > 4500 \text{ psi}$$

A_s calculated must be greater than A_{sh} (shrinkage). However, if $A_s < A_{\text{sf}}$, it is recommended to use $A_s = A_{\text{sf}}$ and then choose bars and spacings.

g. Check development length: Refer to Tables 7.1, 7.2, 7.3 and 7.4.

h. Calculate secondary reinforcement in the direction of the wall. $A_s = A_{\text{sh}}$ as calculated in step 6d using $b = 12$ in. Choose bars and spacings.

3. *Design of square/rectangular footings:* The design steps are as follows.

a. Assume a total depth h (in.); let d_a (assumed) $= (h - 4.5)$ in. Calculate $q_e = q_a - (h/12)(150) - W_s(H - h/12)$. (Use psf.)

b. Calculate the area of the footing, $AF = (P_D + P_L)/q_e$. Choose either a square footing, side $= \sqrt{AF}$, or a rectangular footing of length L and width B (short length); then round dimensions to the higher half ft.

c. Calculate $q_u = P_u/(LB)$.

d. Check footing depth due to two-way shear first. Maximum V_{u2} occurs at a section located at a distance equal to $d/2$ around the column.

1. Calculate $b_0 = 4(c + d)$ for square columns and $b_0 = 2\,(c_1 + d) + 2\,(c_2 + d)$ for rectangular columns.

$$V_{u_2} = P_u - q_u(c + d)^2 \qquad \text{for square columns}$$

$$V_{u_2} = P_u - q_u(c_1 + d)(c_2 + d) \qquad \text{for rectangular columns}$$

2. Calculate $d_1 = V_{u_2}/4\phi\lambda\sqrt{f_c'}b_0$ when $\beta = L/B \leq 2$.

$$d_1 = \frac{V_{u_2}}{\phi(2 + 4/\beta)\lambda\sqrt{f_c'}b_0} \qquad \text{when } \beta > 2$$

3. Calculate

$$d_2 = \frac{V_{u2}}{\phi(\alpha_s d/b_0 + 2)\lambda\sqrt{f'_c}b_0}$$

Let d = the larger of d_1 and d_2. If d is less than d_a (assumed), increase d_a (or h) and repeat. The required d should be close to the assumed d_a (within 5% or 1 in. higher).

e. Check one-way shear (normally does not control in single footings):

1. $V_{u_{11}} = q_u B(L/2 - c/2 - d)$ in the long direction (or for square footings).

$$d_{11} = \frac{V_{u11}}{2\phi\lambda\sqrt{f'_c}B}$$

2. $V_{u_{12}} = q_u L(B/2 - c/2 - d)$ in the short direction.

$$d_{12} = \frac{V_{u12}}{2\phi\lambda\sqrt{f'_c}L} \quad \text{(for rectangular footings)}$$

3. Let d be the larger of d_{11} and d_{12}; then use the larger d from steps 4 and 5.
4. Determine h = (d + 4.5) in.; round to the nearest higher inch.
5. Calculate the final $d = (h - 4.5)$ in.

f. Calculate the bending moment and the main steel in one direction only for square footings and two directions for rectangular footings.

1. In the long direction (or for square footings)

$$M_{uL} = 0.5q_u\left(\frac{L}{2} - \frac{c}{2}\right)^2 \qquad R_u = \frac{M_{uL}}{Bd^2}$$

2. In the short direction (for rectangular footings):

$$M_{us} = 0.5q_u\left(\frac{B}{2} - \frac{c}{2}\right)^2 \qquad R_{us} = \frac{M_{us}}{Ld^2}$$

3. Calculate the reinforcement in the long direction, A_{sL}, and in the short direction, A_{ss}, using Eq. 13.14.
4. Check that A_{sL} and A_{ss} are greater than the minimum steel reinforcement. Choose bars and spacings. For square footings, the same bars are used in both directions. Distribute bars in the bandwidth of rectangular columns according to Eq. 13.15.

g. Check bearing stress:

1. Calculate N_1 and $N_2 : N_1\,\phi(0.85f'_cA_1)$, where $\phi = 0.65$ and A_1 = area of column section; $N_2 = N_1\sqrt{A_2/A_1} \le 2N_1$, where A_2 = square area of footing under column ($A_2 = B^2$).
2. If $P_u \le N_1$ bearing stress is adequate. Minimum area of dowels is 0.005 A_1. Choose four bars to be placed at the four corners of column section.
3. If $P_u > N_1$, determine the excess load, $P_{ex} = (P_u - N_1)$, and then calculate A_{sd} (dowels) = P_{ex}/f_y. A_{sd} must be equal to or greater than $0.005A_1$. Choose at least four dowel bars.
4. Determine the development length in compression for dowels in the column and in the footing.

h. Calculate the development lengths, l_d, of the main bars in the footings. The calculated l_d must be greater than or equal to l_d provided in the footing. Provided l_d is $(L/2 - c/2 - 3)$ in. in the long direction and $l_d = (B/2 - c/2 - 3)$ in the short direction. Examples 13.2 and 13.3 explain these steps.

Section 13.5

Plain concrete may be used to support walls. The maximum flexural stress in tension should be calculated and must be less than the allowable stress of $5\phi\sqrt{f'_c}$.

Section 13.6

A combined footing is used when a column is located near a property line. Design of such footings is explained in Example 13.6.

Sections 13.7–13.9

Footings under eccentric column loads are explained in Example 13.7.

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PROBLEMS

For all problems in this chapter, use the following:

H_a = distance from bottom of footing to the final grade

h = depth of concrete footing

q_a = allowable soil pressure in ksf

Assume the weight of the soil is 100 pcf and $f_y = 60$ ksi.

13.1 For each problem in Table 13.1, design a wall footing to support the given reinforced concrete wall loads. Design for shear and moment; check the development length requirements. Also, determine the footing bars and their distribution. (Assume $d = h - 3.5$ in.)

13.2 For each problem in Table 13.2, design a square single footing to support the given square and round column loads. Design for moments, shear, load transfer, dowel length, and development lengths for footing main bars. Choose adequate bars and spacings. (Assume $d = h - 4.5$ in. for all problems.)

