Markov Chains

Markov Chains

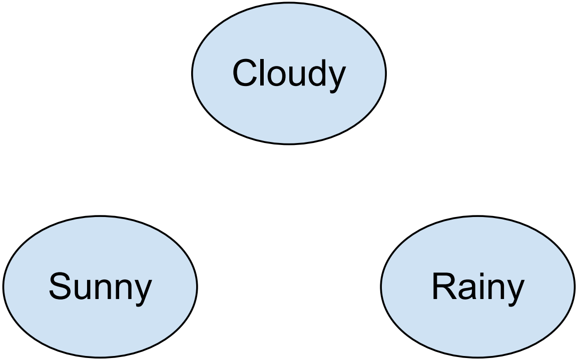
Stochastic Processes and Markov Chains

Stochastic Processes

- It is often possible to represent the behaviour of a system by a collection of “states”.
- The system being modelled is assumed to occupy one and only one state at any moment in time.
- The evolution of the system is represented by transitions from state to state.

Stochastic Processes

An example of this could be the behaviour of the weather:



Formal definition of a Stochastic Process

A stochastic process is defined as a family of random variables $\{X(t),\ t \in T\}$.

- T represents the index set.
  - T can be discrete: $T = \{0, 1, 2, 3, ...\}$: *Discrete time stochastic process*.
  - T can be continuous: $T = \mathbb{R}_{\geq 0}$: *Continuous time stochastic process*.
- The values assumed by the random variables $X(t)$ are called states. The set of all possible values of $X(t)$ is called the state space: Ω.
  - Ω can be discrete: $\{\text{Rainy}, \text{Sunny}, \text{Cloudy}\}$
  - Ω can also be continuous.

Chains

When *Omega* is discrete, the stochastic process is called a *chain*.
For the rest of the course we will be concerned with:

Homogeneous Markov Chains.

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- Discrete
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Applications:

- Biology
- Economics
- (Queueing Theory)
- (Board games)

Discrete Markov Chains

Definition of a Discrete Markov Chain

For a discrete Markov chain we observe the state of a system at a discrete, but infinite set of times. We may take:

$$T = \mathbb{N} = \{0, 1, 2, \dots\}$$

The state of the system is then denoted as $X_0, X_1, X_2, \dots$. A discrete time Markov chain is then a stochastic process that satisfies the following relationship:

$$P(X_{n+1} = x_{n+1} | X_n = x_n, \dots, X_0 = x_0) = P(X_{n+1} = x_{n+1} | X_n = x_n)$$

For ease of notation we write the probability of going from state i to state j at time period n as:

$$p_{ij}(n) = P(X_{n+1} = j | X_n = i)$$

Transition Probability Matrix

$$P(n) = \begin{pmatrix} p_{00}(n) & p_{01}(n) & p_{02}(n) & \dots & p_{0j}(n) & \dots \\ p_{20}(n) & p_{11}(n) & p_{12}(n) & \dots & p_{1j}(n) & \dots \\ p_{20}(n) & p_{21}(n) & p_{22}(n) & \dots & p_{2j}(n) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i0}(n) & p_{i1}(n) & p_{i2}(n) & \dots & p_{ij}(n) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$P(n)$ is a stochastic matrix:

- $P(n)$ is a square matrix.
- $\sum_j p_{ij}(n) = 1$ for all $i \in \Omega$
- $p_{ij}(n) \geq 0$ for all $i, j \in \Omega$

Homogeneous Markov Chains

In a Homogeneous Markov Chain the transition probabilities do not depend on the amount of time that has passed:

$$P(k) = P(0) \text{ for all } k$$

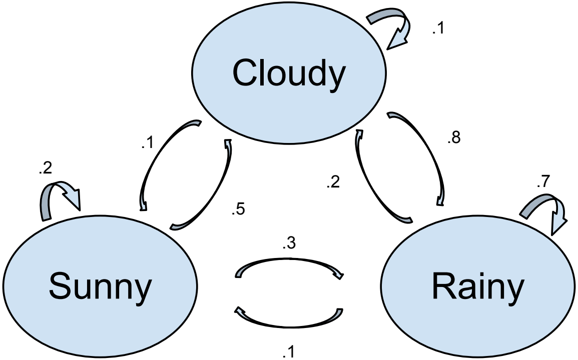
This is what we consider in this course.