Table 13.1 Problem 13.1

Number	Wall Thickness (in.)	Dead Load (K/ft)	Live Load (K/ft)	f'_c (Ksi)	q_a (Ksf)	H (ft)	Part Answers L (ft)	h (in.)
(a)	12	22	12	3	4	5	10	19
(b)	12	18	14	3	5	4	7.5	17
(c)	14	28	16	3	6	6	8.5	20
(d)	14	26	24	3	4	5	14.5	27
(e)	16	32	16	3	5	5	11	23
(f)	16	24	20	4	6	8	9	19
(g)	14	20	18	4	4	6	11.5	19
(h)	14	28	20	4	5	4	10.5	21
(i)	12	18	14	4	6	5	6	14
(j)	14	16	20	4	6	5	7	16

Table 13.2 Problem 13.2

Number	Column (in.)	Column bars	Dead Load (K)	Live Load (K)	f'_c (Ksi)	q_s (Ksf)	H (ft)	Part Answers L (ft)	h (in.)
(a)	16 × 16	8 no. 8	150	115	3	5	6	8	20
(b)	18 × 18	8 no. 9	160	100	3	6	5	7	19
(c)	20 × 20	12 no. 9	245	159	3	6	7	9	23
(d)	12 × 12	8 no. 8	180	140	3	5	8	9	24
(e)	14 × 14	8 no. 9	140	160	4	5	6	8.5	21
(f)	16 × 16	8 no. 9	190	140	4	4	5	10	21
(g)	18 × 18	12 no. 8	200	120	4	6	7	8	20
(h)	20 × 20	12 no. 9	195	195	4	5	8	10	22
(i)	Dia. 20	8 no. 9	120	85	4	5	5	7	16
(j)	Dia. 16	8 no. 8	110	90	3	4	6	8	18

13.3 Repeat Problem 13.2*a–h* using rectangular footings with widths of 6, 6, 8, 8, 7, 8, 6, and 9 ft, respectively.

13.4 Repeat Problem 13.2*a–d* using rectangular columns of 14 × 20 in., 16 × 20 in., 16 × 24 in., and 12 × 18 in., respectively, and rectangular footings with the length equal to about 1.5 times the width.

13.5 Repeat Problem 13.1*a–d* using plain concrete wall footings and one-half the applied dead and live loads.

13.6 Design a rectangular combined footing to support the two columns shown in Fig. 13.28. The center of the exterior column is 1 ft away from the property line and 14 ft from the center of the interior column. The exterior column is square with 18-in. sides, is reinforced with no. 8 bars, and carries an axial dead load of 160 K and a live load of 140 K. The interior column is square with 20-in. sides, is reinforced with no. 9 bars, and carries an axial dead load of 240 K and a live load of 150 K. The bottom of the footing is 5 ft below final grade. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and an allowable soil pressure of 5 ksf.

13.7 Determine the footing areas required for a balanced footing design (equal settlement approach) if the usual load is 25% for all footings. The allowable soil pressure is 5 ksi and the dead and live loads are given in Table 13.3.

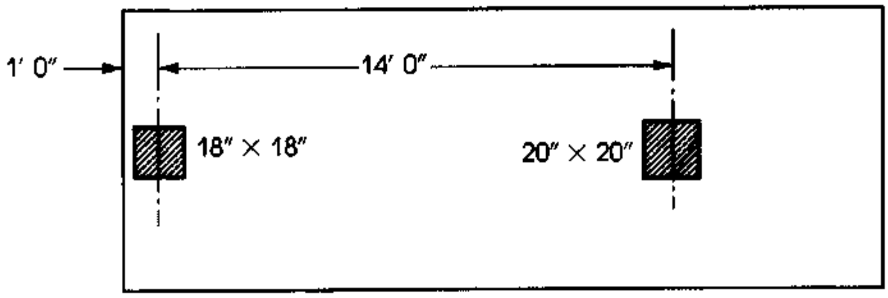


Figure 13.28 Problem 13.6.

Table 13.3 Problem 13.7

	Footing no.					
	1	**2**	**3**	**4**	**5**	**6**
Dead load	130 K	220 K	150 K	180 K	200 K	240 K
Live load	160 K	220 K	210 K	180 K	220 K	200 K

13.8 The 12- by 20-in., (300- by 500-mm) column of the frame shown in Fig. 13.29 is subjected to an axial load $P_D = 200\,\text{K}$ and a moment $M_D = 120\,\text{K·ft}$ due to dead load and an axial load $P_L = 160\,\text{K}$ and a moment $M_L = 110\,\text{K·ft}$ due to live load. The base of the footing is 4 ft below final grade. Design the footing using $f'_c = 4$ ksi, $f_y = 40$ ksi, and an allowable soil pressure of 4 ksi. Use a uniform pressure and eccentric footing approach.

13.9 Repeat Problem 13.8 if both the column and the footing have the same centerline (concentric case).

13.10 Determine the size of a square or round footing for the case of Problem 13.9, assuming that the loads and moments on the footing are for a rotating crane fixed at its base.

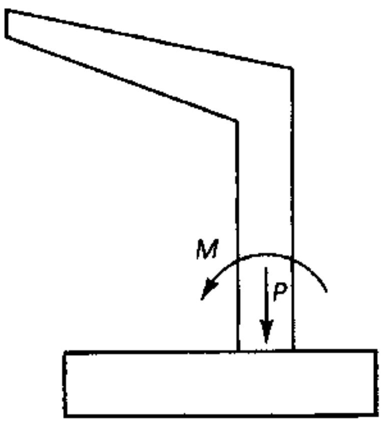


Figure 13.29 Problem 13.8.

CHAPTER 14

RETAINING WALLS

Apartment building, Miami, Florida.

14.1 INTRODUCTION

Retaining walls are structural members used to provide stability for soil or other materials and to prevent them from assuming their natural slope. In this sense, the retaining wall maintains unequal levels of earth on its two faces. The retained material on the higher level exerts a force on the retaining wall that may cause its overturning or failure. Retaining walls are used in bridges as abutments, in buildings as basement walls, and in embankments. They are also used to retain liquids, as in water tanks and sewage-treatment tanks.