Weather Example

This stochastic matrix:

$$\begin{pmatrix} .2 & .5 & .3 \\ .1 & .1 & .8 \\ .1 & .2 & .7 \end{pmatrix}$$

corresponds to:



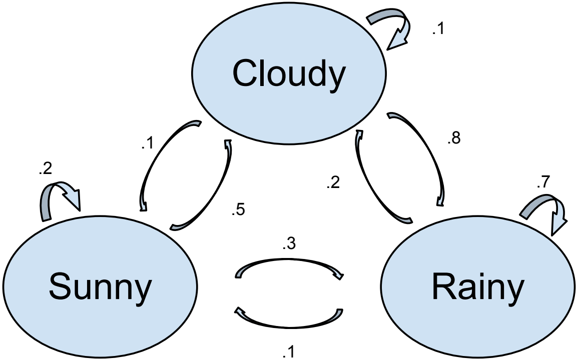
State Vector

We can describe the state of a Markov Chain by a vector: $\pi^{(n)}$.
$\pi_j^{(n)}$ denotes the probability of being in State j at time n:

- $\sum_{j \in \Omega} \pi_j^{(n)} = 1$ for all n
- $\pi_j^{(n)} \geq 0$ for all j, n.

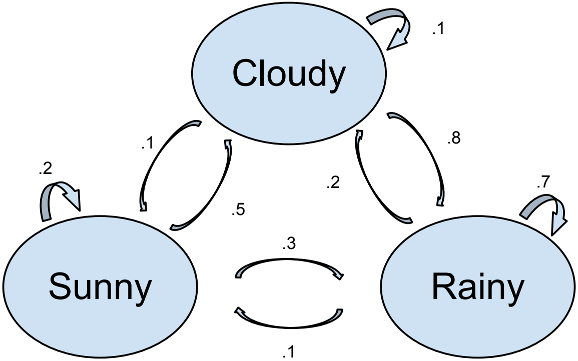
Weather example

Assume $\pi^{(0)} = (1, 0, 0)$, what is $\pi^{(1)}$?



Weather example

Assume $\pi^{(0)} = (1, 0, 0)$, what is $\pi^{(1)}$?



$$\pi^{(1)} = (.2, .5, .3)$$

Powers of Transition Probability Matrix

In general:

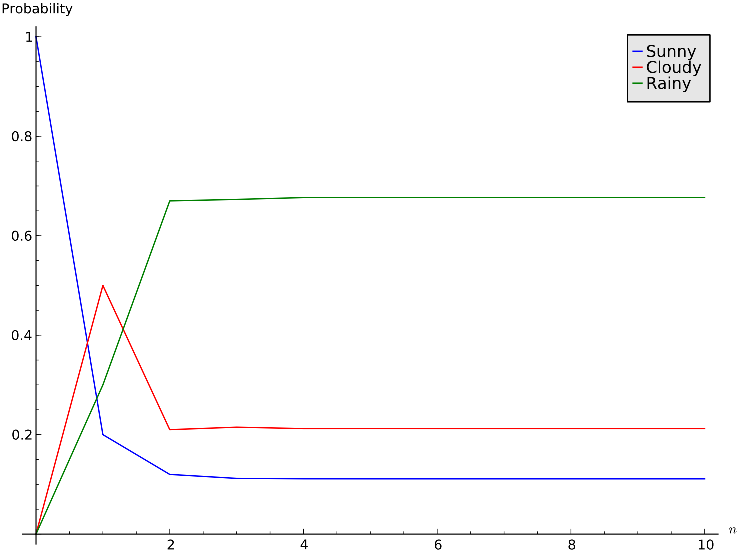
$$\pi^{(n+1)} = \pi^{(n)} P$$

Thus:

$$\pi^{(n)} = \pi^{(0)} P^n$$

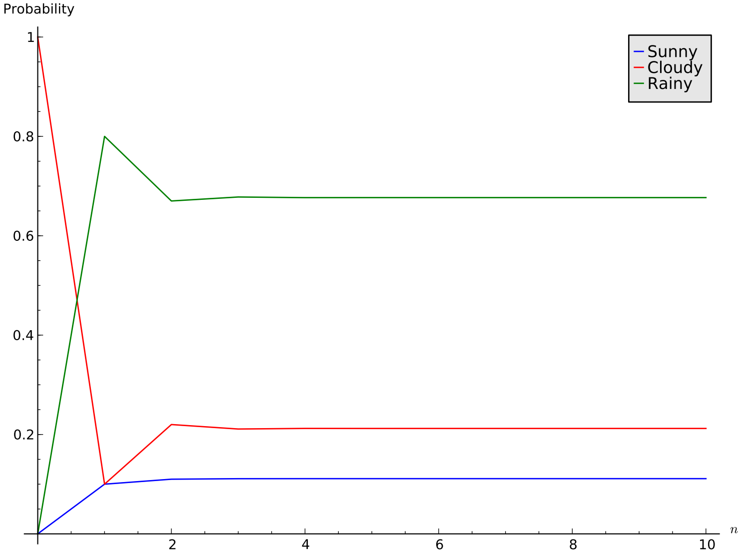
Weather Example

$$\pi^{(0)} = (1, 0, 0)$$



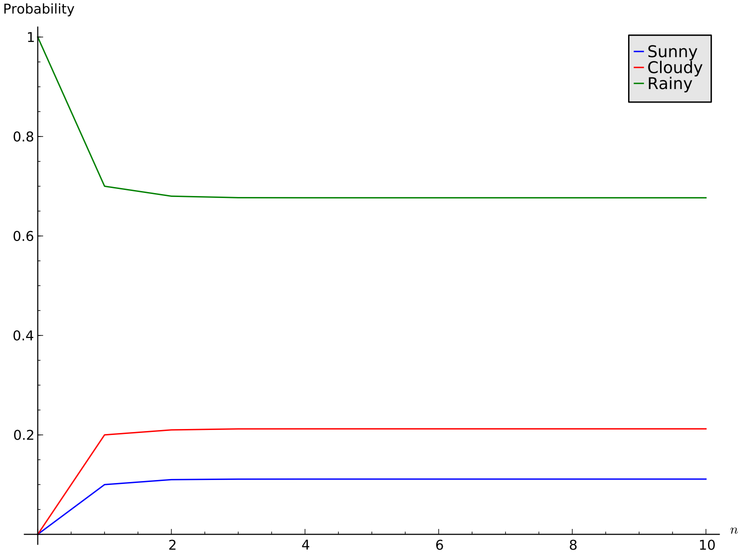
Weather Example

$$\pi^{(0)} = (0, 1, 0)$$



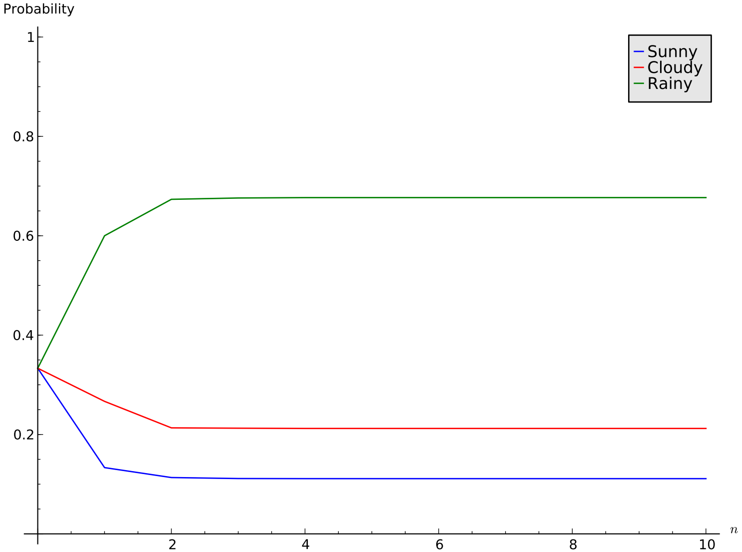
Weather Example

$$\pi^{(0)} = (0, 0, 1)$$



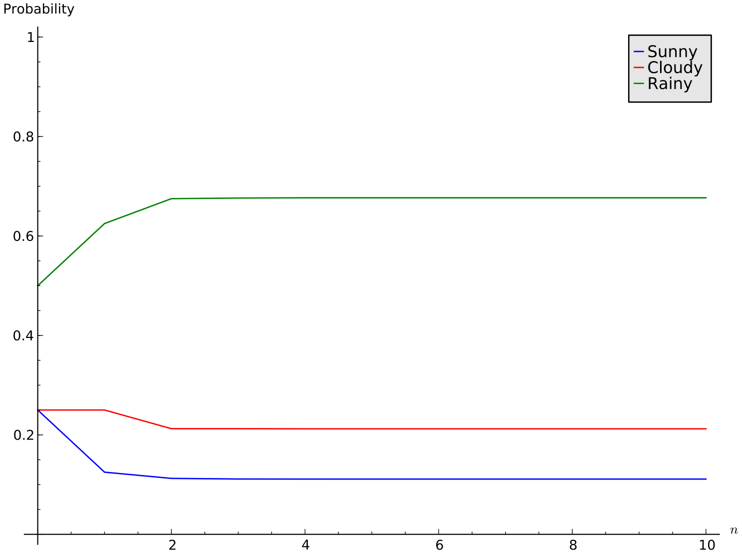
Weather Example

$$\pi^{(0)} = (1/3, 1/3, 1/3)$$



Weather Example

$$\pi^{(0)} = (.25, .25, .5)$$



Limiting and Steady State Distribution

- If the limit:

$$\lim_{n \to \infty} P^n$$

exists, then the probability distribution $\pi = \lim_{n \to \infty} \pi^{(0)} P^n$ is called the limiting distribution. (Note that this can depend on $\pi^{(0)}$).

- If a limiting distribution exists and it is independent of $\pi^{(0)}$ it is called a steady state distribution. Such a distribution satisfies:

$$\pi = \pi P$$

Weather Example

$$\pi = \pi P \Rightarrow \begin{cases} \pi_1 = .2\pi_1 + .1\pi_2 + .1\pi_3 \\ \pi_2 = .5\pi_1 + .1\pi_2 + .2\pi_3 \\ \pi_3 = .3\pi_1 + .8\pi_2 + .7\pi_3 \end{cases}$$

Solving this gives:

$$\begin{cases} \pi_1 = \frac{11}{67} c \\ \pi_2 = \frac{21}{67} c \\ \pi_3 = c \end{cases}$$

For some c. Recalling that $\pi_1 + \pi_2 + \pi_3 = 1$ gives:

$$\begin{cases} \pi_1 = \frac{1}{9} \approx .11 \\ \pi_2 = \frac{7}{33} \approx .21 \\ \pi_3 = \frac{67}{99} \approx .68 \end{cases}$$

Continuous Markov Chains

Transition Rates

In a discrete time Markov chain:

- $T = \{1, 2, 3, \ldots\}$
- Interactions between states given by transition probabilities

In a continuous time Markov chain:

- $T = \mathbb{R}_{\geq 0}$
- Interactions between states given by *rates at which transitions happen*.

Transition Rate Matrix

$$Q(t) = \begin{pmatrix} q_{00}(t) & q_{01}(t) & q_{02}(t) & \dots & q_{0j}(t) & \dots \\ q_{10}(t) & q_{11}(t) & q_{12}(t) & \dots & q_{1j}(t) & \dots \\ q_{20}(t) & q_{21}(t) & q_{22}(t) & \dots & q_{2j}(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{i0}(t) & q_{i1}(t) & q_{i2}(t) & \dots & q_{ij}(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$Q(t)$ is a transition rate matrix:

- $Q(n)$ is a square matrix.
- $q_{ii}(t) = -\sum_{j \neq i} q_{ij}(t)$ for all $i \in \Omega$
- $q_{ij}(n) \geq 0$ for all $i \neq j \in \Omega$

Homogeneous Markov Chains

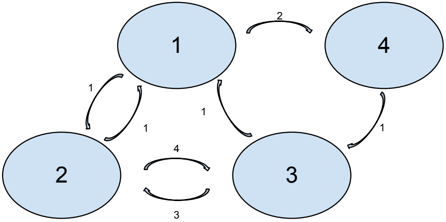
In a Homogeneous Markov Chain the transition rates do not depend on the amount of time that has passed:

$$Q(t) = Q(0) \text{ for all } t$$

This is what we consider in this course.