14.2 TYPES OF RETAINING WALLS

Retaining walls may be classified as follows (refer to Fig. 14.1):

1. *Gravity walls* usually consist of plain concrete or masonry and depend entirely on their own weight to provide stability against the thrust of the retained material. These walls are proportioned so that tensile stresses do not develop in the concrete or masonry due to the exerted forces on the wall. The practical height of a gravity wall does not exceed 10 ft.
2. *Semigravity walls* are gravity walls that have a wider base to improve the stability of the wall and to prevent the development of tensile stresses in the base. Light reinforcement is sometimes used in the base or stem to reduce the large section of the wall.
3. The *cantilever retaining wall* is a reinforced concrete wall that is generally used for heights from 8 to 20 ft. It is the most common type of retaining structure because of economy and simplicity of construction. Various types of cantilever retaining walls are shown in Fig. 14.1.
4. *Counterfort retaining walls* higher than 20 ft develop a relatively large bending moment at the base of the stem, which makes the design of such walls uneconomical. One solution in this case is to introduce transverse walls (or counterforts) that tie the stem and the base together at intervals. The counterforts act as tension ties supporting the vertical walls. Economy is achieved because the stem is designed as a continuous slab spanning

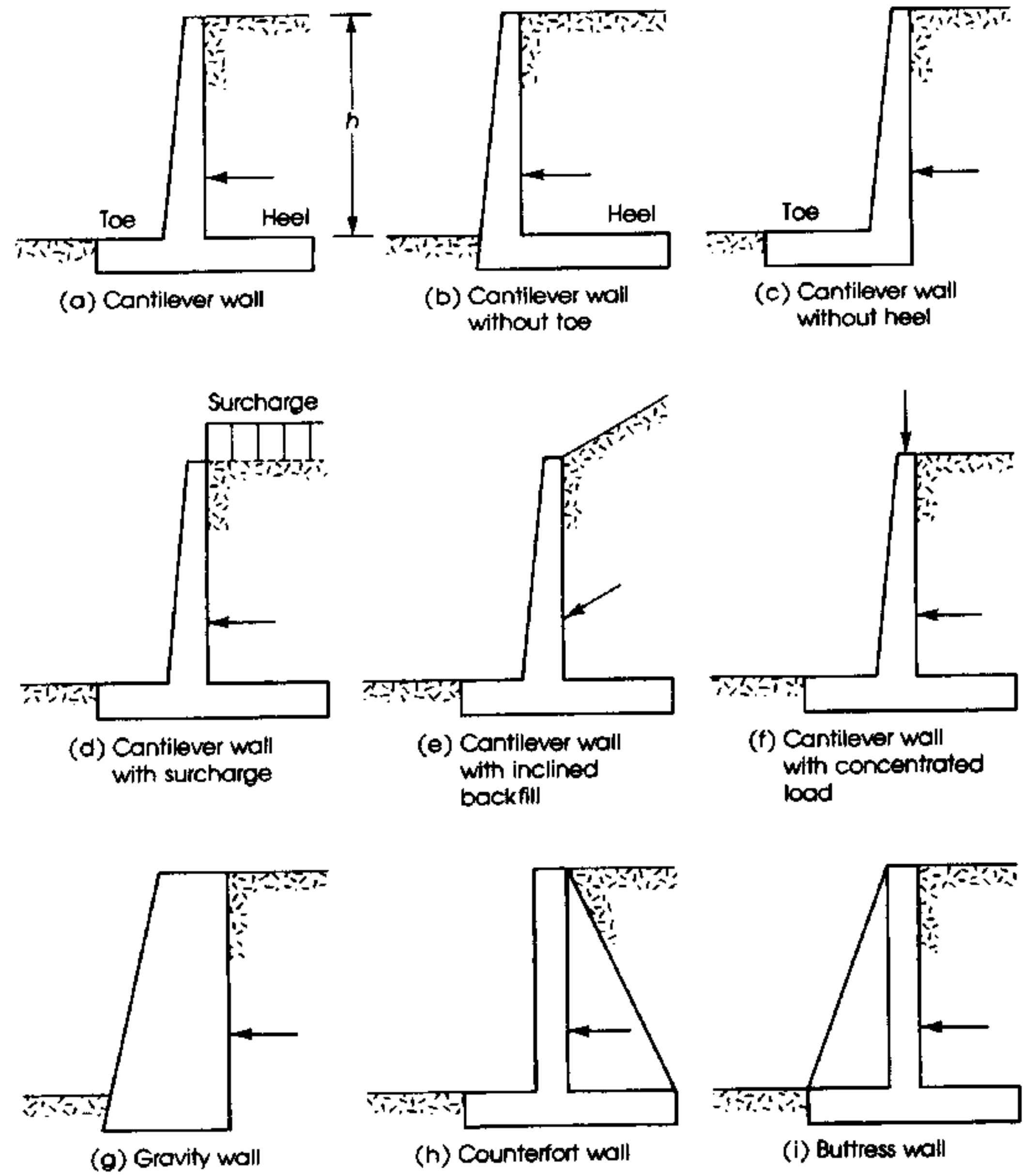


Figure 14.1 Types of retaining walls.

horizontally between counterforts, whereas the heel is designed as a slab supported on three sides (Fig. 14.1*h*).

5. The *buttressed retaining wall* is similar to the counterfort wall, but in this case the transverse walls are located on the opposite, visible side of the stem and act in compression (Fig 14.1*i*). The design of such walls becomes economical for heights greater than 20 ft. They are not popular because of the exposed buttresses.
6. *Bridge abutments* are retaining walls that are supported at the top by the bridge deck. The wall may be assumed fixed at the base and simply supported at the top.
7. *Basement walls* resist earth pressure from one side of the wall and span vertically from the basement-floor slab to the first-floor slab. The wall may be assumed fixed at the base and simply supported or partially restrained at the top.

14.3 FORCES ON RETAINING WALLS

Retaining walls are generally subjected to gravity loads and to earth pressure due to the retained material on the wall. Gravity loads due to the weights of the materials are well defined and can be calculated easily and directly. The magnitude and direction of the earth pressure on a retaining wall depends on the type and condition of soil retained and on other factors and cannot be determined as accurately as gravity loads. Several references on soil mechanics [1,2] explain the theories and procedure for determining the soil pressure on retaining walls. The stability of retaining walls and the effect of dynamic reaction on walls are discussed in Refs. 3 and 4.

Granular materials, such as sand, behave differently from cohesive materials, such as clay, or from any combination of both types of soils. Although the pressure intensity of soil on a retaining wall is complex, it is common to assume a linear pressure distribution on the wall. The pressure intensity increases with depth linearly, and its value is a function of the height of the wall and the weight and type of soil. The pressure intensity, p, at a depth h below the earth's surface may be calculated as follows:

$$p = Cwh \tag{14.1}$$

where w is the unit weight of soil and C is a coefficient that depends on the physical properties of soil. The value of the coefficient C varies from 0.3 for loose granular soil, such as sand, to about 1.0 for cohesive soil, such as wet clay. If the retaining wall is assumed absolutely rigid, a case of earth pressure at rest develops. Under soil pressure, the wall may deflect or move a small amount from the earth, and active soil pressure develops, as shown in Fig. 14.2. If the wall moves toward the soil, a passive soil pressure develops. Both the active and passive soil pressures are assumed to vary linearly with the depth of wall (Fig. 14.2). For dry, granular, noncohesive materials, the assumed linear pressure diagram is fairly satisfactory; cohesive soils or saturated sands behave in a different, nonlinear manner. Therefore, it is very common to use granular materials as backfill to provide an approximately linear pressure diagram and also to provide for the release or drainage of water from behind the wall.

For a linear pressure, the active and passive pressure intensities are determined as follows:

$$P_a = C_a wh \tag{14.2}$$

$$P_p = C_p wh \tag{14.3}$$

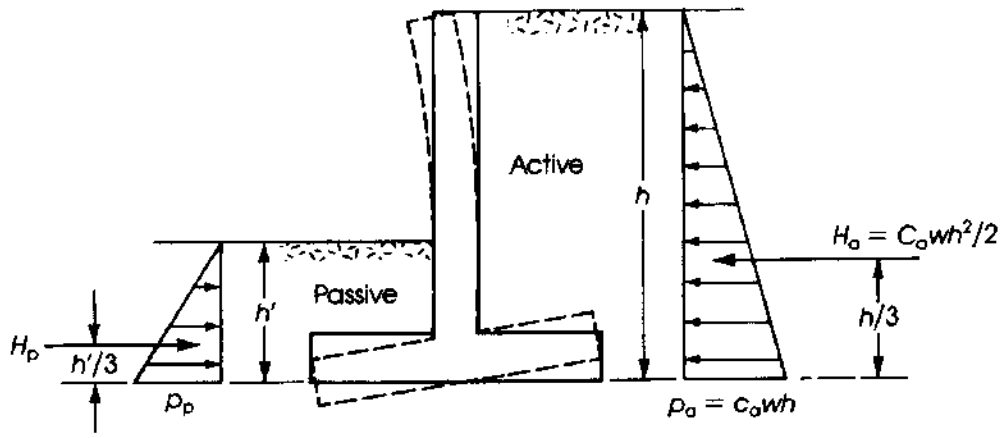


Figure 14.2 Active and passive earth pressure.

where C_a and C_p are the approximate coefficients of the active and passive pressures, respectively.

14.4 ACTIVE AND PASSIVE SOIL PRESSURES

The two theories most commonly used in the calculation of earth pressure are those of Rankine and Coulomb [1,6].

1. In Rankine's approach, the retaining wall is assumed to yield a sufficient amount to develop a state of plastic equilibrium in the soil mass at the wall surface. The rest of the soil remains in the state of elastic equilibrium. The theory applies mainly to a homogeneous, incompressible, cohesionless soil and neglects the friction between soil and wall. The active soil pressure at a depth h on a retaining wall with a horizontal backfill based on Rankine's theory is determined as follows:

$$P_a = C_a wh = wh\left(\frac{1-\sin\phi}{1+\sin\phi}\right) \tag{14.4}$$

where

$$C_a = \left(\frac{1-\sin\phi}{1+\sin\phi}\right)$$

ϕ = angle of internal friction of the soil (Table 14.1)

$$\text{Total active pressure, } H_a = \frac{wh^2}{2}\left(\frac{1-\sin\phi}{1+\sin\phi}\right) \tag{14.5}$$

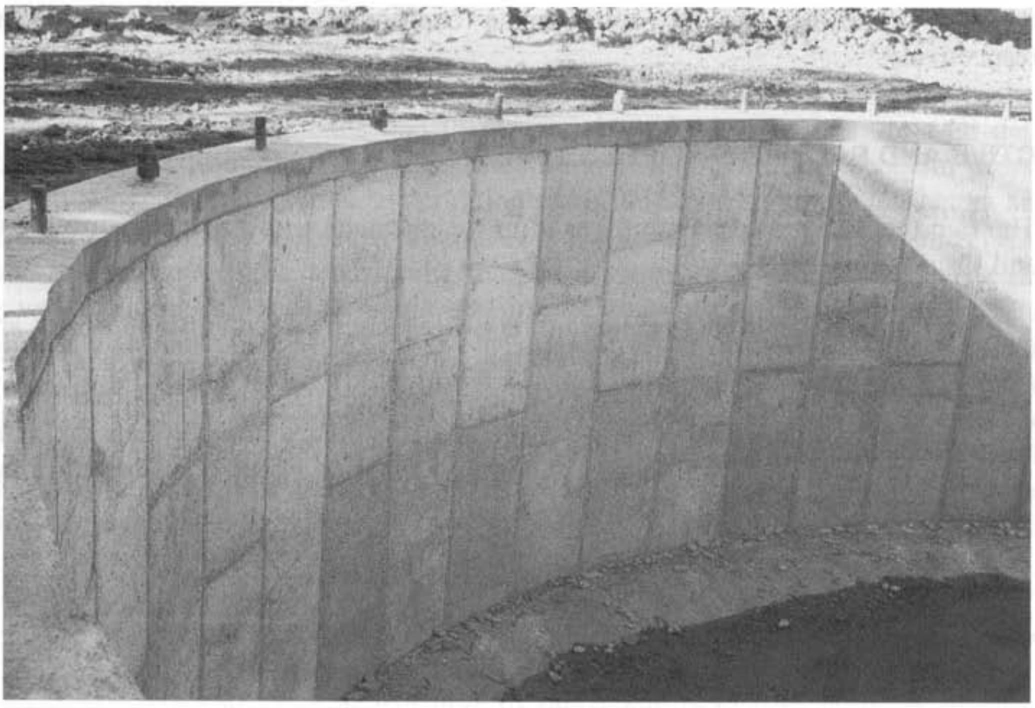
The resultant, H_a, acts at $h/3$ from the base (Fig. 14.2). When the earth is surcharged at an angle δ to the horizontal, then

$$C_a = \cos\delta\left(\frac{\cos\delta - \sqrt{\cos^2\delta - \cos^2\phi}}{\cos\delta + \sqrt{\cos^2\delta - \cos^2\phi}}\right) \tag{14.6}$$