Example

The following continuous Markov chain:



Has transition rate matrix:

$$Q = \begin{pmatrix} -3 & 1 & 0 & 2 \\ 1 & -5 & 4 & 0 \\ 1 & 3 & -4 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Transient and Steady State Distribution

- We have the following expression for the transient distribution:

$$\frac{d\pi(t)}{dt} = \pi(t)Q$$

thus:

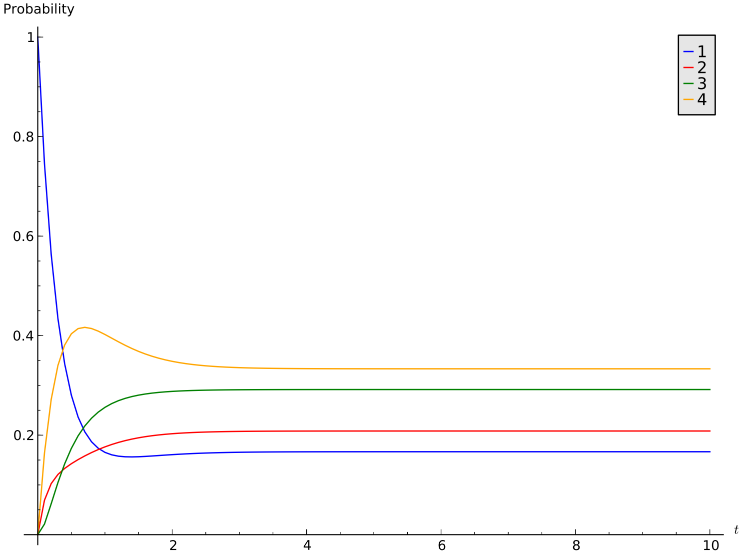
$$\pi(t) = \pi(0)e^{Qt} = \pi(0)\left(\mathbb{I} + \sum_{k=1}^{\infty} \frac{Q^k t^k}{k!}\right)$$

- The steady state distribution (if it exists) may be obtained by solving the following equation:

$$\pi Q = 0$$

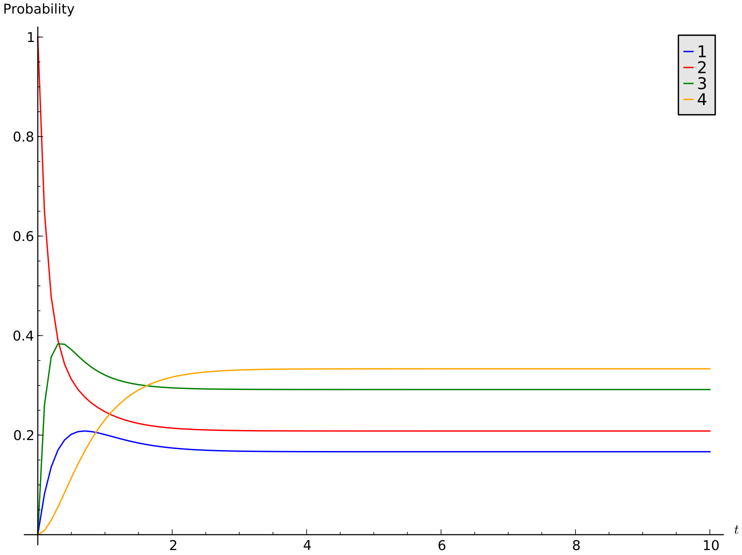
Example

$$\pi(0) = (1, 0, 0, 0)$$



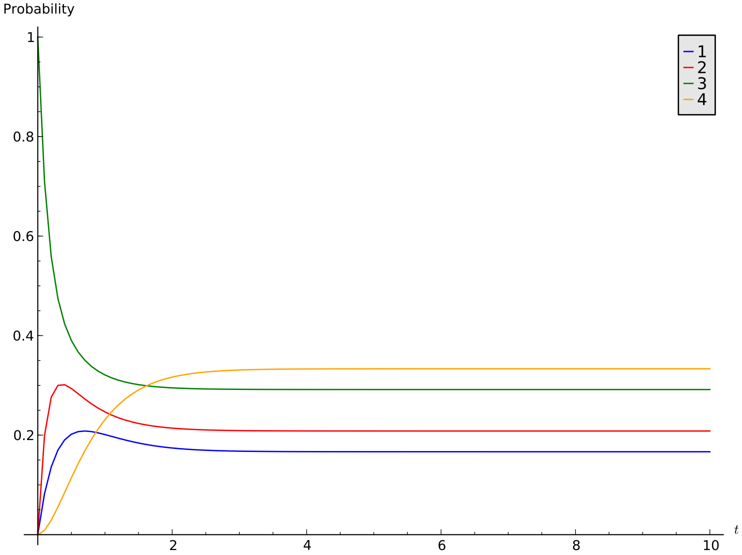
Example

$$\pi(0) = (0, 1, 0, 0)$$



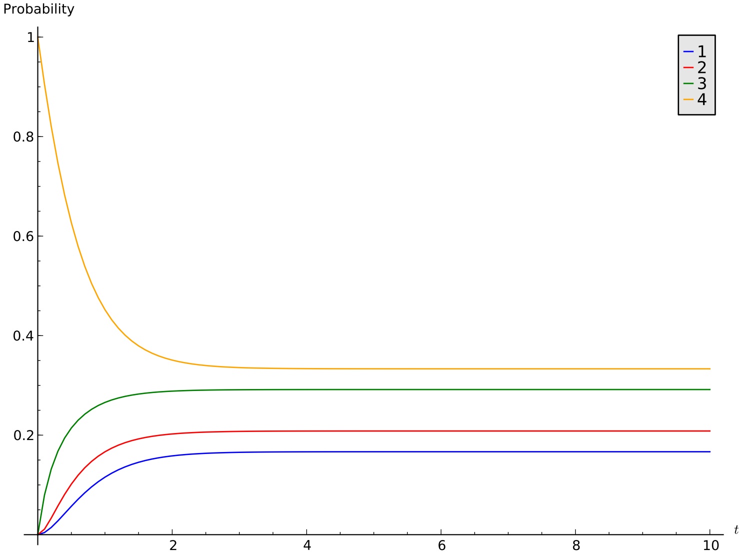
Example

$$\pi(0) = (0, 0, 1, 0)$$



Example

$$\pi(0) = (0, 0, 0, 1)$$



Weather Example

$$\pi Q = 0 \Rightarrow \begin{cases} -3\pi_1 + 1\pi_2 + 1\pi_3 + 0\pi_4 = 0 \\ 1\pi_1 - 5\pi_2 + 3\pi_3 + 0\pi_4 = 0 \\ 0\pi_1 + 4\pi_2 - 4\pi_3 + 1\pi_4 = 0 \\ 2\pi_1 + 0\pi_2 + 0\pi_3 - \pi_4 = 0 \end{cases}$$

Solving this gives:

$$\begin{cases} \pi_1 = c \\ \pi_2 = \frac{5}{4}c \\ \pi_3 = \frac{7}{4}c \\ \pi_4 = 2c \end{cases}$$

For some c. Recalling that $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$ gives:

$$\begin{cases} \pi_1 = \frac{1}{6} \approx .17 \\ \pi_2 = \frac{5}{24} \approx .21 \\ \pi_3 = \frac{7}{24} \approx .29 \\ \pi_4 = \frac{1}{3} \approx .33 \end{cases}$$

Equivalence of Continuous and Discrete Markov Chains

There is an equivalence between Continuous and Discrete Markov Chains:

- If $\pi P = \pi$ then:

  $$\pi(P - \mathbb{I}) = 0$$

  $(P - \mathbb{I})$ has all the properties of a transition rate matrix (check this)

- If $\pi Q = 0$ then:

  $$\pi(Q\Delta t + \mathbb{I}) = \pi$$

  If we take Δt to be *sufficiently* small (so that the probability of 2 transitions occurring in 1 time period is negligible) then $(Q\Delta t + \mathbb{I})$ is a stochastic matrix corresponding to the discretized Markov chain. 1 possibility is to take:

  $$\Delta t \leq \frac{1}{\max_i |q_{ii}|}$$