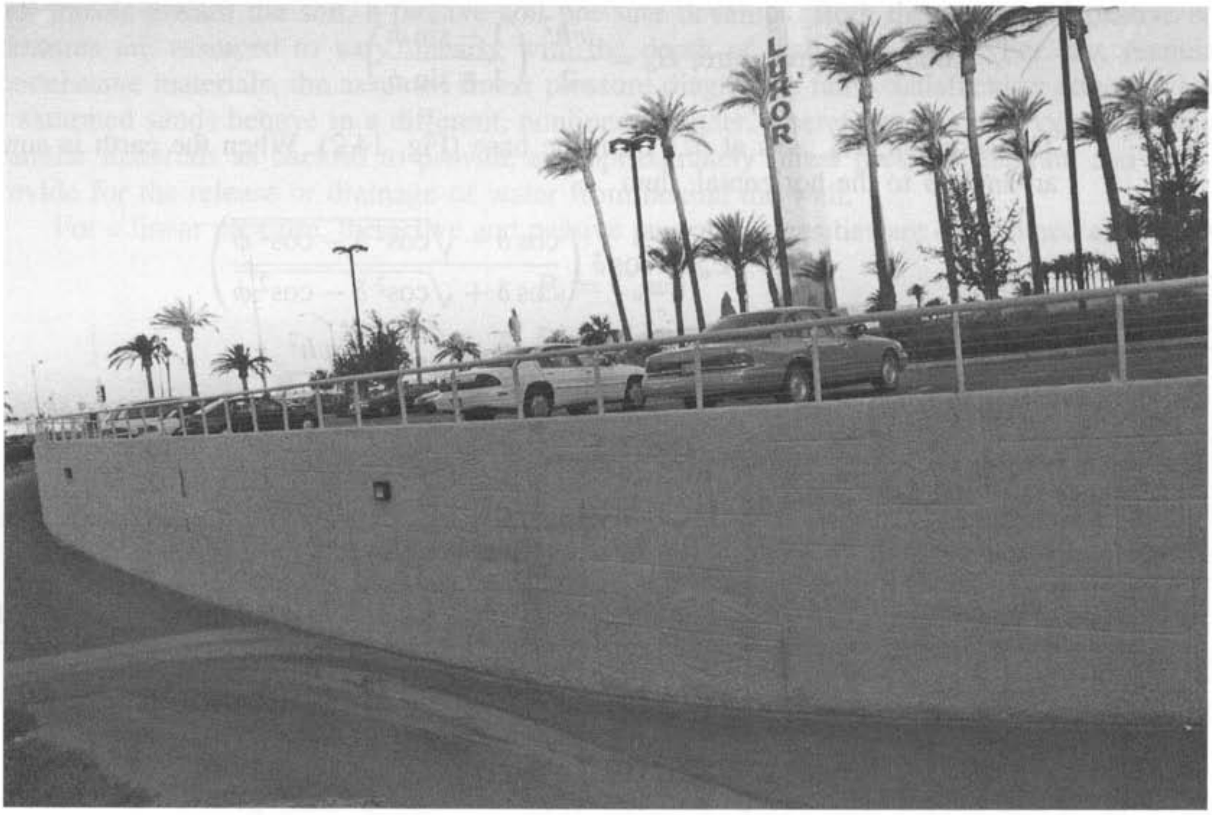
$$P_a = C_a wh \quad \text{and} \quad H_a = C_a\frac{wh^2}{2}$$

Table 14.1 Values of w and ϕ

Type of Backfill	Unit Weight, W		Angle of Internal Friction, ϕ
	pcf	*kg/m³*	
Soft clay	90–120	1440–1920	0°–15°
Medium clay	100–120	1600–1920	15°–30°
Dry loose silt	100–120	1600–1920	27°–30°
Dry dense silt	110–120	1760–1920	30°–35°
Loose sand and gravel	100–130	1600–2100	30°–40°
Dense sand and gravel	120–130	1920–2100	25°–35°
Dry loose sand, graded	115–130	1840–2100	33°–35°
Dry dense sand, graded	120–130	1920–2100	42°–46°



Reinforced concrete retaining wall.



Retaining wall in a parking area.

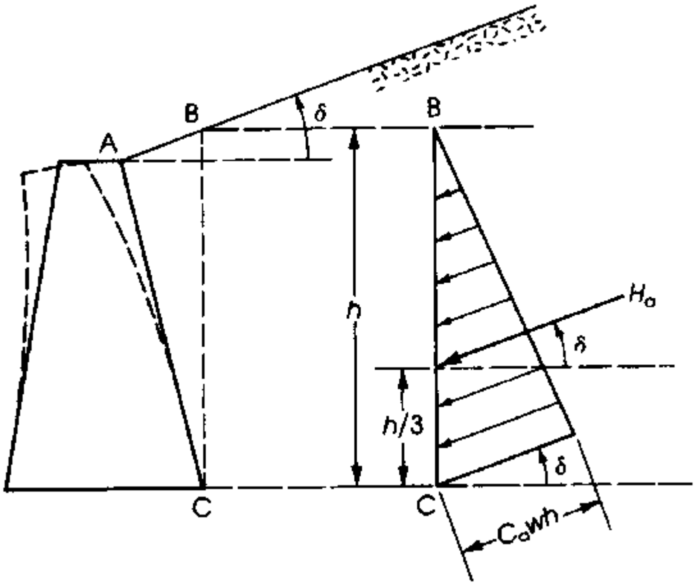


Figure 14.3 Active soil pressure with surcharge.

Table 14.2 Values of C_a

	ϕ (Angle of Internal Friction)						
Δ	28°	30°	32°	34°	36°	38°	40°
0°	0.361	0.333	0.307	0.283	0.260	0.238	0.217
10°	0.380	0.350	0.321	0.294	0.270	0.246	0.225
20°	0.461	0.414	0.374	0.338	0.306	0.277	0.250
25°	0.573	0.494	0.434	0.385	0.343	0.307	0.275
30°	0	0.866	0.574	0.478	0.411	0.358	0.315

The resultant, H_a, acts at $h/3$ and is inclined at an angle δ to the horizontal (Fig. 14.3). The values of C_a expressed by Eq. 14.6 for different values of δ and ϕ are shown in Table 14.2.

Passive soil pressure develops when the retaining wall moves against and compresses the soil. The passive soil pressure at a depth h on a retaining wall with horizontal backfill is determined as follows:

$$P_p = C_p wh = wh\left(\frac{1+\sin\phi}{1-\sin\phi}\right) \tag{14.7}$$

where

$$C_p = \left(\frac{1+\sin\phi}{1-\sin\phi}\right) = \frac{1}{C_a}$$

Total passive pressure is

$$H_p = \frac{wh^2}{2}\left(\frac{1+\sin\phi}{1-\sin\phi}\right) \tag{14.8}$$

The resultant, H_p, acts at $h'/3$ from the base (Fig. 14.2). When the earth is surcharged at an angle δ to the horizontal, then

$$C_p = \cos\delta \left(\frac{\cos\delta + \sqrt{\cos^2\delta - \cos^2\phi}}{\cos\delta - \sqrt{\cos^2\delta - \cos^2\phi}} \right) \tag{14.9}$$

$$P_p = C_p w h \quad \text{and} \quad H_p = C_p \frac{wh^2}{2}$$

H_p acts at $h'/3$ and is inclined at an angle δ to the horizontal (Fig. 14.4). The values of C_p expressed by Eq.14.9 for different values of δ and ϕ are shown in Table 14.3.
The values of ϕ and w vary with the type of backfill used. As a guide, common values of ϕ and w are given in Table 14.1.

2. In Coulomb's theory, the active soil pressure is assumed to be the result of the tendency of a wedge of soil to slide against the surface of a retaining wall. Hence, Coulomb's theory is referred to as the wedge theory. While it takes into consideration the friction of the soil on the retaining wall, it assumes that the surface of sliding is a plane, whereas in reality it is slightly curved. The error in this assumption is negligible in calculating the active soil pressure. Coulomb's equations to calculate the active and passive soil pressure are as follows:

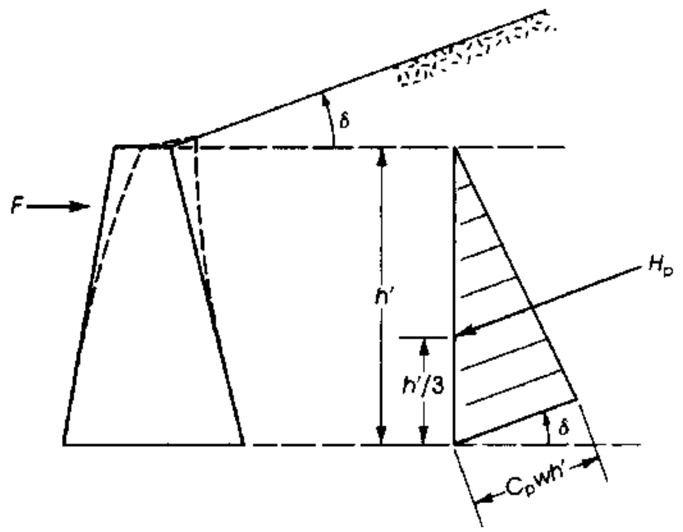


Figure 14.4 Passive soil pressure with surcharge.

Table 14.3 Values of C_p

	ϕ (Angle of Internal Friction)						
Δ	**28°**	**30°**	**32°**	**34°**	**36°**	**38°**	**40°**
0°	2.77	3.00	3.25	3.54	3.85	4.20	4.60
10°	2.55	2.78	3.02	3.30	3.60	3.94	4.32
20°	1.92	2.13	2.36	2.61	2.89	3.19	3.53
25°	1.43	1.66	1.90	2.14	2.40	2.68	3.00
30°	0	0.87	1.31	1.57	1.83	2.10	2.38

The active soil pressure is

$$P_a = C_a wh$$

where

$$C_a = \frac{\cos^2(\phi - \theta)}{\cos^2\theta\cos(\theta + \beta)\left[1 + \sqrt{\dfrac{\sin(\phi + \beta)\sin(\phi - \delta)}{\cos(\theta + \beta)\cos(\theta - \delta)}}\right]^2} \qquad (14.10a)$$

where

ϕ = angle of internal friction of soil
θ = angle of the soil pressure surface from the vertical
β = angle of friction along the wall surface (angle between soil and concrete)
δ = angle of surcharge to the horizontal

The total active soil pressure is

$$H_a = C_a \frac{wh^2}{2} = p_a \frac{h}{2}$$

When the wall surface is vertical, $\theta = 0°$, and if $\beta = \delta$, then C_a in Eq. 14.10*a* reduces to Eq. 14.6 of Rankine.

Passive soil pressure is

$$P_p = C_p wh' \quad \text{and} \quad H_p = \left(\frac{wh'^2}{2}\right) C_p = P_p \frac{h\prime}{2}$$

where

$$C_p = \frac{\cos^2(\phi + \theta)}{\cos^2\theta\cos(\theta - \beta)\left[1 - \sqrt{\dfrac{\sin(\phi + \beta)\sin(\phi + \delta)}{\cos(\theta - \beta)\cos(\phi - \delta)}}\right]^2} \qquad (14.10b)$$

The values of ϕ and w vary with the type of backfill used. As a guide, common values of ϕ and w are given in Table 14.1.

3. When the soil is saturated, the pores of the permeable soil are filled with water, which exerts hydrostatic pressure. In this case the buoyed unit weight of soil must be used. The buoyed unit weight (or submerged unit weight) is a reduced unit weight of soil and equals w minus the weight of water displaced by the soil. The effect of the hydrostatic water pressure must be included in the design of retaining walls subjected to a high water table and submerged soil. The value of the angle of internal friction may be used, as shown in Table 14.1.

14.5 EFFECT OF SURCHARGE

Different types of loads are often imposed on the surface of the backfill behind a retaining wall. If the load is uniform, an equivalent height of soil, h_s, may be assumed acting on the wall to account for the increased pressure. For the wall shown in Fig. 14.5, the horizontal pressure due to the surcharge is constant throughout the depth of the retaining wall.

$$h_s = \frac{w_s}{w} \qquad (14.11)$$

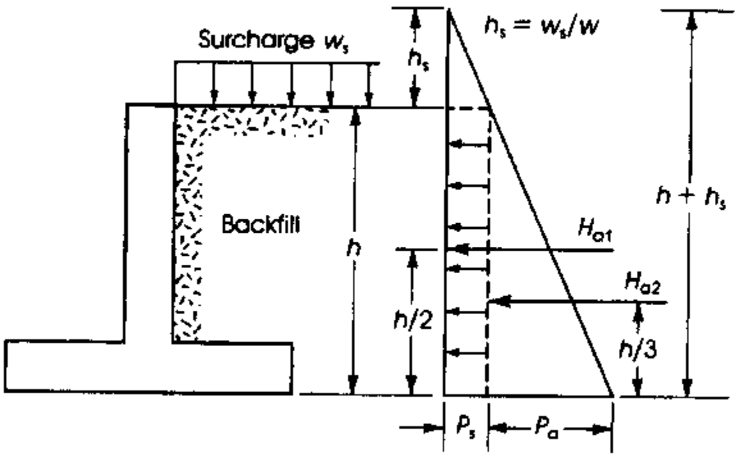


Figure 14.5 Surcharge effect under a uniform load.

where

h_s = equivalent height of soil (ft)
w_s = pressure of the surcharge (psf)
w = unit weight of soil (pcf)

The total pressure is

$$H_a = H_{a1} + H_{a2} = C_a w \left(\frac{h^2}{2} + hh_s\right) \tag{14.12}$$

In the case of a partial uniform load acting at a distance from the wall, only a portion of the total surcharge pressure affects the wall (Fig. 14.6).

It is a common practice to assume that the effective height of pressure due to partial surcharge is h', measured from point B to the base of the retaining wall [1]. The line AB forms an angle of 45° with the horizontal.

In the case of a wheel load acting at a distance from the wall, the load is to be distributed over a specific area, which is usually defined by known specifications such as AASHTO and AREA [4] specifications.

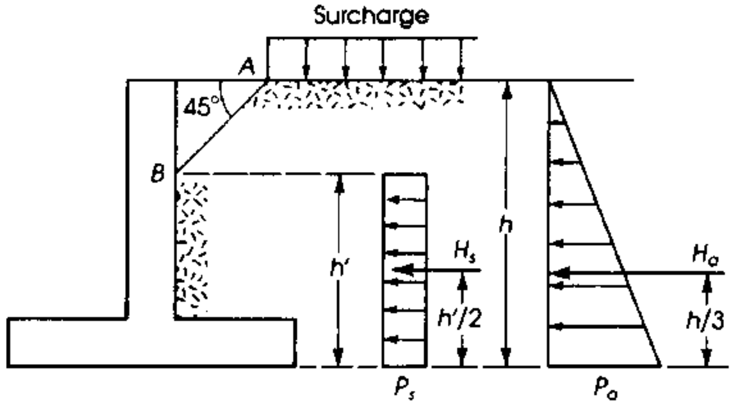


Figure 14.6 Surcharge effect under a partial uniform load at a distance from the wall.

14.6 FRICTION ON THE RETAINING WALL BASE

The horizontal component of all forces acting on a retaining wall tends to push the wall in a horizontal direction. The retaining wall base must be wide enough to resist the sliding of the wall. The coefficient of friction to be used is that of soil on concrete for coarse granular soils and the shear strength of cohesive soils [4]. The coefficients of friction μ that may be adopted for different types of soil are as follows:

- Coarse-grained soils without silt, $\mu = 0.55$
- Coarse-grained soils with silt, $\mu = 0.45$
- Silt, $\mu = 0.35$
- Sound rock, $\mu = 0.60$

The total frictional force, F, on the base to resist the sliding effect is

$$F = \mu R + H_p \tag{14.13}$$

where

μ = the coefficient of friction
R = the vertical force acting on the base
H_p = passive resisting force

The factor of safety against sliding is

$$\text{Factor of safety} = \frac{F}{H_{\text{ah}}} = \frac{\mu R + H_p}{H_{\text{ah}}} \geq 1.5 \tag{14.14}$$

where H_{ah} is the horizontal component of the active pressure, H_a. The factor of safety against sliding should not be less than 1.5 if the passive resistance H_p is neglected and should not be less than 2.0 if H_p is taken into consideration.

14.7 STABILITY AGAINST OVERTURNING

The horizontal component of the active pressure, H_a, tends to overturn the retaining wall about the point zero on the toe (Fig. 14.7). The overturning moment is equal to $M_0 = H_a h/3$. The weight of the concrete and soil tends to develop a balancing moment, or rightening moment, to resist the overturning moment. The balancing moment for the case of the wall shown in Fig. 14.7 is equal to

$$M_b = w_1 x_1 + w_2 x_2 + w_3 x_3 = \sum wx$$

The factor of safety against overturning is

$$\text{Factor of safety} = \frac{M_b}{M_o} = \frac{\sum wx}{\frac{H_a h}{3}} \geq 2.0 \tag{14.15}$$

This factor of safety should not be less than 2.0.

The resultant of all forces acting on the retaining wall, R_A, intersects the base at point C (Fig. 14.7). In general, point C does not coincide with the center of the base, L, thus causing eccentric loading on the footing. It is desirable to keep point C within the middle third to get the whole footing under soil pressure. (The case of a footing under eccentric load was discussed in Chapter 13.)

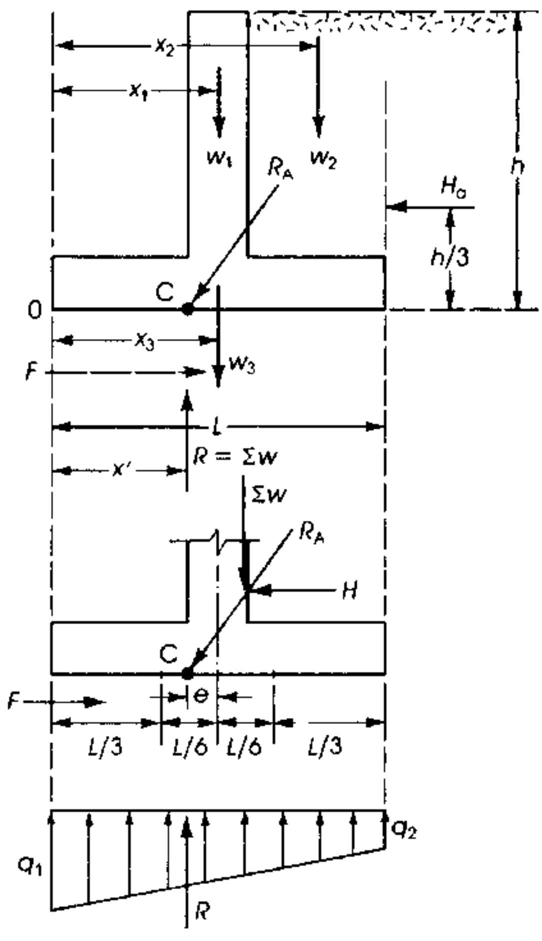


Figure 14.7 Overturning of a cantilever retaining wall.

14.8 PROPORTIONS OF RETAINING WALLS

The design of a retaining wall begins with a trial section and approximate dimensions. The assumed section is then checked for stability and structural adequacy. The following rules may be used to determine the approximate sizes of the different parts of a cantilever retaining wall.

1. *Height of the wall:* The overall height of the wall is equal to the difference in elevation required plus 3 to 4 ft, which is the estimated frost penetration depth in northern states.
2. *Thickness of the stem:* The intensity of the pressure increases with the depth of the stem and reaches its maximum value at the base level. Consequently the maximum bending moment and shear in the stem occur at its base. The stem base thickness may be estimated as $\frac{1}{12}$ to $\frac{1}{10}$ of the height, h. The thickness at the top of the stem may be assumed to be 8 to 12 in. Because retaining walls are designed for active earth pressure, causing a small deflection of the wall, it is advisable to provide the face of the wall with a batter (taper) of $\frac{1}{4}$ in. per foot of height, h, to compensate for the forward deflection. For short walls up to 10 ft high, a constant thickness may be adopted.
3. *Length of the base:* An initial estimate for the length of the base of $\frac{2}{5}$ to $\frac{2}{3}$ of the wall height, h, may be adopted.

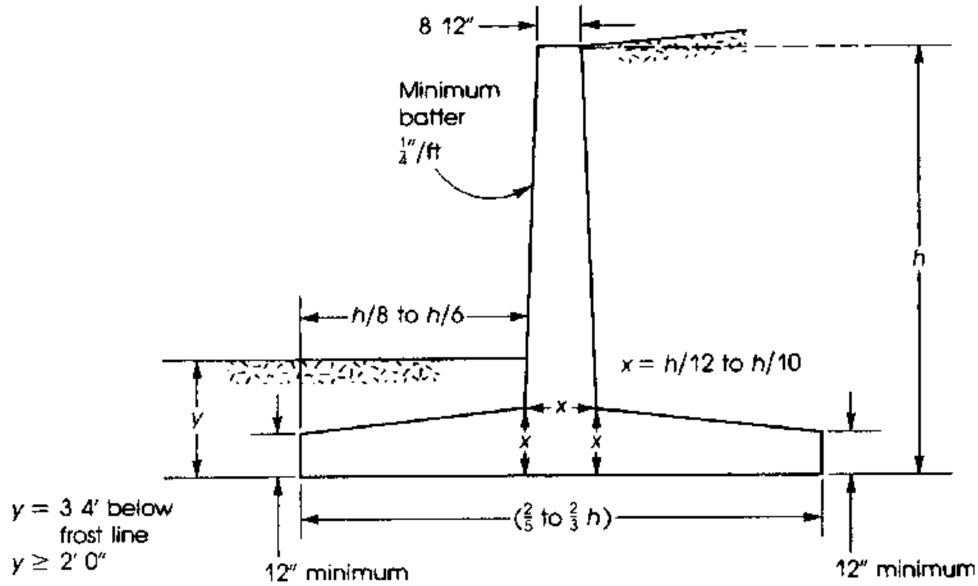


Figure 14.8 Trial proportions of a cantilever retaining wall.

4. *Thickness of the base:* The base thickness below the stem is estimated as the same thickness of the stem at its base, that is, $\frac{1}{12}$ to $\frac{1}{10}$ of the wall height. A minimum thickness of about 12 in. is recommended. The wall base may be of uniform thickness or tapered to the ends of the toe and heel, where the bending moment is 0.

The approximate initial proportions of a cantilever retaining wall are shown in Fig. 14.8.

14.9 DESIGN REQUIREMENTS

The ACI Code, Chapter 14, provides methods for bearing wall design. The main requirements are as follows:

1. The minimum thickness of bearing walls is $\frac{1}{25}$ the supported height or length, whichever is shorter, but not less than 4 in.
2. The minimum area of the horizontal reinforcement in the wall is $0.0025bh$, where bh is the gross concrete wall area. This value may be reduced to $0.0020bh$ if no. 5 or smaller deformed bars with $f_y \geq 60$ ksi are used. For welded wire fabric (plain or deformed), the minimum steel area is $0.0020bh$.
3. The minimum area of the vertical reinforcement is $0.0015bh$, but it may be reduced to $0.0012bh$ if no. 5 or smaller deformed bars with $f_y \geq 60$ ksi are used. For welded wire fabric (plain or deformed), the minimum steel area is $0.0012bh$.
4. The maximum spacing of the vertical or the horizontal reinforcing bars is the smaller of 18 in. or three times the wall thickness.
5. If the wall thickness exceeds 10 in., the vertical and horizontal reinforcement should be placed in two layers parallel to the exterior and interior wall surfaces, as follows:

   For exterior wall surfaces, at least $\frac{1}{2}$ of the reinforcement A_s (but not more than $\frac{2}{3}A_s$) should have a minimum concrete cover of 2 in. but not more than $\frac{1}{3}$ of the wall thickness.